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Some Properties of Fuzzy Dot PS-Subalgebras of PS-Algebra

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Abstract. In this paper, we introduce the notion of fuzzy dot PS-subalgebras in PS-algebras and establish its various properties.

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1. Introduction

The concept of fuzzy set was initiated by L.A.Zadeh in 1965 [11]. Since then these ideas have been applied to other algebraic structures such as groups, rings, modules, vector spaces and topologies. K.Iseki and S.Tanaka [1] introduced the concept of BCK-algebras in 1978 and K.Iseki [2] introduced the concept of BCI-algebras in 1980. It is known that the class of BCK –algebras is a proper subclass of the class of BCI algebras. T.Priya and T.Ramachandran [5,6.7] introduced the class of PS-algebras , which is an another generalization of BCI / BCK/Q / KU algebras. In this paper, we introduce the concept of fuzzy dot PS-subalgebras of PS-algebras as a generalization of a fuzzy PS-subalgebra of a PS-algebra and we investigate few basic properties related to fuzzy dot PS-subalgebra in detail.

2. Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.

Definition 2.1.[1] A BCK- algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) $(x * y) * (x * z) \leq (z * y)$
ii) $x * (x * y) \leq y$
iii) $x \leq x$
iv) $x \leq y$ and $y \leq x \Rightarrow x=y$
v) $0 \leq x \Rightarrow x=0$, where $x \leq y$ is defined by $x * y = 0$,for all x, y, z $\in$ X.

Definition 2.2.[2] A BCI- algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) $(x * y) *(x * z) \leq (z*y)$

ii) $x * (x * y) \leq y$

iii) $x \leq x$

iv) $x \leq y$ and $y \leq x \Rightarrow x = y$

v) $x \leq 0 \Rightarrow x = 0$, where $x \leq y$ is defined by $x * y = 0$,for all x, y, z $\in$ X.

Definition 2.3. A Q-algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) $x * x = 0$

ii) $x * 0 = x$

iii) $(x * y)*z = (x * z) * y$, where $x \leq y$ is defined by $x * y = 0$,for all x, y, z $\in$ X.

Definition 2.4.[3] A d-algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) $x * x = 0$

ii) $0 * x = 0$

iii) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all x, y $\in$ X.

Definition 2.5. [4,10] A KU-algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:

i) $(x * y) * ((y * z) * (x * z)) =0$

ii) $x * 0 = 0$

iii) $0 * x = x$

iv) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all x, y, z $\in$ X.

Definition 2.6. [5] A nonempty set X with a constant 0 and a binary operation ‘ * ‘ is called PS – Algebra if it satisfies the following axioms.

1. $x * x = 0$
2. $x * 0 = 0$
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$, $\forall$ x ,y $\in$ X.

Definition 2.7. [8] Let S be a non empty sub set of a PS-algebra X. Then S is called a subalgebra of X if $x*y \in S$, for all x , y $\in$ S.

Definition 2.8. [6,9] A map $f : X \to Y$ is called a homomorphism if $f (x * y) = f(x)*f(y)$, for all x ,y $\in$ X, where X and Y are PS-algebras.

Definition 2.9. [11] Let X be a non-empty set. A fuzzy subset μ of the set X is a mapping $\mu : X \to [0,1]$.

Definition 2.10. [11] For any subsets λ and μ of a set X , $(\lambda \cap \mu)(x) = \min \{ \lambda(x) , \mu(x) \}$.

Definition 2.11. [3,11] A fuzzy relation μ on a set X is of a fuzzy subset of X x X, that is , a map $\mu : X \times X \rightarrow [0,1]$.

Definition 2.12. [6] A fuzzy set μ in a PS-algebra X is called a fuzzy PS- sub algebra of X if $\mu(x * y) \geq \min\{\mu(x),\mu(y)\}$,for all $x,y \in X$.

Remark :

(i) For any fuzzy subsets λ and μ of a set X, we define $\lambda \subseteq \mu \Leftrightarrow \lambda(x) \leq \mu(x)$.

(ii) Let $f : X \rightarrow Y$ be a function from a set X to a set Y and let μ be a fuzzy subset of X. Then the fuzzy subset λ of Y is defined by

$$\lambda(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \phi, \forall y \in Y \\ 0 & \text{Otherwise} \end{cases}$$

is called the image of μ under f, denoted by f(μ).If λ is a fuzzy subset of Y , then the fuzzy subset μ of X is given by $\mu(x) = \lambda(f(x))$, for all $x \in X$, is called the pre-image of λ under f and is denoted by $f^{-1}(\lambda)$.

4. Fuzzy Dot PS-Subalgebras of PS-algebras

For braveity, here X denotes PS-algebra, unless otherwise specified.

Definition 3.1. A fuzzy subset μ of X is called a fuzzy dot PS-subalgebra of a PS-algebra X, if $\mu(x * y) \geq \mu(x).\ \mu(y)$, for all $x,y \in X$.

Example 3.2. Consider a PS-algebra X = { 0,a,b} having the following Cayley table.

*	0	a	b
0	0	b	a
A	0	0	b
B	0	b	0

Define a fuzzy set μ in X by $\mu(0) = 0.8$, $\mu(a) = \mu(b) = 0.7$. It is easy to verify that μ is a fuzzy dot PS-subalgebra of a PS-algebra X.

Example 3.3. Consider a PS-algebra X = { 0,1,2,3}having the following Cayley table.

*	0	1	2	3
0	0	2	1	3
1	0	0	0	2
2	0	0	0	2
3	0	2	2	0

Define a fuzzy set μ in X by $\mu(0) = 0.8$, $\mu(1) = 0.5$, $\mu(2) = 0.4$ and $\mu(3) = 0.7$. It is easy to verify that μ is a fuzzy dot PS-subalgebra of a PS-algebra X.

Remark :

1. Every fuzzy PS-subalgebra is a fuzzy dot PS-subalgebra of a PS-algebra but the converse is not true.
2. From the above example 3.3, it is seen that , the fuzzy dot PS-subalgebra μ is not a fuzzy PS-subalgebra, because $\mu(1*3) = \mu(2) = 0.4 < \mu(1) = \min\{ \mu(1) , \mu(3) \}$.

Theorem 3.4. If λ and μ are fuzzy dot PS-subalgebras of a PS-algebra X , then $\lambda \cap \mu$ is also a fuzzy dot PS-subalgebra of X.

Proof : Let x , y $\in$ X. Then

$$\begin{aligned}(\lambda\cap\mu)(x * y) &= \min \{ \lambda (x * y) , \mu(x * y) \} \\ &\geq \min \{\lambda (x) . \lambda (y) , \mu(x) . \mu(y) \} \\ &\geq (\min \{\lambda (x), \mu(x)\}) .(\min \{ \lambda (y), \mu(y) \}) \\ &= ((\lambda\cap\mu)(x)). ((\lambda\cap\mu)(y))\end{aligned}$$

Thus $(\lambda \cap \mu)$ is also a fuzzy dot PS-subalgebra of X.

Theorem 3.5. If μ is a fuzzy dot PS-subalgebra of a PS-algebra X , then $\mu(0) \geq (\mu(x))^3$, $\forall x \in X$.

Proof : For every $x \in X$, we have

$$\begin{aligned}\mu(0) &= \mu(x * 0) \\ &\geq \mu(x). \mu(0) \\ &= \mu(x). \mu(x *x) \\ &\geq \mu(x). \mu(x). \mu(x) \\ &= (\mu(x))^3 , \text{ which completes the proof.}\end{aligned}$$

Definition 3.6. The characteristic function of a non-empty subset A of a PS-algebra X , denoted by χ_A , is defined by

$$\chi_A(x) = \begin{cases} 1 , & x\in A \\ 0 , & x\notin A \end{cases}$$

Theorem 3.7. Let $A \subseteq X$. Then A is a subalgebra of a PS-algebra X if and only if χ_A is a fuzzy dot PS-subalgebra of a PS-algebra X.

Proof : Let x , y $\in$ A. Then $x*y \in A$. Hence we get $\chi_A(x *y) = 1 \geq \chi_A(x). \chi_A(y)$.
If $x \in A$ and $y \notin A$ (or $x \notin A$ and $y \in A$) ,then we get $\chi_A(x *y) \geq \chi_A(x). \chi_A(y) = 1.0 = 0$, since , $\chi_A(x) = 1$ and $\chi_A(y) = 0$.
Thus χ_A is a fuzzy dot PS-subalgebra of a PS-algebra X.
Conversely, assume that χ_A is a fuzzy dot PS-subalgebra of a PS-algebra X.
Now let x , y $\in$ A. Then $\chi_A(x *y) \geq \chi_A(x). \chi_A(y) = 1.1 = 1$, hence by definition, we have $x*y \in A$, which completes the proof.

Remark :

1. A fuzzy subset μ of X is a fuzzy subalgebra of PS-algebra X iff the level subset $\mu^t = \{x \in X \mid \mu(x) \geq t\}$, the upper level subset of μ, is a subalgebra of X , for every $t \in [0 , 1]$.
2. But from example 3.2, it is clear that , if μ is the fuzzy dot PS-subalgebra of X, then there exists $t \in [0 , 1]$ such that $\mu^t = \{x \in X \mid \mu(x) \geq t\}$ is not a subalgebra of X . μ is a fuzzy dot PS-subalgebra of X in example 3.2, then consider $\mu^{0.5} = \{x \in X \mid \mu(x) \geq 0.5\} = \{0,1,3\}$, is not a subalgebra of X , since $1 * 3 = 2 \notin \mu^{0.5}$.

Theorem 3.8. Let $f: X \to Y$ be a homomorphism of a PS-algebra X into a PS-algebra Y. If μ is a fuzzy dot PS subalgebra of Y, then the pre- image of μ denoted by $f^{-1}(\mu)$, defined as $\{f^{-1}(\mu)\}(x) = \mu(f(x))$, $\forall x \in X$, is a fuzzy dot PS subalgebra of X.

Proof: Let μ be a fuzzy dot PS-subalgebra of Y. Let x , y $\in$ X.

Now,
$$\begin{aligned}\{f^{-1}(\mu)\}(x * y) &= \mu(f(x * y)) \\ &= \mu(f(x) * f(y)) \\ &\geq \mu(f(x)) \,.\, \mu(f(y)) \\ &= \{f^{-1}(\mu)\}(x) \,.\, \{f^{-1}(\mu)\}(y)\end{aligned}$$

$\Rightarrow$ $f^{-1}(\mu)$ is a fuzzy dot PS-subalgebra of X.

Theorem 3.9. Let $f: X \to Y$ be an onto homomorphism of PS-algebras. If μ is a fuzzy dot PS -subalgebra of X, then the image $f(\mu)$ of μ under f is a fuzzy dot PS subalgebra of Y.

Proof: For any $y_1, y_2 \in Y$, Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1 * y_2)$.

Consider the set $A_1 * A_2 = \{ x \in X / x = a_1 * a_2$ for some $a_1 \in A_1$ and $a_2 \in A_2 \}$

If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that

$f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2$,

that is, $x \in f^{-1}(y_1 * y_2) = A_{12}$. Hence $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned}f[\mu](y_1 * y_2) &= \sup_{x \in f^{-1}(y_1 * y_2)} \mu(x) = \sup_{x \in A_{12}} \mu(x) \\ &\geq \sup_{x \in A_1 * A_2} \mu(x) = \sup_{x_1 \in A_1,\, x_2 \in A_2} \mu(x_1 * x_2) \\ &\geq \sup_{x_1 \in A_1,\, x_2 \in A_2} \mu(x_1) . \mu(x_2)\end{aligned}$$

since . : [0,1] x [0,1] is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$\check{x}_1 \geq \sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\check{x}_2 \geq \sup_{x_2 \in A_2} \mu(x_2) - \delta$, then

$\check{x}_1 \,.\, \check{x}_2 \geq \sup_{x_1 \in A_1} \mu(x_1) . \sup_{x_2 \in A_2} \mu(x_2) - \varepsilon$.

Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that

$\mu(a_1) \geq \sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\mu(a_2) \geq \sup_{x_2 \in A_2} \mu(x_2) - \delta$, then

$\mu(a_1) . \mu(a_2) \geq \sup_{x_1 \in A_1} \mu(x_1) . \sup_{x_2 \in A_2} \mu(x_2) - \varepsilon$.

Consequently,

$$\begin{aligned}f[\mu](y_1 * y_2) &\geq \sup_{x_1 \in A_1,\, x_2 \in A_2} \mu(x_1) . \mu(x_2) \\ &\geq \sup_{x_1 \in A_1} \mu(x_1) \,.\, \sup_{x_2 \in A_2} \mu(x_2) = f[\mu](y_1) . f[\mu](y_2)\end{aligned}$$

Hence $f(\mu)$ is a fuzzy dot PS-subalgebra of Y.

Theorem 3.10. Let $f : X \to X$ be an endomorphism on a PS-algebra X. If μ be a fuzzy dot PS- subalgebra of X. Define a fuzzy set $\mu_f : X \to [0,1]$ by $\mu_f(x) = \mu(f(x))$, $\forall\, x \in X$. Then μ_f is a fuzzy dot PS-subalgebra of X.

Proof: Let μ be fuzzy dot PS- subalgebra of X. Let $x, y \in X$.

Now, $\mu_f(x * y) = \mu(f(x * y))$
$= \mu(f(x) * f(y))$
$\geq \mu(f(x)) . \mu(f(y))$
$= \mu_f(x) . \mu_f(y)$

$\Rightarrow$ μ_f is a fuzzy dot PS-subalgebra of X.

Definition 3.11. Let λ and μ be two fuzzy sets in a set X.. The Cartesian Product $\lambda \times \mu$: $X \times X \to [0,1]$ is defined by $(\lambda \times \mu)(x,y) = \lambda(x) . \mu(y)$.

Theorem 3.12. If λ and μ are fuzzy dot PS-subalgebras of a PS-algebra X , then $\lambda \times \mu$ is also a fuzzy dot PS- subalgebra of X x X.

Proof : For any $x_1, x_2, y_1, y_2 \in X$.

Then $(\lambda \times \mu)((x_1, y_1) * (x_2, y_2)) = (\lambda \times \mu)(x_1 * x_2, y_1 * y_2)$
$= \lambda(x_1 * x_2) . \mu(y_1 * y_2)$
$\geq ((\lambda(x_1). \lambda(x_2)) . (\mu(y_1). \mu(y_2))$
$= ((\lambda(x_1). \mu(y_1)) . (\lambda(x_2). \mu(y_2))$
$= (\lambda \times \mu)(x_1, y_1) . (\lambda \times \mu)(x_2, y_2))$

This completes the proof.

Definition 3.13. Let β be a fuzzy subset of X. The strongest fuzzy β-relation on PS-algebra X is the fuzzy subset μ_β of X x X given by $\mu_\beta(x, y) = \beta(x).\beta(y)$, for all $x, y \in X$.

Theorem 3.14. Let μ_β be the strongest fuzzy β-relation on PS-algebra X , where β is a fuzzy subset of a PS-algebra X. If β is a fuzzy dot PS-subalgebra of a PS-algebra X , then μ_β is a fuzzy dot PS-subalgebra of X x X.

Proof : Let β be a fuzzy dot PS-subalgebra of a PS-algebra X and let $x_1, x_2, y_1, y_2 \in X$.

Then $\mu_\beta((x_1, y_1) * (x_2, y_2)) = \mu_\beta(x_1 * x_2, y_1 * y_2)$
$= \beta(x_1 * x_2) . \beta(y_1 * y_2)$
$\geq (\beta(x_1). \beta(x_2)) . (\beta(y_1). \beta(y_2))$
$= (\beta(x_1). \beta(y_1)) . (\beta(x_2). \beta(y_2))$
$= \mu_\beta(x_1, y_1) . \mu_\beta(x_2, y_2)$

Therefore μ_β is a fuzzy dot PS-subalgebra of X x X.

Theorem 3.15. Let μ_β be the strongest fuzzy β-relation on PS-algebra X , where β is a fuzzy subset of a PS-algebra X. If μ_β is a fuzzy dot PS-subalgebra of X x X, then β is a fuzzy dot PS-subalgebra of a PS-algebra X.

Proof : Let $x, y \in X$.

Now, $(\beta(x * y))^2 = \beta(x * y) . \beta(x * y)$
$= \mu_\beta((x * y) * (x * y))$

$$\geq \mu_\beta\ (x * y) . \mu_\beta (x * y)$$
$$= [\beta(x) . \beta(y)]^2$$

$\Rightarrow \beta (x * y) \geq \beta(x). \beta(y)$, which completes the proof.

Definition 3.16. Let β be a fuzzy subset of a PS-algebra X. A fuzzy relation μ on PS-algebra X is called a fuzzy β-product relation if $\mu(x,y) \geq \beta(x).\beta(y)$, for all $x , y \in X$.

Definition 3.17. Let β be a fuzzy subset of a PS-algebra X. A fuzzy relation μ on PS-algebra X is called a right fuzzy relation on β-product if $\mu(x,y) = \beta(y)$, for all $x , y \in X$.

Remark :

1. Let β be a fuzzy subset of a PS-algebra X. A fuzzy relation μ on PS-algebra X is called a left fuzzy relation on β-product if $\mu(x,y) = \beta(x)$, for all $x , y \in X$.
2. Left (right) fuzzy relation on β is a fuzzy β-product relation.

Theorem 3.18. Let μ be a right fuzzy relation on a fuzzy subset β of a PS-algebra X. If μ is a fuzzy dot PS-subalgebra of X x X , then β is fuzzy dot PS-subalgebra of a PS algebra X.

Proof:

$$\begin{aligned}\beta(y_1 * y_2) &= \mu (x_1 * x_2 , y_1 * y_2)\\ &= \mu ((x_1,y_1) * (x_2 , y_2))\\ &\geq \mu (x_1,y_1) .\mu (x_2 , y_2)\\ &= \beta (y_1). \beta (y_2) , \text{ for all } x_1, x_2 , y_1, y_2 \in X.\end{aligned}$$

Hence, β is a fuzzy dot PS-subalgebra of a PS-algebra.

Theorem 3.19. Let X and Y be PS-algebras. Let μ be a fuzzy dot PS subalgebra of X x Y.Define a fuzzy set $\beta_x (\mu)$ of X such that $\beta_x (\mu)(x) = \mu (x,0), \forall x \in X$. Then $\beta_x (\mu)$ is a fuzzy dot PS- subalgebra of X.

Proof: Let $x , y \in X$.

$$\begin{aligned}\beta_x (\mu) (x * y) &= \mu (x * y ,0)\\ &= \mu ((x * y),(0 * 0))\\ &= \mu ((x,0) * (y,0))\\ &\geq \mu (x,0) . \mu (y,0)\\ &= \beta_x (\mu) (x) . \ \beta_x (\mu) (y)\end{aligned}$$

$\therefore \beta_x (\mu)$ is a fuzzy dot PS-subalgebra of X.

Theorem 3.20. Let X and Y be PS-algebras. Let μ be a fuzzy dot PS-subalgebra of X x Y. Define a fuzzy set $\beta_y (\mu)$ of Y such that $\beta_y (\mu)(y) = \mu (0,y), \forall y \in Y$. Then $\beta_y (\mu)$ is a fuzzy dot PS subalgebra of Y.

Proof: Let $x , y \in Y$.

$$\begin{aligned}\beta_y (\mu) (x * y) &= \mu (0, x * y)\\ &= \mu ((0 * 0), (x * y))\\ &= \mu ((0, x) * (0,y))\end{aligned}$$

$$\geq \mu\,(0,x)\,.\,\mu\,(0,y)$$
$$= \beta_y\,(\mu)\,(x)\,.\ \beta_y\,(\mu)\,(y)$$

$\therefore\ \beta_y\,(\mu)$ is a fuzzy dot PS-subalgebra of Y.

4. Conclusion

In this article authors have been discussed fuzzy dot PS-subalgebra in fuzzy PS-algebra. The relationship between fuzzy dot PS-subalgebra and fuzzy subalgebra also established. It has been observed that PS-algebra as a generalization of BCK/BCI/Q/d/TM/KU-algebras. This concept can further be generalized to Intuitionistic fuzzy sets, interval valued fuzzy sets, Anti fuzzy sets for new results in our future work.

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