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Some Properties of Glue Graph

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Abstract. Let $G = (V,E)$ be a graph. The *glue graph* of G is the graph denoted by G_g, is a graph with vertex set $V(G_g) = V(G)$ and (u,v) is an edge if and only if $e_G(u) = e_G(v)$, where $e(G)$ is the eccentricity of a graph G. In this paper we have studied miscellaneous properties of glue graph.

Keywords: Eccentricity, equi-eccentric point set graph, glue graph

AMS Mathematics Subject Classification (2010): 05C62

1. Introduction

Graph theory is a very important topics due to its wide applications in science, engineering and technology, medical science even in social sciences. Different special types of graphs along with general graphs are studies for specific application, for example interval graph [6,8], permutation graph [7], trapezoidal graph [9], circular-arc graph [10] and many others. Glue graph is also another important subclass of general geraph. In this paper, we consider glue graph and investigated some of its properties.

The graphs considered in this paper are finite, undirected without loops or multiple edges. Let G be such a graph with $V = V(G)$ as its vertex set, $E= E(G)$ its edge set and let its vertices whose number is n, be labeled by v_1,v_2,v_3 - - - - - - v_n. The *distance*, which is the length of a shortest path between the vertices v_i and v_j of G is denoted by $d(v_i ,v_j /G)$.The distance of a graph was first introduced by Entringer, Jackson and Snyder [4]. *Eccentricity* of a vertex $u\in V(G)$ is defined as, $e(u)=max\{d(u,v): v\in V(G)\}$,where $d(u,v)$ is the distance between u and v in G. The minimum and maximum eccentricity is the *radius* r and *diameter* d of G respectively. *When* $d(G) = r(G)$, G is called a *self-centered graph* with diameter d or r. A vertex u is said to be an *eccentric point of* v when $d(u,v)=e(v)$. An investigation to compute diameter and center of an interval graph has been done in [11].

In general, u is called an *eccentric point*, if it is an eccentric point of some vertex, otherwise noneccentric. For any graph G, the *equi-eccentric point set graph* is a graph with vertex set $v(G)$ and two vertices are adjacent if and only if they correspond to two vertices of G with equal eccentricities [5]. The vertex v is a *central vertex* if $e(v)=r(G)$ and is denoted by ξ_r. Let $\{\xi_r\}$ be the set of vertices having minimum eccentricity. A graph is self centered if every vertex is in the center.

The minimum degree of G is the minimum degree among the vertices of G and is denoted by $\delta(G)$; the maximum degree of G denoted by $\Delta(G)$.The *girth* of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth [3].

A vertex and an edge are said to cover each other if they are incident. A set of vertices which covers all the edges of a graph G is called a *vertex cover for G*, while a set of edges which covers all the vertices is *an edge cover.*

The smallest number of vertices in any vertex cover for G is called its *vertex covering number* and is denoted by $\alpha_{o.}$ Similarly, α_1 is the smallest number of edges in any edge cover of G and is called its *edge covering number*. A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in such a set is called the *point independent number* of G and is denoted by β_o. An independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the *edge independence number* β_1 [3].

The minimum cardinality of minimal dominating set is called *domination number* and is denoted by $\gamma(G)$.The smallest number of colors in any coloring of a graph G is called the *chromatic number of G* and is denoted by $\chi(G)$[1].The definitions and details not furnished here may be found in Buckley and Hararay [4].

Definition 1.1. Let $G= (V,E)$ be a graph and G_g is the glue graph of G. The *glue graph* G_g is a graph with $V(G_g) = V(G)$ and $u,v \in V(G)$ are adjacent in $V(G_g)$ if and only if $e_G(u) = e_G(v)$.

2. Results

The following will be useful in the proof of our results

Observation 2.1. For any nontrivial connected graph G, $r(G) \geq r(G_g)$ and $d(G) \geq d(G_g)$.

Observation 2.2. For any graph G, there exist atleast two vertices having same eccentricity.

Proposition 2.1. For any tree T, T_g contains cycle if and only if $|V(T)| \geq 3$.

Proof: Let G_g contains a cycle and $|V(T)| \leq 2$.Then T is either K_1 *or* K_2. Since K_1 and K_2 are isomorphic to their glue graphs respectively, is a contradiction. Therefore $|V(T)| \geq 3$.

Conversely, let $|V(T)| \geq 3$, then we know that there exists at least two non adjacent vertices say u and v having same eccentricity because distance matrix is symmetrical and also distance is a metric. Therefore there exist at least two vertices of same eccentricity which will form a cycle. Therefore u and v will form a cycle of length at least three. This implies T_g contains cycle if and only if $|V(T)| \geq 3$.

Theorem 2.A. Suppose G be a graph and $r \in \{\xi_r\}$. Then for its G_g,

$$e_{Gg}(v_r) = \begin{cases} r-1, & if \;\; C_4.P_n \quad : n \geq 1 \\ r, & otherwise. \end{cases}$$

Theorem 2.B. Let G be a graph with $\delta(G)$ and $\Delta(G)$ being minimum and maximum degree among the vertices in a graph G respectively, then

(i) $\delta(G) \leq \Delta(G) \leq \Delta(G_g)$
(ii) $\Delta(G) + \delta(G) \leq \Delta(G_g) + \delta(G_g)$.

Proposition 2.2. In a graph G, if $d = r+1$ then G_g is self centered.
Proof: Suppose d= r+1 and G_g is not selfcentered.That is there exist at least one vertex $v_i \in G$ such that e $(v_i) \neq$ k in G_g.
Claim: d$\neq$ r+1.
Let $v_i, v_j \in G$ such that $e(v_i/G)$= r and $e(v_j/G) = r+1$. By Theorem 2.A, $e(v_i/G_g) = r$. v_j are adjacent in G_g,since $e(v_j/G)$ = r+1, $e(v_j/G_g)$= r+1-1 = r. This implies $e(v_j/G_g)$ = e (v_j/G_g) = r. Hence G_g is self centered, a contradiction.

Proposition 2.3. If G = $K_{1,p-1}$ then $G_g = K_p$.
Proof: Let $v_1, v_2, v_3 \ldots\ldots\ldots\ldots v_i \in K_{1,p-1}$. Let $v_r \in \{\xi_r\}$. Then $d\ (v_i, v_r)$ = 1 and $d(v_i, v_j) = 2$, $for\ all\ \ v_i, v_j \in G$ and $i, j \neq r$. This implies v_i, v_j are adjacent in glue graph of $K_{1,p-1}$. This implies $e(v_i) = e(v_r) = 1$. Hence $G_g = K_p$.

Theorem 2.C. For any graph G, glue graph G_g is complete if G is self centered.
Proof: Let G be a self-centered graph. Then $e(v_i/G) = k$, $for\ all\ v_i \in G$. Therefore in G_g, $there\ exist$ an edge (v_i, v_j), $for\ all\ i,j \in G$. This implies $e(v_i) = e(v_j) =$ 1,hence G_g is selfcentered. Converse is not true. By proposition 2.3, star graph is not self centered but its glue graph is complete.

Theorem 2.D. For any graph G, g(G_g)=3 if and only if G satisfies the following conditions
(1) G contains C_3 as a sub graph, or
(2) There exist any two vertices u,v$\in G$ such that $d(u,v)=2$ and $e(u)=e(v)$.
Proof: Suppose G satisfies either of the above conditions, then $g(G_g)=3$.
Conversely, let $g(G_g) = 3\ and$ if G does not satisfies any of the above conditions.
In particular, if G does not satisfy condition (i) then G is either a tree of order at least three or connected graph containing a cycle of length at least four which is impossible. Therefore G must be a tree. By observation there exist at least two *vertices* $u,v \in V(G)$ such that $e(u)=e(v)$. By our assumption $d(u,v) \geq 3$.Therefore u and v will form a cycle of length at least four which is a contradiction.

Proposition 2.4. The glue graph G_g is self centered if and only if $r(G) \leq 2$.
Proof: Let G be a graph and G_g is its glue graph. Let $\{\xi_r\}$, $\{\xi_{r+1}\}\{\xi_{r+2}\}$,- - - -$\{\xi_d\}$ be the set of vertices with eccentricity $r, r+1, r+2$ - - - - d respectively. Suppose $r(G) \leq 2$. If r=1, then G is either K_p or $K_{1,p-1,}$ by proposition 2.3.
Suppose r=2, we consider the following two cases.
Case 1: r=2 and uni-central.
In G the eccentric vertex for v_r is v_{r+2}, for v_{r+1} the vertex belongs to v_d which is farthest and for V_d is the farthest vertex belongs to u_d. This is because v_r is adjacent to all v_{r+1} but v_{r+1} is not adjacent to all v_{r+1} or V_d. But it is adjacent to at least one vertex of v_d. In G_g equi eccentric vertices are adjacent.

In G_g, $e(v_{r+1})= d(v_{r+1},u_{r+1})+ d(u_{r+1},v_d)=2$
Similarly, $e(v_d)=2$. This implies G_g is self centered with $r=2$.
Case 2: $r=2$ and bi-central.
By applying the above method, it is evident that G_g is self centered with $r=2$.
In G $e(v_{r+1})= d(v_{r+1},v_r)+ d(v_r,v_{r+1})+ d(u_{r+1},v_d)=3$. Similarly $e(u_{r+1})=3$.
In G_g, $e(v_{r+1})= d(v_{r+1},u_{r+1})+ d(u_{r+1},v_d)=2$ and $e(v_d)=d(v_d, v_{r+1})+ d(v_{r+1},v_r)=2$. This implies $e(v_i)= 2$, for all v_i in G_g.
This implies G_g is self centered with $r=2$ if $r(G)\leq 2$.
Now suppose G_g is self centered and $r(G)> 2$.
In G, $e(v_r)=$ path $v_r\to v_{r+1}\to v_{r+2}\to - - - -\to v_d =r$
In G_g, $e(v_r)=$ path $v_r\to v_{r+1}\to v_{r+2}\to - - - -\to v_d =r$ --------(1)
In G, $e(v_{r+1})=$ path $v_{r+1}\to v_{r+2}\to v_r -\to u_{r+1}-\to v_{r+2}\to - - - -, \to v_d =r+1$
In G_g, $e(v_{r+1})=$ path $v_{r+1}\to u_{r+1}\to v_{r+2} -\to \to - - - -, \to v_d =r+1-1=r$- - - (2)
In G, $e(v_{r+2})=$ path $v_{r+2}\to v_{r+1}\to v_r\to u_{r+1} -\to u_{r+2}\to - - - -\to v_d =r+2$
In G_g, $e(v_{r+2})=$ path $v_{r+2}\to - - - -\to v_d =r+2-3=r-1$ - - - (3)
From (2) and (3) we observe that $e(v_{r+1})$ and $e(v_{r+2})= r-1$, is contradiction to our assumption.
Hence the proof.
.

3. Results on glue graph of a path

Theorem 3.A. Suppose $G = P_n$, then

i. $r[(P_n)g] =\begin{cases} \left\lceil \frac{n+2}{4} \right\rceil, & if\ n\ is\ even \\ \left\lceil \frac{n+1}{4} \right\rceil, & otherwise. \end{cases}$

ii. $d[(P_n)g] =\begin{cases} \left\lceil \frac{n-1}{2} \right\rceil, & if\ n\ is\ even \\ \frac{n-1}{2}, & otherwise. \end{cases}$

iii. $d(P_n) \leq r[(P_n)_g]$
iv. $r[(P_n)_g] \leq r(P_n) \leq d[(P_n)_g] \leq d(P_n)$
v. $r(P_n) = d[(P_n)_g]$.

Theorem 3.B. Suppose P_n is a path and $v_i\in P_n$, $i=1,2,3$------n. Let $v_r,u_r\in \{\xi r\}$. Then $e(v_r) = e(u_r) = d(P_n)_g$.
Proof: For P_n, if n is even it is bi-central and if n is odd it is uni-central. We have

$$r(P)=\begin{cases} \frac{n}{2}, & if\ n\ is\ even \\ \left\lfloor \frac{n}{2} \right\rfloor, & otherwise \end{cases}$$

In P_n, $e(v_r)=e(u_r)=r$. $e(v_r)=$ path $v_r\to v_{r+1}\to v_{r+2}\to - - - -\to v_d$. In P_n, (v_r is adj to at the most one v_{r+1}) and (u_r is adj to at the most one u_{r+1}). If n is even then (v_r *is* adj *to* u_r) and (at the most one v_{r+1}) and similarly for u_r. *In* $(P_n)_g$ adjacency of v_r u_r remains same but $(v_{r+1} \to u_{r+1})$, $(v_{r+2} \to u_{r+2})$-------------- $(v_d \to u_d)$.
In $(P_n)_g$, $e(v_r)=$ $v_r\to v_{r+1}\to v_{r+2}\to - - - -\to v_d$ __________________ (1)

This implies P_n and $(P_n)_g$ the path taken by v_r and u_r is same. This implies, $e(v_r)=e(u_r)=r$ in $(P_n)_g$. Now we have to show that $e(v_r)=e(u_r)=r = d(P_n)_g$. The eccentric vertex of v_{r+1} will be v_d in P_n.

In P_n , $d(v_{r+1}, v_d)$= path $v_{r+1}\rightarrow v_r \rightarrow v_{r+1}\rightarrow v_{r+2}\rightarrow$ - - -$\rightarrow v_d$ ________(2)

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That is, $d(v_{r+1}, v_d)$ in $(P_n)g < d(v_{r+1}, v_d)$ in p_n

his implies $e(v_{r+1})$ in$(p_n)(g) < e(v_{r+1})$ *in* p_n

Now let us prove that $e(v_r)=d(p_n)g$.

That is, prove that $e(v_{r+1}), e(v_{r+2})\ldots\ldots\ldots\ldots < e(v_r)$ *in* $(p_n)g$.

In $(p_n)g$ $e(v_{r+1}) = d(v_{r+1}, v_d)$.

Comparing (1) and (2) we have $e(v_{r+1}) < e(v_r)$.

Similarly we can prove for other vertices, $e(v_r) = (p_n)g$.

Theorem 3.E. *In* $(p_n)g$, $1\leq \delta(G) \leq 2$ *and* $1\leq \Delta(G) \leq 3$.

Remark: *For* P_2, P_3 *and* P_4 , $\delta(G) = \Delta(G)$

or

$\delta(G) = \Delta(G), for\ (p_n)g$ *where* $2\leq n \leq 4$.

Theorem 3.F. Suppose G= P_n , then $\chi(P_n)= \begin{cases} 2, if\ n\ is\ even \\ 3, otherwise \end{cases}$

Proof: Let G be a path P_n. Now we consider the following cases. Consider a path graph P_n

***Case1**:*

Let n be even. Let v_1,v_2,v_3,----v_{2n} be the vertices of P_n . P_n is bi-central. We can partition the vertices based on their respective eccentricities as below. Central vertices v_r,,$u_r\in$ e_r,vertices at distance $r+1$:$v_{r+1}, u_{r+1}, \in e_{r+1}$, vertices at distance $r+2$:$v_{r+2}, u_{r+2}, \in e_{r+2}$ vertices at distance d :$v_d, u_d, \in e_d$

But in $(p_n)g$ the vertices belongs to the sets $e_r, e_{r+1}, e_{r+2}\ldots\ldots\ldots e_d$ are adjacent to their respective set elements.*This* implies $v_1, v_3, v_5\ldots\ldots\ldots v_r\ldots\ldots\ldots v_{d-1}$ can be assigned with a single color and $v_2, v_4, v_6\ldots\ldots\ldots v_r\ldots\ldots\ldots v_d$ with another color.

Hence $(p_n)g$ is two colorable.

Case 2:

Let n be odd $v_1, v_2, v_3\ldots\ldots\ldots v_n$ be the vertices of P_n.. *If* P_n is unicentral then proceeding as above in case1, , we get odd cycle.

ince $\chi(C_{2n+1})$= 3 .Therefore $\chi(P_n)=3$.

4. Conclusions

The glue graphs are basically the transformation graphs. It has many applications in computer network and distance related problems. Our contribution in this paper helps to minimize the time complexity to solve the distance related problems in glue graphs.

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