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On the Existence and Uniqueness of Holder Solutions of Nonlinear Singular Integral Equations with Carleman Shift

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Abstract. The paper is concerned with the applicability of the generalized Kantorovich majorization principle to a class of nonlinear singular integral equations with Carleman shift. The abstract results are illustrated in the generalized Holder space.

***Keywords*:** Nonlinear singular integral equations, Kantorovich majorization principle, Carleman shift

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1. Introduction

The successful development of the theory of singular integral equations (SIE) naturally stimulated the study of singular integral equations with shift (SIES). (see [9,11,13,14], [15-18] and others). Existence results and approximate solutions have been studied for certain classes of nonlinear singular integral equations (NSIE) and nonlinear singular integral equations with shift (NSIES) by many authors among them we mention [1-6, 12, 20]. The classical and more recent results on the solvability of NSIE should be generalized to corresponding equations with shift,(see [22]). The theory of SIES is an important part of integral equations because of its recent applications in many fields of physics and engineering, (see [8,15,17]).

In the present paper a class of NSIES has been investigated for the case of finite group of iterations generated by preserving orientation Carleman shift, we discuss the existence and uniqueness for Holder solutions of NSIES by application of a generalized Kantorovich majorization principle. This majorization principle reduces the problem of finding fixed points of abstract nonlinear operators in Banach spaces to that of finding fixed points of simple convex functions on the real line. In this technique, we do not require Frechet differentiability of nonlinear operator involved, but only a suitable Lipschitz condition.

2. Formulation of the problem

Let Γ be a simple smooth closed Lyapunov contour, which divides the plane of the complex variable Z into two domains, the interior domain D^+ and the exterior domain D^-, and let $G:\Gamma\times\Gamma\times R\to R$ be given Caratheodory function (i.e. function which are continuous in the last variable and measurable in the other variables) in the Lebesgue space $L_p=L_p(\Gamma),(1\le p\le\infty)$. and assume that G satisfies the following Lipschitz condition

$$|G(t,s,u_1)-G(t,s,u_2)|\le A_1(r)\,|u_1-u_2|,\quad (|u_1|,|u_2|\le r) \tag{1.1}$$

where $A_1(r)$ denotes the Lipschitz constant for G. The purpose of this paper is to investigate NSIES:

$$u(t)=\sum_{i=0}^{m-1}\left(\frac{a_i(t)}{\pi i}\int_\Gamma\frac{u(\tau)}{\tau-\alpha_i(t)}d\tau+\lambda b_i(t)\int_\Gamma\frac{G(\alpha_i(t),\tau,u(\tau))}{\tau-\alpha_i(t)}d\tau\right),\ t\in\Gamma. \tag{1.2}$$

By means of the generalized Kantorovich majorization principle under the assumption that $\alpha(t)$ homeomorphically maps Γ into itself with preservation orientation and satisfies the Carleman condition:

$$\alpha_m(t)=t,\ \alpha_i(t)\ne t,\ 1\le i\le m-1,$$

where

$$\alpha_i(t)=\alpha[\alpha_{i-1}(t)],\alpha_0(t)=t,$$

and $m\ge 2$. Moreover assume that $\alpha'(t)$ satisfies the Holder condition and the coefficients $a_i(t),b_i(t),i=0,1,...,m-1$ belong to the generalized Holder space $H_\Gamma(\omega)$ and $\lambda\in(-\infty,\infty)$, is a numerical parameter, the function $G(t,\tau,u(\tau))$ is a given function and $u(t)$ is an unknown function.

Our problem has been studied when, $a_i(t)=0,i=0,1,...,m-1$ by applicability of Banach fixed point theorem in [3], also it has been studied under same preceding condition and Γ is a real segment , (without shift), in usual Holder space in [21] .

3. The generalized Kantorovich majorization principle

Let X be a Banach space, and let $\mathrm{P}:\overline{B}(u_0,R)\to X$ be a nonlinear operator where $\overline{B}(u_0,R)$ denotes the closure of the ball $B(u_0,R)=\{u:u\in X,\ \|u-u_0\|<R\}$.

Suppose that the operator P satisfies the local Lipschitz condition

$$\|\mathrm{P}(u_1)-\mathrm{P}(u_2)\|\le k(r)\|u_1-u_2\|,\quad (u_1,u_2\in\overline{B}(u_0,R);r\le R), \tag{2.1}$$

where $k(r)$ denotes the minimal Lipschitz constant for P on the ball $\overline{B}(u_0,R)$, i.e.

$$k(r)=\sup\left\{\frac{\|\mathrm{P}(u_1)-\mathrm{P}(u_2)\|}{\|u_1-u_2\|}:u_1,u_2\in\overline{B}(u_0,R);u_1\ne u_2\right\}.$$

Define a scalar function ϕ on $[0,R]$ by

$$\phi(r) = a + \int_0^r k(t)dt\,, \tag{2.2}$$

where

$$a = \| u_0 - \mathrm{P}u_0 \|. \tag{2.3}$$

Theorem 2.1. [7] Let $\mathrm{P}: \overline{B}(u_0, R) \to X$ be an operator satisfying a Lipschitz condition (2.1). Suppose that the scalar equation (2.2) has a unique fixed point $r_* \in [0, R]$ and that $\phi(R) \le R$. Then the operator P has a fixed point u_* in the ball $\overline{B}(u_0, r_*)$, this solution is unique in the ball $B(u_0, R)$, and this fixed point may be obtained as limit of the successive approximations

$$u_n = \mathrm{P}^n u_0 \in \overline{B}(u_0, r_*).$$

We make some remarks on Theorem 2.1. The usefulness of this theorem consists in reducing the hard problem of finding fixed points of a nonlinear operator in a Banach space to the simple problem of finding fixed points of a scalar function. Moreover, in the generic case $r_* < R$ we get much more information on u_* than just existence, the smaller we may choose the fixed point r_* of ϕ, the better we may localize the fixed point u_* of P, and the larger we may choose the invariant interval $[0, R]$, the better we may exclude other fixed points of P. The case $r_* = R$, we may guarantee then only uniqueness in $B(0, R)$ and existence in $\overline{B}(0, R)$.

Since the function k in (2.1) is increasing, the function ϕ in (2.2) is convex. Consequently, existence and uniqueness of fixed points of ϕ essentially depend on the size of the initial value a in (2.3). Of course, in the classical Banach-Caccioppoli fixed point principle we simply have $k(r) = k < 1$. In this case we have existence and uniqueness in $\overline{B}(0, R)$, where $R \ge r_* = (1-k)^{-1} a$ may be chosen arbitrarily large [7].

4. Some notations and auxiliary results

In this section, we introduce some notations and auxiliary results, which will be used in the sequel.

Definition 3.1.[10] We denote by Φ the class of all functions $\omega(\delta)$, defined on $(0, l]$, where l is the length of the curve Γ, which satisfies the following conditions:

1. $\omega(\delta)$ is a modulus of continuity,
2. $\sup\limits_{\delta>0} \dfrac{1}{\omega(\delta)} \displaystyle\int_0^\delta \frac{\omega(s)}{s} ds = I_\omega < \infty$,

3. $\sup\limits_{\delta>0} \dfrac{\delta}{\omega(\delta)} \int\limits_{\delta}^{l} \dfrac{\omega(s)}{s^2} ds = J_\omega < \infty.$

Definition 3.2. [10] The generalized Holder space $H_\Gamma(\omega)$ is the set of all continuous function $u(t)$ such that

$$H_\Gamma^\omega(u) = \sup_{t_1,t_2\in\Gamma} \frac{|u(t_1)-u(t_2)|}{\omega(|t_1-t_2|)} < \infty .$$

For $u \in H_\Gamma(\omega)$ we define the norm:

$$\| u \|_{H_\Gamma} = \| u \|_{c(\Gamma)} + H_\Gamma^\omega(u), \text{ where } \| u \|_{c(\Gamma)} = \max_{t\in\Gamma} |u(t)|.$$

Using the notations

$$(\Lambda_G u)(t) = \lambda \int_\Gamma \frac{G(t,\tau,u(\tau))}{\tau - t} d\tau, \tag{3.1}$$

$$L(u)h(t) = \lambda \int_\Gamma \frac{l(t,\tau,u(\tau))}{\tau - t} h(\tau) d\tau; \quad h(t) \in H_\Gamma(\omega), \tag{3.2}$$

$$(S u)(t) = \frac{1}{\pi i} \int_\Gamma \frac{u(\tau)}{\tau - t} \tag{3.3}$$

For singular integral operators, where

$$l(t,s,u) = \frac{\partial G(t,s,u)}{\partial u},$$

is also a caratheodory function,

$$(Wu)(t) = u(\alpha(t)),$$

for shift operator, and the operators B_1, B_2 are defined by

$$(B_1 u)(t) = \sum_{i=0}^{m-1} a_i(t) W_i u(t), \quad (B_2 u)(t) = \sum_{i=0}^{m-1} b_i(t) W_i u(t). \tag{3.4}$$

where

$$(W_i u)(t) = u(\alpha_i(t)), \qquad i = 0,1,\ldots,m-1.$$

Consequently, The equation (1.2) takes the following operator form:

$$u(t) = (B_1 S u)(t) + (B_2 \Lambda_G u)(t). \tag{3.5}$$

Now, we study the singular integral operator Λ_G defined by the equality (3.1) where the function $G = G(t,\tau,u) : \Gamma\times\Gamma\times R \to R$ satisfies the following condition:

$$|v(t_1,\tau_1) - v(t_2,\tau_2)| \le A_2\, \omega^*(|t_1 - t_2|) + A_3 \omega(|\tau_1 - \tau_2|); \tag{3.6}$$

where $v(t,\tau) = G(t,\tau,u)$, and for $\omega(\delta), \omega^*(\delta) \in \Phi$, we have

$$\omega^*(\delta)\ln(l/\delta) \le A_4\omega(\delta), \tag{3.7}$$

where A_2 , A_3 and A_4 are positive constants.

Lemma 3.1. If the function $G(t,s,u)$ satisfies the conditions (3.6), (3.7), then the operator Λ_G defined by (3.1) is bounded on $H_\Gamma(\omega)$.

Proof: Now, putting

$$\bar{f}(t) = \int_\Gamma \frac{v(t,\tau)}{\tau - t} d\tau,\ M_v = \max_{(t,\tau)\in\Gamma\times\Gamma} |v(t,\tau)| \quad ,$$

From Definition (3.1) and inequalities (3.6), (3.7), we get

$$\left|\bar{f}(t)\right| \le \left|\int_\Gamma \frac{v(t,\tau)-v(t,t)}{\tau - t} d\tau\right| + |v(t,t)|\left|\int_\Gamma \frac{d\tau}{\tau - t}\right| \le$$

$$\le m^* A_3 I_\omega \omega(l) + \pi M_v \ ,$$

where $$|d\tau| \le m^* d\theta,$$

m^* is a positive constant [8].

Now, we estimate $\left|\bar{f}(t_1) - \bar{f}(t_2)\right|$ as follows:

Suppose $|t_1 - t_2| < \sigma_0$, fix an arbitrary number n, $1 < n < \sigma_0 / |t_1 - t_2|$. Draw a circle of radius $\sigma = n|t_1 - t_2|$ centered at the point t_1. This circle intersects Γ at two points ε_1 and ε_2. The part of Γ lying within this circle is denoted by $\varepsilon_1\varepsilon_2$.

$$\left|\bar{f}(t_1) - \bar{f}(t_2)\right| = \left|\int_\Gamma \frac{v(t_1,\tau)-v(t_1,t_1)}{\tau - t_1} d\tau + v(t_1,t_1)\int_\Gamma \frac{d\tau}{\tau - t_1} - \int_\Gamma \frac{v(t_2,\tau)-v(t_2,t_2)}{\tau - t_1} d\tau - v(t_2,t_2)\int_\Gamma \frac{d\tau}{\tau - t_2}\right|$$

Therefore, we get

$$\begin{aligned}\left|\bar{f}(t_1) - \bar{f}(t_2)\right| &\le \left|\int_{\varepsilon_1\varepsilon_2} \frac{v(t_1,\tau)-v(t_1,t_1)}{\tau - t_1} d\tau\right| + \left|\int_{\varepsilon_1\varepsilon_2} \frac{v(t_2,\tau)-v(t_2,t_2)}{\tau - t_2} d\tau\right| + \left|\int_{\Gamma\setminus\varepsilon_1\varepsilon_2} \frac{v(t_1,\tau)-v(t_2,\tau)}{\tau - t_1} d\tau\right| + \\ &+ \left|\int_{\Gamma\setminus\varepsilon_1\varepsilon_2} \frac{(t_1-t_2)(v(t_2,\tau)-v(t_2,t_2))}{(\tau-t_1)((\tau-t_2))} d\tau\right| + \left|(v(t_2,t_2)-v(t_1,t_1))\left[\int_{\Gamma\setminus\varepsilon_1\varepsilon_2} \frac{d\tau}{\tau - t_1} - \pi i\right]\right| \\ &= I_1 + I_2 + I_3 + I_4 + I_5 .\end{aligned}$$

From, [20], we obtain

$$I_i \le M_i\omega(|t_1 - t_2|);\ i = 1,2,..,5$$

$M_1 = 2m^* A_3(n+1) I_\omega,\quad M_2 = 2m^* A_3(n+1) I_\omega,\quad M_3 = m^* A_2 A_4,$

$$M_4 = (\frac{n+1}{n})^2 m * A_3 J_\omega, \quad M_5 = (A_2 A_4 (\ln(l/s))^{-1} + A_3) M_6,$$

where

$$M_6 = \left| \int_{\Gamma/\varepsilon_1\varepsilon_2} \frac{d\tau}{\tau - t_1} - \pi_i \right|.$$

Therefore, we have

$$\|\Lambda_G u\|_{H_\Gamma} \leq \wedge_1 + \wedge_2, \tag{3.8}$$

where $\Lambda_1 = |\lambda| \left(m^* A_3 I_\omega \omega(l) + \pi M_v \right)$,

and $\wedge_2 = |\lambda| (M_1 + M_2 + M_3 + M_4 + M_5)$.

Hence, the nonlinear singular integral operator Λ_G defined by the right-hand side of (3.1) is a bounded operator in generalized Holder space $H_\Gamma(\omega)$.

Lemma 3.2. [10,19] Let the function $u(t)$ belong to the space $c(\Gamma)$ and $\int_0^l \frac{\omega_u(\zeta)}{\zeta} d\zeta < \infty$, where $\omega_u(\delta) = \sup_{|t_1 - t_2| < \delta} |u(t_1) - u(t_2)|$. Then the following inequalities

$$\|Su\|_c \leq c_1 \left(\int_0^l \frac{\omega_u(\zeta)}{\zeta} d\zeta + \|u\|_c \right), \qquad \omega_{Su}(\delta) \leq c_2 \left(\int_0^\delta \frac{\omega_u(\xi)}{\xi} d\xi + \delta \int_0^l \frac{\omega_u(\xi)}{\xi^2} d\xi \right),$$

are valid, where c_1 and c_2 are constant.

Lemma 3.3. The singular operator S is a bounded operator on the space $H_\Gamma(\omega)$.

Proof: From Definition 3.2, we have

$$\|S u\|_{H_\Gamma} = \|Su\|_{c(\Gamma)} + H_\Gamma^\omega (Su)$$

From Lemma 3.2, we have

$$\|Su\|_{H_\Gamma} \leq c_1 \left(\int_0^l \frac{\omega_u(\zeta)}{\zeta} d\zeta + \|u\|_c \right) + c_2 \sup_{\delta > 0} \frac{1}{\omega(\delta)} \left(\int_0^\delta \frac{\omega_u(\xi)}{\xi} d\xi + \delta \int_0^l \frac{\omega_u(\xi)}{\xi^2} d\xi \right).$$

$$\|Su\|_{H_\Gamma} \leq c_1 \|u\|_{H_\Gamma} \left(\int_0^l \frac{\omega(\zeta)}{\zeta} d\zeta + 1 \right) + c_2 \|u\|_{H_\Gamma} \sup_{\delta > 0} \frac{1}{\omega(\delta)} \left(\int_0^\delta \frac{\omega(\xi)}{\xi} d\xi + \delta \int_0^l \frac{\omega(\xi)}{\xi^2} d\xi \right)$$

Hence, from Definition 3.1, we get

$$\|Su\|_{H_\Gamma} \leq \rho_0 \|u\|_{H_\Gamma}. \tag{3.9}$$

where ρ_0 is a constant defined as

$$\rho_0 = c_1\left(\int_0^l \frac{\omega(\zeta)}{\zeta}d\zeta + 1\right) + c_2(I_\omega + J_\omega)$$

Thus, the lemma is proved.

Lemma 3.4. [2] Let the function $g(t,\tau) = m(t,\tau)\,h(\tau)$; $h(\tau)$ belong to the generalized Holder space $H_\Gamma(\omega)$. Then the following inequality is valid

$$\omega_g(\delta_1,\delta_2) \le \|h\|_c\,\omega_m(\delta_1,0) + \|h\|_c\,\omega_m(0,\delta_2) + \beta H_\Gamma^\omega(h)\omega(\delta), \quad (3.10)$$

where

$$\omega_g(\delta_1,\delta_2) = \sup_{\substack{|t_1-t_2|<\delta_1 \\ |\tau_1-\tau_2|<\delta_2}} |g(t_1,\tau_1) - g(t_2,\tau_2)|$$

and
$$\beta = \max_{t,\tau\in\Gamma}|m(t,\tau)|.$$

The proof of boundedness of the operator L in the generalized Holder space $H_\Gamma(\omega)$ depends on the classical Zygmund inequality [19],

$$\omega_{Lu}(\delta) \le c\left(\int_0^\delta \frac{\omega_u(\xi)}{\xi}d\xi + \delta\int_0^l \frac{\omega_u(\xi)}{\xi^2}d\xi\right), \quad (3.11)$$

where c is a constant.

In the following Theorem, the function $m = m(t,\tau)$ should carry the following quite restrictive conditions[10]:

1. $\sup_{0<\delta<l} \dfrac{\omega_m(\delta,0)\ln(l/\delta)}{\omega(\delta)} = J_1 < \infty$ (3.12)

2. $\sup_{0<\delta<l} \dfrac{1}{\omega(\delta)}\displaystyle\int_0^\delta \frac{\omega_m(0,\xi)}{\xi}d\xi = J_2 < \infty$ (3.13)

3. $\sup_{0<\delta<l} \dfrac{\delta}{\omega(\delta)}\displaystyle\int_0^\delta \frac{\omega_m(0,\xi)}{\xi^2}d\xi = J_3 < \infty$ (3.14)

Theorem 3.1. The nonlinear singular operator L is a bounded operator on the generalized Holder space $H_\Gamma(\omega)$.

Proof: Let

$$m(t,\tau)h(\tau) = g(t,\tau), \quad (3.15)$$

and
$$\tilde{f}(t) = \lambda\int_\Gamma \frac{g(t,\tau)}{\tau - t}d\tau, \quad (3.16)$$

where

$$l(t,\tau,u(\tau)) = m(t,\tau),$$

from, [2], we have

$$\left\| \tilde{f} \right\|_c = |\lambda| \left(I_4 \|h\|_c + I_5 H_\Gamma^\omega(h) \right) \tag{3.17}$$

and

$$H_\Gamma^\omega(\tilde{f}) \le |\lambda| \left(\tilde{I}_4 \|h\|_c + \tilde{I}_5 H_\Gamma^\omega(h) \right) \tag{3.18}$$

where

$$I_4 = m * \int_0^l \frac{\omega_m(0,\xi)}{\xi} d\xi + \beta\pi, \qquad I_5 = m * \beta \int_0^l \frac{\omega(\xi)}{\xi} d\xi,$$

$$\tilde{I}_4 = c \left(\frac{J_1 I_\omega}{\ln 2} + J_1 + J_2 + J_1 \right), \qquad \tilde{I}_4 = c\,\beta\left(I_\omega + J_\omega\right).$$

From, the inequalities (3.17) and (3.18), we obtain

$$\left\| L(u)h \right\|_{H_\Gamma} \le |\lambda|\, \gamma \, \|h\|_{H_\Gamma} \tag{3.19}$$

where

$$\gamma = \max\left\{ I_4 + \tilde{I}_4, I_5 + \tilde{I}_5 \right\}.$$

Thus, the theorem is proved.

Theorem 3.2. [2] The shift operators $B_i\,; i = 1,2$ are bounded operators on the generalized Holder space $H_\Gamma(\omega)$ and satisfy the inequality

$$\left\| \left(B_i\, u\right)(t) \right\|_{H_\Gamma} \le \Theta_i \left\| u \right\|_{H_\Gamma};\; i = 1,2 \tag{3.20}$$

where $\Theta_i = \max\left\{ M_{B_i} + L_{B_i}, M_{B_i} \right\}$, $M_{B_1} = \sum_{i=0}^{m-1} \left\| a_i(t) \right\|_c$, $M_{B_2} = \sum_{i=0}^{m-1} \left\| b_i(t) \right\|_c$

and $L_{B_1} = \sum_{i=0}^{m-1} H_\Gamma^\omega(a_i)$, $L_{B_2} = \sum_{i=0}^{m-1} H_\Gamma^\omega(b_i)$.

5. Existence and uniqueness of the solution

Our aim is to apply a generalized Kantorovich majorization principle (Theorem 2.1) to the following nonlinear singular integral operator

$$(Pu)(t) = (B_1 Su)(t) + (B_2 \Lambda_G u)(t). \tag{4.1}$$

In order to apply Theorem 2.1 to the operator P defined by (4.1), we have to find an explicit formula, or at least a good upper estimate, for the Lipschitz constant $k(r)$ in (2.1) where we take for simplicity $u_0 = 0$.

Since

$$\left\| \mathrm{P}u_1 - \mathrm{P}u_2 \right\|_{H_\Gamma} \le \left\| B_1 \right\|_{H_\Gamma} \left\| S \right\|_{H_\Gamma} + \left\| B_2 \right\|_{H_\Gamma} \left\| \Lambda_G\, u_1 - \Lambda_G u_2 \right\|_{H_\Gamma}. \tag{4.2}$$

From condition (1.1), we get the nonlinear operator (3.1) whose Lipschitz constant

$$k^*(r) = \sup\left\{ \frac{\| \Lambda_G u_1 - \Lambda_G u_2 \|_{H_\Gamma}}{\| u_1 - u_2 \|_{H_\Gamma}} : \| u_1 \|_{H_\Gamma}, \| u_2 \|_{H_\Gamma} \le r \right\}. \tag{4.3}$$

in the generalized Holder space $H_\Gamma(\omega)$. From the Lagrange formula, we obtain

$$G(t,\tau,u_1(\tau)) - G(t,\tau,u_2(\tau)) = [u_1(\tau) - u_2(\tau)] \int_0^1 l(t,\tau,(1-\theta)u_1(\tau) + \theta u_2(\tau))\, d\theta,$$

hence, we get

$$\| (\Lambda_G u_1)(t) - (\Lambda_G u_2)(t) \|_{H_\Gamma} \le \left\| \lambda \int_\Gamma \frac{(u_1(\tau) - u_2(\tau)) \int_0^1 l\ (t,\tau,(1-\theta)\ u_1(\tau) + \theta\, u_2(\tau)) d\theta}{\tau - t} d\tau \right\|_{H_\Gamma},$$

hence,

$$\| \Lambda_G u_1 - \Lambda_G u_2 \|_{H_\Gamma} \le \| L(u) h \|_{H_\Gamma}, \text{ where } h(\tau) = u_1(\tau) - u_2(\tau).$$

Therefore from the inequality (3.19), we get

$$k^*(r) = |\lambda| \gamma\ . \tag{4.4}$$

hence, from the inequalities (3.9), (3.20) the minimal Lipschitz constant for the operator P on the ball $\overline{B}(0,R)$ is given by

$$k(r) = \Theta_1\, \rho_0 + |\lambda|\, \gamma \Theta_2. \tag{4.5}$$

Choosing

$$|\lambda| < (1 - \Theta_1 \rho_0)\gamma^{-1}\, \Theta_2^{-1},$$

consequently, the relation (4.2) takes the following form:

$$\frac{\| \mathrm{P} u_1 - \mathrm{P} u_2 \|_{H_\Gamma}}{\| u_1 - u_2 \|_{H_\Gamma}} \le \Theta_1 \rho_0 + |\lambda|\, \gamma \Theta_2, \tag{4.6}$$

In this way, we arrive at the following theorem:

Theorem 4.1. Suppose that the operators (3.1)-(3.4) act in the space $H_\Gamma(\omega)$ and are bounded. Moreover, suppose that the scalar function $\phi : [0,\infty) \to [0,\infty)$ defined by

$$\phi(r) = a + (\Theta_1\, \rho_0 + |\lambda|\, \gamma \Theta_2)r$$

($a = \|P(0)\|_{H_\Gamma}$) has a unique fixed point $r_* = a(1-\Omega)^{-1}$, ($\Omega = \Theta_1\, \rho_0 + |\lambda|\, \gamma \Theta_2$ and $\Omega < 1$) in some interval $[0,R]$, $(R \ge r_*)$, and $\phi(R) \le R$. Then equation (3.5) has a fixed point $u_* \in \overline{B}(0,R) \subset H_\Gamma(\omega)$. This fixed point may be obtained as limit of the successive approximations $u_n = (P)^n(0) \in \overline{B}(0,R)$ and is unique in the ball $\overline{B}(0,R)$.

We introduce an elementary example, shows that this principle may be considered as a modification of a classical Banach fixed point principle.

Example. Suppose the nonlinear operator

$$\mathrm{P}(u) = bu^2 + c\sqrt{|u|} + d \; ; -\infty < u < \infty,$$

where

$$(1-\delta)^2 > 4\, b|d| \; ; \quad 0 < \delta < 1, \tag{4.7}$$

and u is a function of t. It is clear that

$$\frac{\| \mathrm{P}u_1 - \mathrm{P}u_2 \|_{H_\Gamma}}{\|u_1 - u_2\|_{H_\Gamma}} \le \left(c + b\|u_1 + u_2\|_{H_\Gamma} \left\| \sqrt{|u_1|} + \sqrt{|u_1|} \right\|_{H_\Gamma} \right) \left\| \sqrt{|u_1|} + \sqrt{|u_1|} \right\|_{H_\Gamma}^{-1} \le \delta + 2br$$

From inequality (2.1), we get $k(r) = \delta + 2br$, obviously, the scalar function (2.2) takes the form:

$$\phi(r) = br^2 + \delta r + |\, d \,| \; .$$

From condition (4.7), the function $r - \phi(r)$ has two positive roots

$$R_\pm = \frac{1}{2}\left\{ (1-\delta) \pm \left[(1-\delta)^2 - 4b|d| \right]^{1/2} \right\}.$$

Consequently, the condition $\phi(R) \le R$ holds for any $R \in [R_-, R_+]$, and Theorem 2.1 applies. Observe that $k(r) \to \infty, as\ r \to \infty$, in this example, therefore the classical Banach fixed-point principle does not apply.

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