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Fixed Point Results in Cyclic Contractions of Generalized Dislocated Metric Spaces

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Abstract. In this paper, two types of the cyclic contraction mappings on generalized types of dislocated metric space have been introduced. Also, we establish some fixed point theorems in these spaces.

Keywords: fixed point, ld-metric, rd-metric, cyclic contraction

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1. Introduction

In 1922, S. Banach [5] proved a fixed point theorem for contraction mapping in complete metric space. In 1986, S. G. Matthews [13] initiated the concept of dislocated metric space under the name of metric domains. In 1994, S. Abramski and A. Jung [2] presented some facts about dislocated metric in the context of domain theory. In 2000, P. Hitzler and A. K. Seda [4] generalized the celebrated Banach contraction principle in complete dislocated metric space. In 2003, Kirk et.al. [12] introduced the notion of cyclic contraction and established fixed point results for such contractions. Since then many authors proved fixed point results in cyclic contraction mappings of metric space. In 2013, M.A. Ahmed et.al [3] introduced the notion of generalized types of dislocated metric space so called left and right dislocated metric spaces. The purpose of this paper is to establish fixed point theorems in cyclic contractions of left and right dislocated metric spaces.

2. Preliminaries

We Start with the following definitions, lemmas and theorems.

Definition 1. [15] *Let X be a set. A distance on X is a map* $d : X \times X \to [0, \infty)$. *A pair (X, d) is called a distance space,if d satisfies the following conditions*
DM1: if $d(x, y) = d(y,x) = 0$ then $x= y$.
DM2: $d(x, y) = d(y, x)$
DM3: $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$ then it is called dislocated metric(or simply d-metric)on X. It is obvious that if d satisfies **DM1 - DM3** and
DM4: $d(x, x) = 0$ for all $x \in X$ then d is a metric on X.

Definition 2. [3] *A distance function d is called left dislocated metric(or ld-metric) if it*

satisfies **DM1** *and the condition,*
(LD) $d(x,y) \le d(z,x) + d(z,y) \quad \forall x,y,z \in X$.

Definition 3. [3] *A distance function d is called right dislocated metric(or rd-metric) if it satisfies* **DM1** *and the condition,*
(RD) $d(x,y) \le d(x,z) + d(y,z) \quad \forall x,y,z \in X$.

It is clear that any d-meric space is ld-metric and rd-metric but the converse may not be true.

Definition 4. [3] *A sequence* $\{x_n\} \subseteq X$ *is ld-convergent iff there exists a point* $x \in X$ *such that* $lim_{n\to\infty} d(x_n, x) = 0$. *In this case, x is said to be ld-limit of* $\{x_n\}$.

Definition 5. [3] *A sequence* $\{x_n\}$ *in ld -metric space (X , d) is called Cauchy sequence if for given* $\varepsilon > 0$, *there corresponds* $n_0 \in N$, *such that for all* $m, n \geqslant n_0$, *we have* $d(x_m, x_n) < \varepsilon$.

Definition 6. [3] *A ld-metric space (X, d) is called complete if every Cauchy sequence in it is a ld- convergent.*

We state the following lemmas without proof.

Lemma 1. [3] *Every subsequence of ld-convergent sequence to a point* x_0 *is ld-convergent to* x_0 .
Definition 7. [3] *Let(X, d)be a ld-metric space. A map* $f : X \to X$ *is called contraction if there exists* $0 \le \lambda < 1$ *such that d(fx, fy)* $\le \lambda$ *d(x, y).*

Lemma 2. [3] *Let (X, d) be a ld-metric space. If* $f : X \to X$ *is a contraction function, then* $\{f^n(x_0)\}$ *is a Cauchy sequence for each* $x_0 \in X$.

Lemma 3. [3] *ld-limits in a ld-metric space are unique.*

Definition 8. [3] *Let (X, d) be a ld-metric space. If* $f : X \to X$ *is a contraction function, then f isld-continuous.*

Similarly, we can have definitions for rd-metric space also. For detail, please do refer [3].

Definition 9. [12] *Let A and B be non empty subsets of a metric space (X, d). A map* $f : A \cup B \to A \cup B$ *is called a cyclic map iff* $f(A) \subseteq B$ *and* $f(B) \subseteq A$.

Definition 10. [12] *Let A and B be non empty subsets of a metric space (X, d). A map*

$f: A \cup B \to A \cup B$ *is called a cyclic contraction if there exists* $k \in [0,1)$ *such that* $d(fx, fy) \le kd(x, y)$ *for all* $x \in A$ *and* $y \in B$.

3. Main results

Definition 11. *Let A and B be non empty closed subsets of a ld-metric space (X, d). A map* $f: A \cup B \to A \cup B$ *is called a ld-cyclic contraction if there exists* $k \in [0, \frac{1}{2})$ *such that* $d(fx, fy) \le kd(x, y)$ *for all* $x \in A$ *and* $y \in B$.

Theorem 1. *Let (X, d) be complete ld-metric space. Let A and B be two non empty closed subsets of X and* $f: A \cup B \to A \cup B$ *be a ld-cyclic contraction in X, then T has a unique fixed point in* $A \cap B$.

Proof: Let $\{f^{2n}x\}$ be a sequence in A and $\{f^{2n-1}x\}$ be a sequence in B.Fix $x \in A$. By above definition there exists $k \in [0, \frac{1}{2})$ such that

$$d(f^{n+1}x, f^n x) \le kd(f^n x, f^{n-1}x) \le k^2 d(f^{n-1}x, f^{n-2}x) \le ... \le k^n d(fx, x)$$

Now for any integer $r \in \mathbb{N} \cup \{0\}$, by LD property we have,

$$\begin{aligned} d(f^n x, f^{n+r}x) &\le d(f^{n+1}x, f^n x) + d(f^{n+1}x, f^{n+r}x) \\ &\le d(f^{n+1}x, f^n x) + d(f^{n+2}x, f^{n+1}x) + d(f^{n+2}x, f^{n+r}x) \\ &\le d(f^{n+1}x, f^n x) + d(f^{n+2}x, f^{n+1}x) + \ldots + d(f^{n+r}x, f^{n+r}x) \\ &\le (k^n + k^{n+1} + \ldots k^{n+r-1}) d(fx, x) + k^{n+r} d(x, x) \\ &\le \frac{k^n}{1-k} d(fx, x) + k^{n+r} d(x, x) \end{aligned}$$

Now taking limit as $n \to \infty$ the right hand expression tend to 0.

Similarly,

$$d(f^{n+r}x, f^n x) \le \frac{k^n}{1-k} d(x, fx) + k^{n+r} d(x, x)$$

the right hand expression tend to 0 as $n \to \infty$. Hence $\{f^n x\}$ is a Cauchy sequence. Since (X, d) is complete, so $\{f^n x\}$ converges to some point $z \in X$. Since, $\{f^{2n}x\}$ is a sequence in A and $\{f^{2n-1}x\}$ is a sequence in B, so $z \in A \cap B$.

We claim that $fz = z$.

$$d(fz, z) = d(fz, \lim_{n\to\infty} f^{2n-1}x) \le kd(z, \lim_{n\to\infty} f^{2n-2}x) \le 0$$

Again, $d(z, fz) = d(\lim_{n\to\infty} f^{2n}x, fz) \le kd(\lim_{n\to\infty} f^{2n-1}x, z) \le 0$

Hence, $fz = z$.

Uniqueness: Let u and v be two fixed points of f. Then,

$d(u,v) = d(fu, fv) \le k d(u,v) \Rightarrow d(u,v) = 0$

Similarly, $d(v,u) = d(fv, fu) \le k d(v,u) \Rightarrow d(v,u) = 0$

Thus, $d(u,v) = d(v,u) = 0$. Hence $u = v$. This completes the proof of the theorem.

Example 1. *Let* $X = \mathbb{R}$. *Let* $d(x,y) = \max\{|x|,|y|\}$ *for all* $x, y \in X$.

Let $A = [-1,0]$ and $B = [0,1]$ and define $f : A \cup B \to A \cup B$ by

$$f(x) = \begin{cases} \dfrac{-x}{3} & \text{for } x \in [-1,0] \\ \dfrac{-x^2}{5} & \text{for } x \in [0,1] \end{cases}$$

Now,

$$d(fx, fy) = \max\{|\frac{-x}{3}|,|\frac{-y^2}{5}|\} = \max\{\frac{-x}{3}, \frac{y^2}{5}\} \le \max\{\frac{-x}{3}, \frac{y}{3}\} = \frac{1}{3}\max\{-x, y\}$$
$$= \frac{1}{3} d(x,y)$$

Here, x = 0 is the unique fixed point.

Now we prove a fixed point theorem for ld- metric space.

Definition 12. *Let A and B be non empty closed subsets of a ld-metric space* (X, d). *We say that a cyclic mapping* $f : A \cup B \to A \cup B$ *is Kannan type ld-cyclic contraction if there exists* $\lambda \in [0, \frac{1}{2})$ *such that* $d(fx, fy) \le \lambda[d(x, fx) + d(y, fy)]$ *for all* $x \in A$ *and* $y \in B$.

Theorem 2. *Let (X, d) be complete ld- metric space. A and B be non empty closed subsets of X and* $f : A \cup B \to A \cup B$ *be continuous mapping satisfying Kannan type ld-cyclic contraction in X. Then, f has a unique fixed point in* $A \cap B$

Proof: Let $\{f^{2n-1}x\}$ be a sequence in A and $\{f^{2n}x\}$ be a sequence in B. Fix $x \in A$. By above definition there exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(fx, f^2x) \le \lambda[d(x, fx) + d(fx, f^2x)$$

or,
$$(1-\lambda) d(fx, f^2x) \le \lambda d(x, fx)$$
$$d(fx, f^2x) \le \frac{\lambda}{1-\lambda} d(x, fx)$$

Put $k = \dfrac{\lambda}{1-\lambda} < 1$, then $d(fx, f^2x) \le kd(x, fx)$.

By induction,

$$d(f^n x, f^{n+1}x) \le k^n d(x, fx).$$

More generally for $m > n$ we have,

$$\begin{aligned} d(f^m x, f^n x) &\le d(f^{m-1}x, f^m x) + d(f^{m-2}x, f^{m-1}x) + \ldots + d(f^{n-1}x, f^n x) \\ &\le (k^{m-1} + k^{m-2} + \ldots + k^{n-1}) d(x, fx) \\ &= k^{n-1}(1 + k + k^2 + \ldots + k^{m-n}) d(x, fx) \end{aligned}$$

since, $k < 1$, So as $m, n \to \infty$ we have $k^{n-1}(1 + k + k^2 + \ldots + k^{m-n}) \to 0$. Hence, $d(f^m x, f^n x) \to 0$. Similarly $d(f^n x, f^m x) \to 0$.

So $\{f^n x\}$ is a Cauchy sequence. Since (X, d) is complete, so $\{f^n x\}$ converges to some $z \in X$. Note that $\{f^{2n}x\}$ is a sequence in A and $\{f^{2n-1}x\}$ is a sequence in B therefore, $z \in A \cap B$.

Since f is continuous, so $f(z) = f(\lim_{n\to\infty}\{f^n x\}) = \lim_{n\to\infty}\{f^{n+1}x\} = z$

Uniqueness: Let u and v be two fixed points of f. Let u be a fixed, then

$d(u,u) = d(fu, fu) \le k[d(u, fu) + d(u, fu)] \le 2kd(u,u)$

a contradiction, so $d(u,u) = 0$. Similarly we show that $d(v,v) = 0$. Now,

$$d(u,v) = d(fu, fv) \le k[d(u,u) + d(v,v)]$$

which implies $d(u,v) \le 0$. But $d(u,v) \ge 0$. Hence $d(u,v) = 0$.

Similarly we show that $d(v,u) = 0$ .Hence, $u = v$. This completes the proof.

Again for right dislocated metric space we have the following definitions and theorems. One can follow the similar process to prove the theorems.

Definition 13. *Let A and B be non empty closed subsets of a rd-metric space (X, d). A map* $f : A \cup B \to A \cup B$ *is called a rd-cyclic contraction if there exists* $k \in [0, \frac{1}{2})$ *such that* $d(fx, fy) \le kd(x, y)$ *for all* $x \in A$ *and* $y \in B$.

Theorem 3. *Let* (X, d) *be complete rd-metric space. Let A and B be two non empty closed subsets of X and* $f : A \cup B \to A \cup B$ *be a rd-cyclic contraction in X, then T has a unique fixed point in* $A \cap B$.

Definition 14. *Let A and B be non empty closed subsets of a rd-metric space* (X, d). *We say that a cyclic mapping* $f : A \cup B \to A \cup B$ *is Kannan type rd-cyclic contraction if there exists* $k \in [0, \frac{1}{2})$ *such that* $d(fx, fy) \le k[d(x, fx) + d(y, fy)]$ *for all* $x \in A$ *and*

$y \in B$.

Theorem 4. *Let (X, d) be complete rd- metric space. A and B be non empty closed subsets of X and* $f : A \cup B \to A \cup B$ *be continuous mapping satisfying Kannan type rd-cyclic contraction in X. Then, f has a unique fixed point in* $A \cap B$.

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REFERENCES

1. C.T. Aage and J.N. Salunke, The Results on fixed points in dislocated and dislocated quasi-metric Space, *Applied Math. Sci.*, 2(59) (2008) 2941 - 2948.
2. S. Abramsky and A. Jung, *Domain theory in hand book of logic in computer sciences*, 3, Oxford Univ. Press, 1994.
3. M.A. Ahamad, F.M. Zeyada and G.F. Hasan, Fixed point theorems in generalized types of dislocated metric spaces and its applications, *Thai J. Math.*, 11 (2013) 67-73.
4. P. Hitzler and A. K. Seda, Dislocated topologies, *J. Electronic. Engg.*, 51 (12) (2000) 3 –7.
5. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fundamental Mathematicae*, 3(7) (1922) 133-181.
6. George Reny, R.Rajgpoalan and S. Vinayagam, Cyclic contractions and fixed points in dislocated metric space, *Int. J. Math. Anal.*, 7(9) (2013) 403 -411.
7. A. Isufati, Fixed point theorem in dislocated quasi-metric space, *Applied Math.Sci.*, 4(5) (2010) 217-223.
8. K. Jha and D. Panthi, A common fixed point theorem in dislocated metric space, *Applied Math. Sci.*, 6(91) (2012) 4497-4503.
9. K. Jha, R. P. Pant and K. B. Manandhar, A common fixed point theorem for reciprocal continuous compatible mappings in metric space, *Annals of Pure and Applied Math.*, 5(2) (2014) 120 -124.
10. K. Jha, M Imdad and U. Upadhyaya, Fixed point theorems for occasionally weaklycompatible mappings in semi metric space, *Annals of Pure and Applied Math.*, 5 (2) (2014) 153 -157.
11. R. Kannan, Some results on fixed points, *Bull. Cal. Math. Soc.*, 60(1968) 71-76.
12. W. A. Kirk , P. S. Shrinivasan and P. Veeramani, Fixed points for mapping satisfying cyclic contractive condition, *Fixed Point theory*, 4 (2003) 79 -89.
13. S. G. Matthews, *Metric domains for completeness, Technical report 76* (*Ph.D. Thesis*), Department of Computer Science, University of Warwick, Coventry, UK, 1986.
14. D. Panthi, K. Jha and G. Porru, A fixed point theorem in dislocated quasi-metric space, *Amer. J. Math. Stat.*, 3 (3) (2013) 153-156.
15. P. Waszkiewicz, The local traingle axiams in topology and domain theory, *Applied General Topology*, 4(1) (2003) 47-70.