

On a Class of Nonlinear Finite-Part Singular Integral Equations with Carleman Shift

Samah M. Dardery

Department of Mathematics, Faculty of Science, Zagazig University, Egypt
Department of Mathematics, Faculty of Science and Arts Al-Mithnab
Qassim University, KSA
E-Mail: sd.1974@hotmail.com

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Abstract. In this study, the existence and uniqueness of the solution of a class of nonlinear finite-part singular integral equations with Carleman shift preserving orientation has been investigated in the generalized Holder space $H_{\varphi}(\Gamma)$.

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1. Introduction

Many important problems of engineering mechanics like elasticity, plasticity, and fracture mechanics and aerodynamics can be reduced to the solution of a non-linear singular integral equation or non-linear finite-part singular integral equations. Hence, since these are connected with a wide range of problems of an applied character. The theory of non-linear singular integral equations and non-linear finite-part singular integral equations seems to be particularly complicated if closely linked with applied mechanics problems.

Having in mind the implications for different problems of engineering mechanics, E.G.Ladopoulos [15-17] and E.G.Ladopoulos and V.A.Zisis [12-14] introduced and investigated non-linear singular integral equations and non-linear finite-part singular integral equations. This type of non-linear equations has been applied to many problems of structural analysis, fluid mechanics and aerodynamics.

The theory of nonlinear singular integral equations with Hilbert and Cauchy kernel and its related Riemann-Hilbert problems have been developed in works of Pogorzelski W. [19], Guseinov A. I and Mukhtarova Kh. Sh. [7], Wolfersdorf L.V. [22], Ladopoulos E.G [15] and others.

The successful development of the theory of singular integral equations naturally stimulated the study of singular integral equations with shift. The Noether theory of singular integral operators with shift is developed for a closed and open contour (see [9-11], [18] and others).

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The classical and more recent results on the solvability of non-linear singular integral equation and non-linear finite-part singular integral equations should be generalized to corresponding equations with shift, (see[21]). The theory of singular integral equations with shift is an important part of integral equations because of its recent applications in many field of physics and engineering,(see [10,11]).

Existence results and approximate solutions have been studied for nonlinear singular integral equations and nonlinear singular integral equations with shift by many authors among them we mention (1-6, 8, 14).

In this study, the existence and uniqueness of the solution of a class of nonlinear finite-part singular integral equations with Carleman shift preserving orientation has been investigated in the generalized Holder space $H_\varphi(\Gamma)$.

2. Formulation of the problem

In the present paper we construct an approximate solution of the following nonlinear finite-part singular integral equation with Carleman shift:

$$u(t) = (Pu)(t), \quad t \in \Gamma, \quad (2.1)$$

where

$$(Pu)(t) = \lambda AG(t), \quad t \in \Gamma \quad (2.2)$$

and

$$G(t) = F(t, u(t), \Lambda_1 k_1(., u())(t), \Lambda_2 k_2(., u())(t)) \quad (2.3)$$

with

$$\Lambda_1 k_1(t, u(t)) = \frac{1}{\pi i} \int_{\Gamma} \frac{k_1(\tau, u(\tau))}{\tau - t} d\tau \quad (2.4)$$

$$\Lambda_2 k_2(t, u(t)) = \frac{1}{\pi i} \int_{\Gamma} \frac{k_2(\tau, u(\tau))}{(\tau - t)^2} d\tau \quad (2.5)$$

where Γ is a simple smooth closed Lyapunov contour which divide the plane of the complex variable Z into two domains the interior domain D^+ and the exterior domain D^- , $F(t, u, v, w), k_r(t, u(t)), r = 1, 2$, are continuous functions on the domains

$$D_1 = \{(t, u, v, w) : t \in \Gamma, u, v, w \in (-\infty, \infty)\},$$

$$D_2 = \{(t, u) : t \in \Gamma, u \in (-\infty, \infty)\}$$

respectively, where $u(t)$ is an unknown function that has continuous derivative belongs to the space $H_\varphi(\Gamma)$, and the shift operator A is defined by

$$(Au)(t) = \sum_{i=0}^{m-1} a_i(t) W_i u(t), \quad (2.6)$$

where $(W_i u)(t) = u(\alpha_i(t)), \quad i = 0, 1, \dots, m-1$.

Under the assumption that $\alpha(t)$ homeomorphically maps Γ into itself with preservation orientation and satisfies the Carleman condition:

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$$\alpha_m(t) = t, \quad \alpha_i(t) \neq t, \quad 1 \leq i \leq m-1,$$

where $\alpha_i(t) = \alpha[\alpha_{i-1}(t)], \alpha_0(t) = t,$

and $m \geq 2$. Moreover assume that $\alpha'(t)$ satisfies the Holder condition and the coefficients $a_i(t), i = 0, 1, \dots, m-1$ belong to the generalized Holder space $H_\varphi(\Gamma)$ and $\lambda \in (-\infty, \infty)$, is a numerical parameter.

3. Basic assumptions and auxiliary results

In this section, we introduce some definitions, assumptions and results which will be used in the sequel.

Definition 3.1. [7] We denote by Φ the class of all functions $\varphi(\delta)$ defined for sufficiently small non-negative value δ and having the following properties:

i) $\varphi(\delta)$ is a continuous monotonic increasing function.

ii) $\varphi(0) = 0$ and $\varphi(\delta) > 0; \delta > 0$.

iii) $\varphi(\delta)\delta^{-1}$ is almost decreasing function.

Definition 3.2. [9] We denote by $c(\Gamma)$ the space of all continuous functions $u(t)$ defined on Γ with the norm:

$$\|u\|_{c(\Gamma)} = \max_{t \in \Gamma} |u(t)|. \quad (3.1)$$

Definition 3.3. [4] We denote by $H_\varphi(\Gamma)$ the space of all functions $u(t) \in c(\Gamma)$,

$\varphi \in H\Phi$, with the norm:

$$\|u\|_\varphi = \|u\|_{c(\Gamma)} + \|u\|; \quad (3.2)$$

$$\|u\| = \sup_{t_1, t_2 \in \Gamma} \frac{|u(t_1) - u(t_2)|}{\varphi(|t_1 - t_2|)};$$

$$H\Phi = \left\{ \varphi \in \Phi : \int_0^\delta \frac{\varphi(\xi)}{\xi} d\xi + \delta \int_\delta^1 \frac{\varphi(\xi)}{\xi^2} d\xi \leq \tilde{c} \varphi(\delta) \right\}, \quad \tilde{c} \text{ is a positive constant.}$$

Definition 3.4. We denote by $H_{\varphi,1,1,1}(D_1)$ to be the space of all functions $F(t, u, v, w)$, which satisfy the following condition:

$$|F(t_1, u_1, v_1, w_1) - F(t_2, u_2, v_2, w_2)| \leq l_1 \varphi(|t_1 - t_2|) + l_2 |u_1 - u_2| + l_3 |v_1 - v_2| + l_4 |w_1 - w_2|, \quad (3.3)$$

where $(t_i, u_i, v_i, w_i) \in D_1, i = 1, 2, \varphi \in \Phi$ and l_1, l_2, l_3, l_4 are constants.

Definition 3.5. We denote by $H_{\varphi,1}(D_2)$ to be the space of all functions

$k_r(t, u(t)), r = 1, 2$, which have partial derivatives up to first order with respect to t, u and satisfy the following condition:

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$$\left| \frac{\partial^l k_r(t_1, u_1)}{\partial t^i \partial u^j} - \frac{\partial^l k_r(t_2, u_2)}{\partial t^i \partial u^j} \right| \leq m_1^r \varphi(|t_1 - t_2|) + m_2^r |u_1 - u_2|, \quad (3.4)$$

where $(t_r, u_r) \in D_2$, $r = 1, 2$, $\varphi \in \Phi$, $i + j = l$, $l = 0, 1$ and m_1^r, m_2^r are positive constants; the functions $k_r(t, u(t))$, $r = 1, 2$ and their first derivative with respect to t and u belong to the space $H_\varphi(\Gamma)$ for any $u \in H_\varphi(\Gamma)$ [20].

Definition 3.6. Let R be positive number and the function φ satisfies the assumptions of Definition 3.1 we define the convex compact subset $H_\varphi^R(\Gamma)$ as

$$H_\varphi^R(\Gamma) = \{u \in H_\varphi(\Gamma) : |u| \leq R, |u(t_2) - u(t_1)| \leq R\varphi(|t_2 - t_1|), t_1, t_2 \in \Gamma\}$$

Lemma 3.1. [4] The singular operator $S : H_\varphi(\Gamma) \rightarrow H_\varphi(\Gamma)$ denoted by the operator of singular integration

$$(Su)(t) = \frac{1}{\pi i} \int_\Gamma \frac{u(\tau)}{\tau - t} d\tau \quad (3.5)$$

is bounded operator on the space $H_\varphi(\Gamma)$ and satisfies the inequality

$$\|Su\|_\varphi \leq \rho_0 \|u\|_\varphi, \quad (3.6)$$

where ρ_0 is a constant defined as follows :

$$\rho_0 = c_1 \left(\int_0^\delta \frac{\varphi(\xi)}{\xi} d\xi + 1 \right) + c_2 \tilde{c},$$

where c_1, c_2, \tilde{c} are constants.

Lemma 3.2. The shift operator $A : H_\varphi(\Gamma) \rightarrow H_\varphi(\Gamma)$ is a linear bounded continuously invertible operator on the space $H_\varphi(\Gamma)$ and satisfies the inequality

$$\|Au\|_\varphi \leq \gamma_0 \|u\|_\varphi, \quad (3.7)$$

where $\gamma_0 = \sum_{i=1}^{m-1} \|a_i\|_\varphi$.

Proof of the Lemma 3.2 is obvious from the definitions of the shift operator A and the space $H_\varphi(\Gamma)$.

Theorem 3.1. [3] The shift operator A is a linear bounded continuous invertible operator on the space $L_p(\Gamma)$, $1 < p < \infty$ and satisfies the inequality

$$\|Au\|_p \leq \gamma_1 \|u\|_p, \quad (3.8)$$

where

$$\gamma_1 = \sum_{i=1}^{m-1} \|a_i\|_{c(\Gamma)} \left\| (\alpha_i^{-1}) \right\|_{c(\Gamma)}^{1/p}.$$

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For every $u(t), v(t) \in H_\varphi(\Gamma)$, the following equality

$$(uv)(t) = u(t)v(t), \quad t \in \Gamma \quad (3.9)$$

is satisfied.

Lemma 3.3. Let the functions $u(t)$ and $v(t)$ belong to the generalized Holder space $H_\varphi(\Gamma)$ and the equality (3.9) is satisfied. Then $(uv)(t)$ belongs to the space $H_\varphi(\Gamma)$, where

$$\|uv\| \leq \|u\|_{c(\Gamma)} \|v\| + \|v\|_{c(\Gamma)} \|u\| \quad (3.10)$$

Proof:

From the Definition 3.3, we have

$$\begin{aligned} \|uv\| &= \sup_{t_1, t_2 \in \Gamma} \frac{|u(t_1)v(t_1) - u(t_2)v(t_2)|}{\varphi(|t_1 - t_2|)} \\ &\leq \sup_{t_1, t_2 \in \Gamma} \frac{|u(t_1)(v(t_1) - v(t_2)) - v(t_2)(u(t_2) - u(t_1))|}{\varphi(|t_1 - t_2|)} \\ &\leq \sup_{t_1, t_2 \in \Gamma} |u(t_1)| \frac{|v(t_1) - v(t_2)|}{\varphi(|t_1 - t_2|)} + \sup_{t_1, t_2 \in \Gamma} |v(t_2)| \frac{|u(t_1) - u(t_2)|}{\varphi(|t_1 - t_2|)} \end{aligned}$$

Consequently, we can say $\|uv\| \leq \|u\|_{c(\Gamma)} \|v\| + \|v\|_{c(\Gamma)} \|u\|$.

Hence the is proved.

Lemma 3.4. If the functions $k_r(t, u(t)) \in H_{\varphi,1}(D_2)$, $r = 1, 2$ and $u(t) \in H_\varphi^R(\Gamma)$ then the following inequality is valid

$$\|k_r(t, u(t))\|_\varphi \leq \Omega_r, \quad r = 1, 2 \quad (3.11)$$

where $\Omega_r = m_0^r + m_1^r + 2m_2^r R$, $r = 1, 2$, $m_0^r = \max_{t \in \Gamma} |k_r(t, 0)|$, $r = 1, 2$

Proof of the Lemma 3.4 is obvious from Definitions 3.3, 3.5 and 3.6.

Lemma 3.5. If the assumptions of the Lemmas 3.1 and 3.3 are satisfied, $u'(t) \in H_\varphi^R(\Gamma)$ then the operators $\Lambda_r : H_\varphi(\Gamma) \rightarrow H_\varphi(\Gamma)$, $r = 1, 2$ satisfy the following inequalities

$$\|\Lambda_r k_r(t, u(t))\|_\varphi \leq \Theta_r \quad r = 1, 2 \quad (3.12)$$

where

$$\begin{aligned} \Theta_1 &= \rho_0 \Omega_1, \quad \Theta_2 = \rho_0 (\Omega_1^* + \Omega_2^*), \quad \Omega_1^* = m_0^{*2} + m_1^2 + 2m_2^2 R, \\ \Omega_2^* &= (2m_1^{*2} + m_1^2 + 3m_2^2 R) R, \quad m_0^{*2} = \max_{t \in \Gamma} |k_{2r}(t, 0)|, \quad m_1^{*2} = \max_{t \in \Gamma} |k_{2u}(t, 0)|. \end{aligned}$$

Proof:

From inequality (3.6) we get

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$$\|\Lambda_1 k_1(t, u(t))\|_\varphi \leq \rho_0 \|k_1(t, u(t))\|_\varphi$$

Hence from Lemma 3.4 we obtain

$$\|\Lambda_1 k_1(t, u(t))\|_\varphi \leq \Theta_1 \quad \text{where } \Theta_1 = \rho_0 \Omega_1$$

It is well know that the first derivative of the Cauchy singular integral defined by (3.5) is valid as

$$(S'u)(t) = \frac{1}{\pi i} \int_\Gamma \frac{u(\tau)}{(\tau-t)^2} d\tau \quad (3.13)$$

By integrating the right-hand side of (3.13) once by parts, and assuming that the contour Γ is closed, we obtain

$$(S'u)(t) = \frac{1}{\pi i} \int_\Gamma \frac{u'(\tau)}{\tau-t} d\tau = (S u')(t) \quad (3.14)$$

From the relations (3.13) and (3.14) and inequality (3.6) we can say that

$$\|\Lambda_2 k_2(t, u(t))\|_\varphi \leq \rho_0 \|k_{2t}(t, u(t)) + k_{2u}(t, u(t))u'(t)\|_\varphi$$

From the inequalities (3.4), (3.10) and the assumption $u'(t) \in H_\varphi^R(\Gamma)$, it is seen that

$$\|\Lambda_2 k_2(t, u(t))\|_\varphi \leq \Theta_2$$

where

$$\begin{aligned} \Theta_2 &= \rho_0 (\Omega_1^* + \Omega_2^*), & \Omega_1^* &= m_0^{*2} + m_1^2 + 2m_2^2 R, \\ \Omega_2^* &= (2m_1^{*2} + m_1^2 + 3m_2^2 R)R, & m_0^{*2} &= \max_{t \in \Gamma} |k_{2t}(t, 0)|, & m_1^{*2} &= \max_{t \in \Gamma} |k_{2u}(t, 0)| \end{aligned}$$

Thus the lemma is valid.

Lemma 3.6. If the functions $F(t, u, v, w) \in H_{\varphi,1,1,1}(D_1)$ and $u(t) \in H_\varphi^R(\Gamma)$ then the function $G : \Gamma \rightarrow (-\infty, \infty)$ satisfies the following inequality

$$\|G(t)\|_\varphi \leq \Omega_3 \quad (3.15)$$

where $\Omega_3 = l_0 + l_1 + 2l_2 R + l_3 \Theta_1 + l_4 \Theta_2$, $l_0 = \max_{t \in \Gamma} |F(t, 0, 0, 0)|$.

Proof of the Lemma 3.6 is obvious from Definition 3.4, 3.6 and Lemma 3.5.

Corollary 3.1. If the assumptions of the Lemmas 3.2 and 3.6 are satisfied then the operator P that is defined by (2.2) transforms the subset $H_\varphi^R(\Gamma)$ to the subset $H_\varphi^{|\lambda|M}(\Gamma)$ where $M = \gamma_0 \Omega_3$.

Corollary 3.2. If the assumptions of the Lemmas 3.2 and 3.6 are satisfied and if $|\lambda|M \leq R$ then the operator P that is defined with (2.2) transforms the subset $H_\varphi^R(\Gamma)$ to itself.

Theorem 3.2. If the assumptions of Corollary 3.2 are valid then the operator P acts on $H_\varphi^R(\Gamma)$.

4. The main results

In this section we investigate an approximate method for solving the nonlinear finite-part singular integral equation with Carleman shift (2.1).

Now, we prove that the operator P defined by (2.2) is a contraction mapping.

Lemma 4.1. Let the functions $F(t, u, v, w), k_r(t, u(t)), r = 1, 2$ satisfy the relations (3.3) and (3.4) respectively then the operator P defined by (2.1) satisfies the following inequality

$$\rho_{L_p}(Pu, P\tilde{u}) \leq \eta \rho_{L_p}(u, \tilde{u}) \quad (4.1)$$

where
$$\eta = |\lambda| \sum_{i=1}^{m-1} \|a_i\|_{c(\Gamma)} \left\| (\alpha_i^{-1}) \right\|_{c(\Gamma)}^{1/p} \left(l_2 + l_3 m_2^1 \|\Lambda_1\|_p + l_4 m_2^2 \|\Lambda_2\|_p \right)$$

and
$$\rho_{L_p}(u, \tilde{u}) = \left(\int_{\Gamma} |u(t) - \tilde{u}(t)|^p dt \right)^{1/p}$$
 for $u(t), \tilde{u}(t) \in H_{\varphi}^R(\Gamma)$.

Proof: For $u(t), \tilde{u}(t) \in H_{\varphi}^R(\Gamma)$ we get

$$\rho_{L_p}(Pu, P\tilde{u}) \leq |\lambda| \left(\int_{\Gamma} |AF(t, u, v, w) - AF(t, \tilde{u}, \tilde{v}, \tilde{w})|^p dt \right)^{1/p}$$

From relation (3.3) we obtain

$$\rho_{L_p}(Pu, P\tilde{u}) \leq |\lambda| \sum_{i=1}^{m-1} \|a_i\|_{c(\Gamma)} \left\| (\alpha_i^{-1}) \right\|_{c(\Gamma)}^{1/p} \left(\int_{\Gamma} (l_2 |u(t) - \tilde{u}(t)| + l_3 |v(t) - \tilde{v}(t)| + l_4 |w(t) - \tilde{w}(t)|)^p dt \right)^{1/p}$$

From relation (3.4) we get

$$\rho_{L_p}(Pu, P\tilde{u}) \leq |\lambda| \sum_{i=1}^{m-1} \|a_i\|_{c(\Gamma)} \left\| (\alpha_i^{-1}) \right\|_{c(\Gamma)}^{1/p} \left(l_2 + l_3 m_2^1 \|\Lambda_1\|_p + l_4 m_2^2 \|\Lambda_2\|_p \right) \|u(t) - \tilde{u}(t)\|_p$$

Hence $\rho_{L_p}(Pu, P\tilde{u}) \leq \eta \rho_{L_p}(u, \tilde{u})$

where
$$\eta = |\lambda| \sum_{i=1}^{m-1} \|a_i\|_{c(\Gamma)} \left\| (\alpha_i^{-1}) \right\|_{c(\Gamma)}^{1/p} \left(l_2 + l_3 m_2^1 \|\Lambda_1\|_p + l_4 m_2^2 \|\Lambda_2\|_p \right)$$

Thus the Lemma 4.1 is valid.

Theorem 4.1. If the assumptions of the Lemma 4.1 are satisfied and $|\lambda|M \leq R$. Then the operator $P : H_{\varphi}^R(\Gamma) \rightarrow H_{\varphi}^R(\Gamma)$ is a continuous operator.

Proof: Let $u_0, u_n \in H_{\varphi}^R(\Gamma), n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \|u_n - u_0\|_{\varphi} = 0$

We want to show that $\lim_{n \rightarrow \infty} \|Pu_n - Pu_0\|_{\varphi} = 0$.

From the inequality (3.7) we have

$$\|Pu_n - Pu_0\|_{\varphi} \leq |\lambda| \gamma_0 \|F(t, u_n, v_n, w_n) - F(t, u_0, v_0, w_0)\|_{\varphi} \quad (4.2)$$

From the inequalities (3.3) and (3.6) we get

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$$\|F(t, u_n, v_n, w_n) - F(t, u_0, v_0, w_0)\|_\varphi \leq (l_2 + l_3 m_2^1 \rho_0 + l_4 m_2^2 \rho_0 c^*) \|u_n - u_0\|_\varphi \quad (4.3)$$

where c^* is appositive constant.

From the inequalities (4.2) and (4.3) we obtain

$$\|Pu_n - Pu_0\|_\varphi \leq |\lambda| \gamma_0 (l_2 + l_3 m_2^1 \rho_0 + l_4 m_2^2 \rho_0 c^*) \|u_n - u_0\|_\varphi \quad (4.4)$$

From the inequality (4.4) it can easily be seen that if

$$\lim_{n \rightarrow \infty} \|u_n - u_0\|_\varphi = 0$$

then

$$\lim_{n \rightarrow \infty} \|Pu_n - Pu_0\|_\varphi = 0.$$

Hence the operator P is continuous operator.

From Theorems 3.2, 4.1 and Arzela-Ascoli theorem, the image $H_\varphi^R(\Gamma)$ under the operator P is compact set, therefore we can apply Schauder's fixed-point theorem.

Theorem 4.2. If the assumptions of Theorem 4.1 are satisfied and If

$$k_r(t, u(t)) \in H_{\varphi,1}(D_2), r = 1, 2, \quad F(t, u, v, w) \in H_{\varphi,1,1,1}(D_1), \quad |\lambda| M \leq R$$

Then the nonlinear finite-part singular integral equation with Carleman shift (2.1) has a fixed point in the subset $H_\varphi^R(\Gamma)$.

Now, we will investigate the uniqueness of the solution of the equation (2.1) and the problem that how can we find the approximate solution. For this, we will use a more useful modified version of the Banach fixed-point principle for the uniqueness of the solution of operator equations [7].

Theorem 4.2. Suppose that all assumptions of Theorem 3.2 and Lemma 4.1 are satisfied, $\eta < 1$. Then Equation (2.1) has in $H_\varphi^R(\Gamma)$ exactly one solution u^* . This solution can be obtained by taking limit of the sequence of successive approximations $u_0, u_1, u_2, \dots, u_m, \dots$ uniformly convergent on the set Γ , where

$$u_{m+1}(t) = \lambda AF(t, u_m(t), \Lambda_1 k_1(\cdot, u_m(\cdot))(t), \Lambda_2 k_2(\cdot, u_m(\cdot))(t)), t \in \Gamma.$$

For $m = 0, 1, 2, \dots$ and $u_0 \in H_\varphi^R(\Gamma)$ any initial point and the following estimate for the rate of convergence holds:

$$\rho_{L_p}(u_n, u^*) \leq \frac{\eta^n}{1 - \eta} \rho_{L_p}(u_1, u_0), n = 1, 2, \dots$$

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