

Set-Valuations of Graphs and their Applications: A Survey

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Abstract. A set-valuation of a graph $G = (V, E)$ assigns to the vertices or edges of G elements of the power set 2^X of a given nonempty set X subject to certain conditions and set-valuations have a variety of origins. Acharya defined a set-indexer of G to be an injective set-valuation $f : V(G) \rightarrow 2^X$ such that the induced set-valuation $f^\oplus : E(G) \rightarrow 2^X$ on the edges of G defined by $f^\oplus(uv) = f(u) \oplus f(v), \forall uv \in E(G)$ is also injective, where \oplus denotes the operation of taking the symmetric difference of the subsets of X . In particular, he studied variety of set-valued graphs such as set-graceful graphs, topological set-graceful graphs, set-sequential graphs, set-magic graphs, etc. In 2006, Acharya and Germina defined the concept of distance pattern distinguishing set of a graph (open-distance pattern distinguishing set of a graph). Let $G = (V, E)$ be a given connected simple (p, q) -graph with diameter d_G , $\emptyset \neq M \subseteq V(G)$ and for each $u \in V(G)$, let $f_M(u) = \{d(u, v) : v \in M\}$ be the distance-pattern of u with respect to the marker set M . If f_M is injective (uniform) then the set M is a DPD-set (ODPU-set) of G and G is a DPD-graph (ODPU-graph). Following a suggestion made by Michel Deza, Acharya and Germina, who had been studying topological set-valuations, introduced the particular kind of set-valuations for which a metric, especially the cardinality of the symmetric difference, is associated with each pair of vertices in proportion to the distance between them in the graph. Particular cases of set-valuations of graphs are also being studied in detail by many authors. In this paper, we give a brief report of the existing results, new challenges, open problems and conjectures that are abound in this area of set-valuations of graphs.

Keywords: set-valuations, distance-compatible set-labeling, set-graceful graphs, set-sequential graphs, topologically set-graceful graphs, dcsl graphs, DPD graphs, ODPU graphs, ODPC graphs, CDPU graphs.

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1. Introduction

Labeling is a term used in technical sense for naming objects using symbolic format drawn from any universe of discourse such as the set of numbers, algebraic groups or the power set 2^X of a ‘ground set’ X . The objects requiring labeling could come from a variety of fields of human interest such as chemical elements, radio antennae, spectral bands and plant/animal species. Further, categorization of objects based on certain clustering rules might lead to derived labels from the labels of objects in each cluster; for instance labels a and b of two individual elements in a dyad $\{A, B\}$ could be used to derive a labeling for the dyad in a way that could reflect a relational combination of the labels a and b . To be specific, A and B are assigned labels a, b from an algebraic group, whence the dyad $\{A, B\}$ is assigned the label $a*b$ where $*$ is the group operation. Such assignments are generally motivated by a need to optimize on the number of symbols used to label the entire discrete structure so that the structure could be effectively encoded for handling its computerized analysis.

In general, graph labelings, where the *basic elements* (i.e., vertices and/or edges) of a graph are assigned elements of a given set or subsets of a nonempty ‘ground set’ subject to given conditions, have often been motivated by practical considerations such as the one mentioned above. They are also of theoretical interest on their own right. ‘Graph labeling’ as an independent notion using numbers was first introduced in the mid sixties. Most graph labeling methods trace their origin to one introduced by Rosa in 1967 [95].

Even though the study of graceful graphs and graceful labeling methods was introduced by Rosa [95] the term *graceful graph* was used first by Golomb in 1972 [69]. Rosa defined a β -valuation of a (p, q) -graph G as an injection f from the vertices of G to the set $\{0, 1, \dots, q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are all distinct. In a graceful labeling of a graph G the resulting edge labels must be distinct and take values $1, 2, \dots, q$. The study of graceful labelings of a graph is a prolific area of research in graph theory. The graceful labeling problem is to determine which graphs are graceful. Proving a graph G is or is not graceful involves either producing a graceful labeling of G or showing that G does not admit a graceful labeling. While the graceful labeling of graphs is perceived to be a primarily theoretical topic in the field of graph theory, gracefully labelled graphs often serve as models in a wide range of applications. Such applications include coding theory and communication network addressing. Bloom and Golomb [27] give a detailed account of some of the important applications of gracefully labelled graphs. That ‘all trees are graceful’ is a long-standing conjecture known as the “Ringel–Kotzig Conjecture” [9].

A seminal departure from assigning numbers to the basic elements of a given graphs G was made in [3] by suggesting to consider set-valued functions instead, motivated by certain considerations in social psychology. Interpersonal relationships depend on personal attitudes of the individuals in any social group. When opinions are expressed by the individuals to others in the group, the types of interpersonal interactions get affirmed and/or modified. On the other hand, such affirmations and/or modifications in various types of interpersonal interaction in the group could induce change in the attitudes of the persons in the group. In fact, it was this revisory socio-psychological phenomenon that motivated a study of *total set-valuations* $h : V(G) \cup EG \rightarrow 2^X$, viz. assignment of subsets of a given set to the basic elements of a given graph with a variety

of constraints motivated either by theoretical or by practical considerations [1]. In this paper, we give a brief report of the existing results, new challenges, open problems and conjectures that are abound in this area, of set-valuations of a finite simple graph.

For standard terminology and notation in graph theory, hypergraph theory and signed graph theory not given here, the reader may refer respectively to [72], [34] and [75]. In this paper, by a *graph* we shall mean a finite undirected graph without loops or multiple edges.

2. Set-valuations

A *set-valuation* of a graph $G=(V, E)$ is simply an assignment of elements of the *power set* 2^X of a given nonempty ‘ground set’ X to the basic elements of G ; set-valuations have a variety of origins [3]. In particular, a *set-indexer* of G is defined to be an injective ‘vertex set-valuation’ $f : V(G) \rightarrow 2^X$ such that the induced ‘edge set-valuation’ $f^\oplus : E(G) \rightarrow 2^X$ on the edges of G defined by $f^\oplus(uv) = f(u) + f(v), \forall uv \in E(G)$, is also injective, where ‘ \oplus ’ denotes the operation of taking the symmetric difference of the subsets of X . Further, G is said to be *set-graceful* if there exists a set-indexer $f : V(G) \rightarrow 2^X$ such that $f^\oplus E((G)) := \{f^\oplus(e) : e \in E(G)\} = 2^X - \{\emptyset\}$, such a set-indexer being called a *set-graceful labeling* of G . In [3], it is proved that for every graph G there exists a *topological set-indexer* (or, a *T-set-indexer*), which is a set-indexer $f : V(G) \rightarrow 2^X$ such that the family $f(V(G)) = \{f(u) : u \in V(G)\}$ is a topology on X , thereby establishing a link between graph theory and point-set topology.

In [1], a set-indexer f of a given graph $G=(V, E)$ is called a *segregation* of X on G if $f(V(G)) \cap f^\oplus(E(G)) = \emptyset$ and if, in addition, $f(V(G)) \cup f^\oplus(E(G)) = 2^X$ then f is called a *set-sequential labeling* of G . A graph is then called *set-sequential* if it admits a set-sequential labeling with respect to some set X . Recently, it has been proved that the path P_{2^n-1} is not set-sequential for $n \in \{2, 3\}$ and is set-sequential for every value of $n \geq 4$. In general, the problem of determining set-sequential trees is open [1]. Also, it has been shown that K_5 is the only set-sequential Eulerian graph [5].

Since, by their very definitions, set-graceful, topologically set-graceful and set-sequential graphs have exponential orders or sizes and it is not hard to see that most graphs do not fall under any of these classes of graphs. Even within the classes of graphs satisfying the order or size conditions for a graph to be in any of these classes, we surmise that similar conclusion holds. Hence, it becomes important to have many infinite families of such graphs towards gaining deeper insight into the properties of these very special graphs. Special investigations have been initiated in this area [13,15,16,17,18]; in one of these, given an arbitrary graph G , a method has been described to generate infinite ascending chains of set-graceful graphs, topologically set-graceful graphs and set-sequential graphs with G as an *initializing graph* and such that at each stage of construction the preceding graph is an induced subgraph of the succeeding ‘host’ graph.

Given any set-valuation of a graph $G=(V, E)$, the hypergraphs $H^V=(X, f(V))$, $f(V)=\{f(v) : v \in V\}$ and $H^E=(X, f^\oplus(E))$, $f^\oplus(E)=\{f^\oplus(v) : v \in E\}$ are called respectively the *vertex set-valuation (V_f)-hypergraph* of G and *edge set-valuation (E_f) – hyper*

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-graph of G ; recently these hypergraphs are being studied (see [20, 109, 110]).

Next, for any set $F \subseteq E$, let $\sum_{x \in F} h(x)$ stand for any expression

$$(h(u_x) \oplus h(v_x)) \oplus (h(u_y) \oplus h(v_y)) \oplus \dots$$

when F is expressed as $\{u_x v_y \in E : x \in F\}$. The following result is an analogue of a well known property of arbitrary networks, called *Kirchhoff's Voltage Law* (KVL); the analogy could be seen by treating 2^X as an additive 'voltage group' where the 'addition' is the binary operation of taking symmetric difference between any two subsets of X .

Proposition 1. [1] *Let $G=(V, E)$ be any graph. Then, for any total set-valuation $h:V \cup E \rightarrow 2^X$ and for any cycle C , $\sum_{x \in E(C)} h(x) = \phi$.*

Corollary 2. [5] *If $G=(V, E)$ is any finite Eulerian graph then for any set-valuation $f:V \rightarrow 2^X$, $\sum_{x \in E} f^\oplus(x) = \phi$.*

Proposition 3. [1] *For any graph $G=(V, E)$ any total set-valuation $h:V \cup E \rightarrow 2^X$ and any path P of length at least three, joining vertices u and v , $\sum_{x \in E(P)} h(x) = h(u) \oplus h(v)$.*

Corollary 4. [5] *Let $G=(V, E)$ be any graph, $f:V \rightarrow 2^X$ be any injective vertex set-valuation of G and u, v be any two arbitrarily given distinct vertices of G . Then, for no $u-v$ path P of length at least three, one has $\sum_{x \in E(P)} f^\oplus(x) = \phi$.*

Example 5. *Let $G=(V, E)$ be isomorphic to a path $P=(u=u_0, u_1, u_2, \dots, u_k=v)$ where $k \geq 1$. Suppose G admits an injective set-valuation $f:V \rightarrow 2^X$ such that $\sum_{x \in E(P)} f^\oplus(x) = \phi$. Then, by Corollary 4, we get $k \leq 2$, whence we have $G \in \{P_1, P_2, P_3\}$ where P_n denotes the path with n vertices (and hence of length $n-1$).*

Example 6. *From the way we are able to give an injective set-valuation of P_n , $n \leq 3$, a set-valuation f of the star $K_{1,n}$ is suggested, which is such that f^\oplus is also injective;*

however, note that $h = f \cup f^\oplus$ is not injective.

Observe in the set-valuations of graphs explained above that the induced set-valuation are also injective on their own right which need not be so in general. This motivated the following definition.

Definition 7. [1] *Let $G=(V, E)$ be a graph, X be a nonempty set and 2^X denote the set of all subsets of X . A set-indexer of G is an injective set-valued function $f:V(G) \rightarrow 2^X$ such that the function $f:E(G) \rightarrow 2^X - \{\phi\}$ defined by $f^\oplus(uv) = f(u) + f(v)$, $\forall uv \in E(G)$ is also injective.*

Definition can be thought of as applicable to infinite graphs as well.

Theorem 8. [1] *Every graph has a set-indexer.*

The infimum of the cardinalities of the sets Y with respect to which G has a set-indexer is hence defined to be the *set-indexing number* of G , denoted $\sigma(G)$. Hence, the problem is actually interesting to find $\sigma(G)$ for any given graph G , especially when G is finite. For example, $\sigma(K_4) = 3$.

Theorem 9. [1] *Let G be any graph, X be a non-empty set and $f:V(G) \rightarrow 2^X$ be any assignment to the vertices of G . Then, the mapping $\bar{f}:V(G) \rightarrow 2^X$ defined by $\bar{f}(u) = \overline{f(u)} (= X - f(u)) \forall u \in V(G)$ is a set assignment of the vertices of G .*

Theorem 10. [1] *For any (p,q) -graph G , $\lceil \log_2(q+1) \rceil \leq \sigma(G) \leq p-1$. where $\lceil r \rceil$ denote the least integer not less than the real number r .*

Remark 11. [1] $|E(G)| \leq 2^{|X|} - 1$ and hence $V(G) - 1 \geq \lceil \log_2 E(G) + 1 \rceil$. Clearly, $\sigma(K_n) = n-1$ for $1 \leq n \leq 5$.

Theorem 12. [10] *If G is a (p, q) -graph with $p \geq 6$ then $\sigma(G) \leq p-2$ and this bound is attained by K_6 and K_7 .*

Corollary 13. [10] *If G is a graph with $\sigma(G) = |V(G)| - 1$ then $|V(G)| \leq 5$.*

Lemma 14. [10] *If f is a set-assignment to the vertices of K_n for $n \geq 3$ such that f^\oplus is injective then, f is also injective.*

Another motivation from social psychology to study assignment problems on graphs comes from voltage graphs (or gain networks) [4]. A *voltage graph (gain graph)* is an ordered triple (G, M, \underline{g}) where G is an undirected graph, M is an arbitrary algebraic group, called the voltage group, and \underline{g} is a function assigning elements of M to the edges of G such that $\underline{g}(u, v) \underline{g}(v, u) = e$, the identity element of M , and each edge uv of G is regarded as a symmetric pair of arcs (u, v) and (v, u) ; \underline{g} is called a voltage assignment of G . If in particular when M is taken to be the group M_n of all n -dimensional vectors of $+1$'s and -1 's under ordinary componentwise multiplication; then the voltage assignment s_n is called an n -signing of G . A particular case is the signed graph introduced by Harary [73]. We can interpret an n -signing s_n of a graph $G = (V, E)$ as an assignment of subsets of a set $Z = \{z_1, z_2, \dots, z_n\}$ to the elements as follows: There is a natural one-to-one correspondence $\psi: 2^Z \rightarrow M_n$ obtained by setting $\psi(A) = (a_1, a_2, \dots, a_n)$ for $A \in 2^Z$ such that $a_i = +1$, if $z_i \in A$ and $a_i = -1$, if $z_i \notin A$, for each $i \in \{1, 2, \dots, n\}$. The interested reader may refer to [4] for more results.

3. Set-graceful graphs

Recall that a graph $G=(V,E)$ is said to be *set-graceful* [1] if there exist a set X and a set indexer $f:V(G) \rightarrow 2^X$ such that $f^\oplus(E(G)) = 2^X - \{\emptyset\}$; and Acharya [1] called the

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minimum size of the set X with respect to which G is set-graceful the set-graceful index $\gamma(G)$.

Following are some fundamental results on set-graceful graphs:

Theorem 15. [1] *If G is a set-graceful (p, q) -graph with $\gamma(G) = m$ then, $q=2^m-1$ and $p \leq q+1$.*

Theorem 16. [1] *Every set-graceful graph G with q edges and r vertices can be embedded in a set-graceful graph, with q edges and $q+1$ vertices.*

Theorem 17. [1] *Every connected set-graceful graph with q edges and $q+1$ vertices is a tree of order $p=2^m$ and for every natural number m such a tree exists.*

Theorem 18. [105] *For any integer $m \geq 2$, the path P_{2^m} with 2^m vertices is not set-graceful.*

Theorem 19. [1] *A necessary condition for a graph $G=(V, E)$ to have a set-graceful labeling with respect to a set X of cardinality n is that it be possible to partition $V(G)$ into two subsets V_e and V_o such that the number of edges joining the vertices of V_o with those of V_e is exactly 2^{n-1} .*

If a (p, q) -graph is set-graceful then $q = 2^m - 1$ for some positive integer m . This implies for *almost all* graphs of order p , and hence almost all graphs are not set-graceful. Further, for every positive integer m , there exists a set-graceful graph of size $q = 2^m - 1$. However, not all (p, q) -graphs with $q = 2^m - 1$ are set-graceful as, for instance, it is not difficult to verify that the complete graph K_5 is not set-graceful. More generally the following more results are well known.

Theorem 20. [1] *If K_n is set-graceful and $\gamma(K_n) = m$ then $n = \frac{1}{2}(1 + \sqrt{2^{m+3} - 7})$.*

Theorem 21. [89] *The complete graph K_n is set-graceful if and only if $n \in \{2, 3, 6\}$.*

Theorem 22. [89] *A necessary condition for a complete graph K_n to be set-graceful with respect to a set X is that $(n-2)$ is a perfect square.*

The condition is not sufficient as K_{11} is not set graceful.

Theorem 23. [89] *The cycle C_n is set-graceful if and only if $n=2^m-1$ for some integer $m \geq 2$.*

Acharya [1] considered special Eulerian circuits which yield a set-graceful labeling of the cycle of length $2^{n+1}-1$ and called such Eulerian circuits of D_n^* successful ones and others as unsuccessful and also raised the following problem.

Problem 24. [1] *Determine (at least one) successful Eulerian circuits in D_n^* if they exist.*

Lemma 25. [1] *There is a unique cycle of length 2 in D_n^* .*

Lemma 26. [1] *In any successful Eulerian circuit in D_n^* , (x, A) , (A, \bar{A}) , and (\bar{x}, \bar{A}) , (\bar{A}, A) cannot occur in that order for any $x \neq A$.*

Lemma 27. [1] *There exists an Eulerian circuit α in D_n^* such that for every $x \neq A$ both the arc pairs (x, A) , (A, \bar{A}) and (\bar{x}, \bar{A}) , (\bar{A}, A) do not occur simultaneously.*

The following conjecture of Acharya [1] is yet to be settled.

Conjecture 28. [1] *In D_n^* , all Eulerian circuits are successful if and only if $n=2$.*

Theorem 29. [1] *Let G be any graph and u be any vertex of G . Then for any set-assignment $f:V(G) \rightarrow 2^X$ to the vertices of G there exists a set-assignment $h:V(G) \rightarrow 2^X$ to the vertices of G such that $h(u)=\emptyset$ and $f^\oplus = g^\oplus$.*

Corollary 30. [1] *Let G be any graph and $O_X(G)$ denote the set of all optimal set-indexers f of G with respect to a set X such that $f(u)=\emptyset$ for some $u \in V(G)$. Then, $O_X(G)$ is non-empty.*

Theorem 31. [1] *If $f:V(G) \rightarrow 2^X$ is an optimal set-indexer of a graph G then $\bigcup_{u \in V(G)} f(u) = \emptyset$.*

Theorem 32. [1,13] *Every connected set-graceful graph with q edges and $q+1$ vertices is a tree of order $p=2^m$ and for every natural number m such a tree exists.*

In fact the star $K_{1,2^n-1}$ was the graph used in the proof of Theorem 32. One takes set X with $|X| = n$, assigns \emptyset to the center of the star and all the nonempty remaining subsets of X are then assigned to the remaining vertices of the star in a one-to-one manner. In fact, one may not limit n to be finite in the labeling whence described work shows that the star whose center has order higher infinite degree than the order of the set X with respect to which one obtains the graceful set-valuation.

Theorem 33. [13] *If a tree is set-graceful with respect to a set X of cardinality m , then its order is 2^m .*

It is important to note here that not every tree of order 2^m need be set-graceful as, for instance, it is not difficult to verify that the path P_4 of length 3 is not set-graceful.

Theorem 34. [13] *For any integer $n \geq 2$ the path P_{2^n} is not set-graceful.*

Following conjecture appeared in [13].

Conjecture 35. *For every natural number n , there exists a set-graceful tree of diameter $n-1$.*

This conjecture is proved for $n = 1, 2, 3$ and 4 in [13].

Theorem 36. [89] *For any natural number n , $C_{2^{n-1}}$ is set-graceful.*

If δ_i denotes the diameter of the cycle $C_{2^{i-1}}$ then $d_i = d_{i-1} + 2^{i-2}$ whence, $diam(C_{2^{n-1}}) = \lfloor (2^n - 1)/2 \rfloor$ for any integer $n \geq 2$. Thus we have, for any integer m of the form $\lfloor (2^2 - 1)/2 \rfloor$ for some natural number n , there exists a set-graceful graph of diameter m .

It is interesting to note that every set-graceful (p, q) -graph $G = (V, E)$ with respect to a set X of cardinality n can be embedded in a set-graceful $(q+1, q)$ -graph H . This may be achieved as follows. Let f be a set-indexer of G . Then $2^X - f(V)$ has $m(G) = 2^n - p$ elements each of which does not appear as a set assigned to a vertex in (G, f) . Then adjoin $m(G)$ isolated vertices to G and assign to them the sets from $2^X - f(V)$.

Mentioned below are some of the important results on set-graceful graphs appeared in [15, 86, 94].

1. If H is a set-graceful graph with n edges ($n \geq 1$) and $n + 1$ vertices then the join of H and \overline{K}_m is set-graceful if and only if $m = 2^{n_1} - 1$, $n_1 \in N$.
2. If S_n denote the star with $2^n - 1$ spokes and $m = 2^{n_1} - 1$, for $n_1 \in N$, then the join $S_n + \overline{K}_m$ is set-graceful.
3. $P_n + \overline{K}_m$ is set-graceful if $n \leq 2$ and $m = 2^{n_1} - 1$, for $n_1 \in N$.
4. $P_n + \overline{K}_m$ is not set-graceful for all $n \neq 2^{x_1}$, and for all $m \neq 2^{x_2} - 1$ for $x_1 > 2$, $x_1, x_2 \in N$.
5. $P_n + \overline{K}_m$ is not set-graceful for $n = 2^2$ and $m = 2^{n_1} - 1$ for $n_1 \in N$.
6. Let T be a caterpillar with the path P_m having $V(P_m) = \{v_1, v_2, \dots, v_m\}$. Then T is set-graceful with respect to a set of cardinality m if $d(v_i) = 2^i + 1$, $1 \leq i \leq m - 1$.
7. Let T be a caterpillar with the path K_2 and let $V(K_2) = \{v_1, v_2\}$. Then T is set-graceful with respect to a set of cardinality m if $d(v_1) = 2^{m-1} + 1$ and $d(v_2) = 2^{m-1} - 1$.
8. Let X be a set with $|X| = m$. A uniform binary tree with one pendant edge added at the root vertex having $2^m - 1$ edges is set-graceful.
9. The splitting graph $S'(G)$ of a set-graceful graph G is not necessarily set-graceful.
10. $S'(P_n)$ is not set-graceful for all n .
11. Let G be a (p, q) -graph. $S'(G)$ is not set-graceful for $q \equiv 0, 2, 3 \pmod{4}$. Further the only possible values for q so that $S'(G)$ could be set-graceful are 21, 85, 341,
12. $K_{3,5}$ is not set-graceful.
13. Let G be a set-graceful graph. Then the *corona* of G_f and K_1 , that is, $G_f \odot K_1$ is set-graceful if G_f is the full augmentation of G .

In [1] it has been proved that any graph G can be embedded in a set-graceful graph; further, he showed that every set-graceful (p, q) -graph $G = (V, E)$ with respect to a set of cardinality n can be embedded in a set-graceful $(q+1, q)$ -graph H , where by an *embedding* of G one means identifying an induced subgraph in H that is isomorphic to G . Such a ‘host’ graph H of G together with its set-graceful labeling is considered as *fully augmented*. Fully augmented set-graceful graph of a set-graceful (p, q) -graph G can be obtained by adding $2^n - p$ isolated vertices with labels as those subsets of X that are not present as vertex labels in the set-graceful labeling given on G .

If G is a set-graceful (p, q) -graph, then G_f will denote the *full augmentation* of a set-graceful labeling f of G in the sense that G_f contains G as an induced subgraph and f taken in its extended form as the set-graceful labeling of $H = G + \overline{K}_m$ defined therein (i.e. f restricted to the vertices of G is the original set-graceful labeling of G). Given any set-graceful $(q+1, q)$ -graph H , there exists an infinite ascending chain $H = H_0 \subset H_1 \subset H_2 \subset \dots$ of set-graceful graphs H_1, H_2, \dots such that H_i is an induced subgraph of H_{i+1} for every nonnegative integer i and H_i is a fully augmented connected graph of order $V(H_i) = q - i + 1 + \sum_{r=1}^i 2^n r$, for every $i \geq 1$. Acharya *et al.*

[15] proved that every graph can be embedded into a connected set-graceful graph and the problems of determining the clique number, the independence number and the chromatic number of a set graceful graph are NP-complete. For more results on set-graceful graphs, the interested reader may refer to [1, 3, 5, 10, 13, 14, 15, 16, 18, 78, 86, 94].

Motivated from the following theorem of Acharya, he then defined the concept of topologically set-graceful graphs.

Theorem 37. [1] *For every graph G , there exists a set-indexer $f : V(G) \rightarrow 2^X$ such that the family $f(G) = \{f(u) : u \in V(G)\}$ is a topology on X .*

3.1. Topologically set-graceful graphs

Acharya proved in [1] that for every graph G , there exists a set-indexer $f : V(G) \rightarrow 2^X$ such that the family $f(G) = \{f(u) : u \in V(G)\}$ is a topology on X , called a *topological set-indexer* (or, a *top-set-indexer* in short) of G with respect to X , thereby establishing another interesting link between the theory of graphs and point-set topology, the earlier known such link being a one-to-one correspondence between the set of all transitive digraphs on a given set V and the set of all topologies on V , pointed out by E. Sampathkumar and Kulkarni [99]; hence, it would be of much independent interest to investigate the inter-linkage between the notions of top-set-indexers of a graph and of transitive digraphs. Further, in [1] it is shown that for a finite graph G , the *top-set-indexing number* of G , denoted $t(G)$, is the smallest cardinality of a set X with respect to

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which G admits a top-set-indexer and any such top-set-indexer (denoted σ_g) of G is said to be *optimal* if $f^\oplus(E(G)) = 2^X - \{\emptyset\}$. It is obvious that, in general $\sigma(G) \leq t(G) \leq \sigma_g(G)$. Further, Acharya [3] called G a *topologically set-graceful* (or, *top-set-graceful*) graph if G satisfies $\sigma_g(G) = t(G)$ and any optimal top-set-indexer of such a graph is a *top-set-graceful labeling* of G . In general, if G is a graph and f is a top-set-indexer of G then the graph G together with f , denoted G^f , is said to be *topologised* by f (or, f *topologises* G). Also, if f is a top-set-indexer of G then the members of $\{\emptyset\} \cup f(V)$ are said to be *f -open* and those subsets of X that are not in $f(V)$ are said to be *f -closed* in the topological sense.

Given a topology τ on a nonempty set X , let \mathcal{G}_τ denote the class of all graphs $G = (V, E)$ that admit a set-indexer $f : V(G) \rightarrow 2^X$ of G such that $f(V) = \tau$; G is then a *realization* of τ . Construct a star whose vertices represent the members of τ in a one-to-one manner, with the center labelled by the empty set \emptyset and all the other (pendant) vertices labelled by the nonempty τ -open sets; thus, $K_{1,|\tau|-1} \in \mathcal{G}_\tau$. We shall call τ *graceful* if a realization G of τ is set-graceful. (See [13,17,86,94])

By definition, for a top-set-graceful graph $G = (V, E)$ together with a top-set-graceful labeling $f : V(G) \rightarrow 2^X$, $\{\emptyset\} \cup f(V)$ forms a topology on X . Hence, it is of interest to see precisely for which class of graphs these two definitions coincide. The following result answers one of the conjectures raised by Acharya [1, 3].

Theorem 38. [15] *The complete graph K_n is set-graceful if and only if $n \in \{1, 2, 3, 6\}$.*

The following answers the same question for T-set-graceful complete graphs.

Theorem 39. [15] *The complete graph K_n is topologically set-graceful if and only if $n \leq 3$.*

Theorem 40. [15] *Every graph can be embedded in a connected topologically set-graceful graph.*

Theorem 41. [15] *Let G be a graph. Then there exists an infinite sequence $(H = G_1, \dots, G_2, \dots)$ of connected topologically set-graceful graphs G_i where H contains G as an induced subgraph and G_i contains G_{i-1} as an induced subgraph for all integers $i \geq 2$.*

Since almost all labelled graphs are not topologically set-graceful it might be fruitful to find some classes of graphs which are not topologically set-graceful. Following are some of the classes of graphs which come under this category.

Theorem 42. [17] *The cycle C_n is topologically set-graceful if and only if $n=3$.*

It is not possible to have a graceful topology with 2^n open sets on a set X of cardinality $n + m$, for $n, m \in \mathbb{N}$, where \mathbb{N} denotes the set of natural numbers. A topology τ with 2^n open sets is a graceful topology of a graph G if and only if the size of G is $2^n - 1$.

Lemma 43. [15,17] *The n -dimensional cube Q_n , $n > 1$, is not T-set-graceful.*

For a T-set-graceful (p, q) -graph G , we have $p = 2^i + 2^{k-i} - 1$ and $q = 2^k - 1$, for some k and i . Hence, the grid $P_m \times P_n$ for $m, n \in \mathbb{N}$ and $C_m \times C_n$ for $m, n \in \mathbb{N}$ are not T-set-graceful. Similarly, if G_1 is a T-set-graceful (p_1, q_1) -graph and G_2 is a T-set-graceful (p_2, q_2) -graph then $G_1 + G_2$ and $G_1 \cup G_2$ are not T-set-graceful and if $G_1 \times G_2$ is T-set-graceful then p_1 and p_2 are both odd.

Lemma 44. [15,16] *An r -regular graph is not T-set-graceful for $r \equiv 0, 1, 3 \pmod{4}$.*

Hence, if G is an r -regular T-set-graceful graph then $r \equiv 2 \pmod{4}$. The converse of this result is not true. For instance, if $r = 2$ then, by Theorem 42, no cycle C_n , $n \geq 4$, is T-set-graceful. Next, for any integer r , $6 \leq r \equiv 2 \pmod{4}$ we have K_{r+1} which is not T-set-graceful.

Problem 45. [15,16] *For $6 \leq r \equiv 2 \pmod{4}$, determine the class of r -regular T-set-graceful graphs.*

Further, we have the following conjecture.

Conjecture 46. [15,18] *Every graph G can be embedded as an induced subgraph in an r -regular T-set-graceful graph for any integer r that satisfies $r - \Delta \equiv (k+2) \pmod{4}$ for some nonnegative integer k .*

Theorem 47. [17, 94] *For trees, the notion of set-gracefulness and the notion of top-set-gracefulness are equivalent.*

The following most fundamental result mimics a well known result of P. Erdős quoted by Golomb [69]. Towards this end, we need to recall the result due to Harary [72] that the number of labelled trees with p vertices is p^{p-2} .

Theorem 48. [17, 94] *Almost all finite labelled trees are not top-set-graceful.*

By virtue of Theorem 47 and the fact that all the trees are covered by the class of labeled trees, we have the following interesting result which is somewhat surprising in view of quite a contrasting analogue of it in the theory of graceful graphs.

Corollary 49. [17] *Almost all finite trees are not top-set-graceful.*

In fact, there do exist exponential order trees that are not even set-graceful. For example, it is known that the path P_{2^n} for $n \geq 2$ is not set-graceful (see [10, 78]) and hence cannot be top-set-graceful. On the other hand, a number of exponential order trees are known to be set-graceful (e.g., see [3, 10, 17, 79]).

Thus, it becomes quite interesting to determine the class of set-graceful trees. Further, which of them are top-set-graceful?

Conjecture 50. [17] *There exists a ‘good’ characterization of (top-)set-graceful trees.*

Given a topology τ on a nonempty set X , let G_τ denote the class of all graphs G

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$=(V, E)$ that admit a set-indexer $f : V(G) \rightarrow 2^X$ of G such that $f(V) = \tau$; G is then a realization of τ . Further, the topology τ is called *graceful topology* if a realization G of τ is set-graceful.

Let τ be a graceful topology on the set X with $|X| = n$. Take $i \in X$. Then, the realization of τ has an edge of label $X \setminus \{i\}$. Correspondingly, there exist two sets A_i and B_i in τ so that $A_i \oplus B_i = X \setminus \{i\}$. Then, it is necessary that if $A_i \cap B_i \neq \emptyset$, then $A_i \cap B_i = \{i\} \in \tau$, and if $A_i \cap B_i = \emptyset$, then $A_i \oplus B_i = A_i \cup B_i = X \setminus \{i\} \in \tau$. Accordingly, let I_0 and I_1 be defined as follows: $I_0 = \{i \in X : A_i \cap B_i \neq \emptyset\}$, $I_1 = \{i \in X : A_i \cap B_i = \emptyset\}$. Then for every $i \in I_0$, $\{i\} \in \tau$ and for every $j \in I_1$, $X \setminus \{j\} \in \tau$. For any graceful topology τ with respect to a set X , I_0 is nonempty, since otherwise, if I_0 is empty then for every $i \in X$, $i \in I_1 = \{i \in X : A_i \cap B_i = \emptyset, i \in X\}$, whence $X \setminus \{i\} \in \tau$. But since τ is a topology, $\bigcup_{i=1}^{n-1} X \setminus \{i\} = \{n\} \in \tau$, a contradiction.

Hence, every graceful topology on an arbitrarily given nonempty set X contains at least one singleton set.

Hence, it is natural to ask whether we can determine the number of singleton sets that are necessarily to be in τ so that τ is a graceful topology. The following theorem gives a complete answer to this question.

Theorem 51. [17] *A topology τ_m in which there are exactly m singleton sets A_1, A_2, \dots, A_m is a graceful topology if and only if it contains all supersets and all subsets of $A_1 \cup A_2 \cup \dots \cup A_m$.*

Corollary 52. [17] *The minimal cardinality of a graceful topology, containing k singleton sets, on a set of cardinality n is $2^k + 2^{n-k} - 1$.*

Theorem 53. [17] *If $l \geq 1$ and $2^l + 2^{n-l} - 1 = S$, then there exist graceful topologies with cardinalities $S, S + 2^1 - 1, S + 2^1, S + 2^2 - 1, S + 2^2, \dots, S + 2^{n-l} - 1$.*

Theorem 54. [17] *Let X be a set of cardinality n and τ_n be the discrete topology on X . Then, at most $2^{(n-1)(2^n-1)}$ labelled set-graceful realizations of τ_n can be constructed.*

Corollary 55. [17] *There exist graceful topologies with cardinality n where $1 \leq n \leq 27$.*

Theorem 56. [17] *Let X be a set of cardinality n and τ_n be the discrete topology on X . Then, at most $2^{(n-1)(2^n-1)}$ labelled set-graceful realizations of τ_n can be constructed.*

Corollary 57. [17] *Let τ_i be a graceful topology on X . Then $\tau_i \times \tau_i$ can be partitioned into $2^n - 1$ equivalence classes.*

Further, in view of Corollary 57, Acharya *et al.* pursued their study in finding newer classes of top-set-graceful graphs and, in that spirit, established some such classes of graphs. Of course, still there could be special classes of top-set-graceful graphs whose characterizations turn out to be NP-complete.

Theorem 58. [17] *The number of distinct graceful topologies on a set X of cardinality n is $2^n - 1$.*

As well known, two topologies τ_1 and τ_2 on X are isomorphic if there exists a bijection $f: X \rightarrow X$ such that $A \in \tau_1$ if and only if $f(A) \in \tau_2$.

Theorem 59. [17] *Almost all labelled graphs are not top-set-graceful.*

The next result gives an infinite class of finite graphs, which are not trees, that are top-set-graceful.

Theorem 60. [17] *$K_2 + \overline{K}_t$ is top-set-graceful with respect to X of cardinality m if and only if $t = 2^{m-1} - 1$.*

By a *topologically full augmentation* of a set-graceful labeling f of a graph G we mean the number of isolated vertices that need to be added to G such that by assigning distinct nonempty subsets from $2^X \setminus f(V(G))$ to the isolated vertices the resulting extension F of f to the so *augmented graph* G^* is its top-set-graceful labeling. More results may be found in [1, 15, 16, 17, 86, 94].

3.2. Topogenic graphs

Let X be any nonempty set, $\mathfrak{T}(X)$ denote the set of all topologies on X and let $\tau \in \mathfrak{T}(X)$. We shall say that τ is ‘graphical’ [19] if there exist a graph $G = (V, E)$ and a set-labeling $f: V(G) \rightarrow 2^X$ of G such that $f(V) \cup f^\oplus(E) = \tau$, where $f^\oplus(E) := \{f^\oplus(e) : e \in E\}$. Construct a graph $G = (V, E)$ with vertex set V such that $f: V \rightarrow \tau$ is a bijection and edge set $E = \{\{A, B\} : A, B \in \tau \text{ and } A \cap B = \emptyset\}$.

Then, by the very definition of a topology on X ,

$$uv \in E \Leftrightarrow f(u) \cap f(v) = \emptyset \Leftrightarrow f^\oplus(uv) = f(u) \oplus f(v)$$

$$= f(u) \cup f(v) - (f(u) \cap f(v)) = f(u) \cup f(v) \in \tau.$$

Since f is a bijection, it is easy to see that $f(V) \cup f^\oplus(E) = \tau$. Thus, we have

Proposition 61. [19] *Every topology on a nonempty set X is graphical.*

This motivated us to initiate a study of the following new notion.

Definition 62. [19] *A graph $G = (V, E)$ is topogenic with respect to a nonempty ‘ground set’ X if it admits a topogenic set-indexer, which is a set-indexer $f: V(G) \rightarrow 2^X$ such that $f(V) \cup f^\oplus(E) = \tau$, is a topology on X .*

Theorem 63. [19] *For every positive integer n , there exists a connected topogenic graph of order n .*

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Consider a topogenic set-indexer $f(V) \cup f^\oplus(E) = \tau$, of a (p, q) -graph $G = (V, E)$ and let $\tau_f = f(V) \cup f^\oplus(E)$. The number of distinct f -open sets, viz., $|\tau_f|$, is called the *topogenic strength* [19] of f over G and if G is finite, the minimum (respectively maximum) of $|\tau_f|$ taken over all possible topogenic set-indexers f of G is denoted $\zeta^0(G)$ (respectively $\zeta^1(G)$). Because of the injectivity of f and f^\oplus we must have $\zeta^0(G) \leq |f(V) \cup f^\oplus(E)| \leq \zeta^1(G) \leq p + q - k$, where k is the number of vertices of G that are adjacent to the vertex w for which $f(w) = \emptyset$ (such a vertex w exists since τ_f is a topology on X). Moreover, $p \leq \zeta^0(G)$. Further, since $\emptyset \notin f^\oplus(E(G))$, $q \leq \zeta^0(G) - 1$ or, equivalently, $q + 1 \leq \zeta^0(G)$. Thus, for a topogenic (p, q) -graph G , $p \leq \zeta^0(G)$ and $q + 1 \leq \zeta^0(G)$. (See [19, 57, 58].)

Since a non-set-graceful graph may still be topogenic, we need to examine whether K_4 , which is not set-graceful as such (cf.: [10]), is topogenic. We shall indeed see that K_4 is not topogenic. In [57] it is proved that K_4, K_5, K_6 are not topogenic. In fact Germina *et al.* [57] proved the following theorem.

Theorem 64. [57] $K_p, p \in \{1, 2, 3, 6\}$ are the only set-graceful complete graphs that are (gracefully) topogenic.

However, as observed above, non-set-graceful graphs could be topogenic, even gracefully topogenic. Therefore, it would be of potential interest to determine such complete graphs.

Conjecture 65. [57] For every integer $p \geq 7, K_p$ is not topogenic.

Finiding the total number of labelled topologies $T(n)$ one can define on a set X of cardinality n is still an open question. Also, there is no known simple formula giving $T(n)$ for at least some specific values of n . For small values of n , $T(n)$ may be found; for example, $T(1) = 1, T(2) = 4$, and $T(3) = 29$. For $n \geq 4$, the calculations are complicated.

The *topogenic index* [19] of a graph G is defined as the least cardinality of a ground set X such that there is a topology τ on X which acts as a topogenic set-indexer of a graph H having the least order and containing G as an induced subgraph; this number is denoted as $Y(G)$. If G is a topogenic graph then $Y(G)$ is just the least cardinality of a ground set X such that there is a topology τ on X which acts as a topogenic set-indexer of G . Topogenic graphs are studied in [19, 57, 58, 82]. We list some of the results.

1. The star $K_{1, 2^{n-1}}$ is gracefully topogenic for any positive integer n .
2. Every graph can be embedded as an induced subgraph of a gracefully topogenic graph.
3. The totally disconnected graph, which is characterized by the relations, $V \neq \emptyset$ and $E = \emptyset$, is topogenic.
4. The complete bipartite graph $K_{m, n}$ is topogenic for all positive integers m and n .
5. The complete tripartite graph $K_{1, m, n}$ is topogenic for all positive integers m and n .

3.3. Bi-set-graceful graphs

Definition 66. [94] A graph G is said to be bi-set-graceful if both G and its line graph are set-graceful. The Line graph of G , denoted by $L(G)$ has $E(G)$ as its vertex set with two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent..

Proposition 67. [94] A uniform Binary tree T_n with one pendant edge added at the root vertex having $2^m - 1$ edges is bi set graceful if and only if $n \leq 2$.

Proposition 68. [94] Star S_n with $2^n - 1$ spokes is bi set-graceful if and only if $n \leq 2$.

Proposition 69. [94] The complete graph K_n on n vertices is bi set-graceful if and only if $n \leq 3$.

Theorem 70. [94] An r -regular connected (p, q) -graph G is bi set-graceful if and only if $r = 2$ and $q = 2^n - 1$ for some $n \in \mathbb{N}$.

4. Set-sequential graphs

A graph G is said to be set-sequential [1] if there exist a nonempty set X and a bijective set-valued function $f: V(G) \cup E(G) \rightarrow 2^X - \{\emptyset\}$ such that $f(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$. We quote here some of the interesting results.

Theorem 71. [1] If $G = (V, E)$ is a connected set-sequential (p, q) -graph, then $G + K_1$ is set-graceful.

Thus, one has the following straight forward result, giving a necessary condition for a (p, q) -graph to be set-sequential.

Theorem 72. [1] If a (p, q) -graph is set-sequential, then $p+q = 2^m - 1$, for some positive integer m .

Corollary 73. [1] No (p, q) -graph with $p+q \not\equiv 0 \pmod{2}$ is set-sequential.

Theorem 74. [21] If G is set-graceful then $G \cup \overline{K}_t$ is set-sequential for some positive integer t .

For every positive integer m , there exists a set-sequential (p, q) -graph with $p+q = 2^m - 1$. For instance, take the star $G = K_{1, 2^{m-1} - 1}$ and assign the non-empty subsets of the set $X = \{1, 2, 3, \dots, m\}$ as follows: Assign X to the central vertex and assign the first $2^{m-1} - 1$ nonempty subsets of X in their natural lexicographic order to the pendant vertices of G in a one-to-one manner. It is easy to verify that this assignment results into a set-sequential labeling of G . The converse is not true as, for instance, the path P_4 of length 3 shows. Using the fact that by the adjunction of one new vertex w with \emptyset as its label to a set-sequentially labelled graph H and then making w adjacent to all the vertices of H yields a set-graceful graph as also a necessary condition for a graph to be set-graceful [1] and the following result gives a necessary condition for a graph to be set-sequential.

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Theorem 75. [77] *If a (p, q) -graph G has a set-sequential labeling with respect to a set X of cardinality $m \geq 2$ then there exists a partition of the vertex set $V(G)$ of G into two non-empty sets A and B such that $|A| \geq |B|$ and the number of edges joining vertices of A with those of B is exactly $2^{m-1} - |B|$.*

Acharya and Hegde [10] have given the following conjecture, which was later on disproved by Hegde [78].

Conjecture 76. [10] *For every integer $m \geq 2$ such that $m' = 2^{m+3} - 7$ is a perfect square, the complete graph K_n of order $n = \frac{m}{2} [\sqrt{m'} - 1]$ is set-sequential.*

Theorem 77. [77] *The complete graph K_n is set sequential with respect to a set X of cardinality $m \geq 2$, if and only if $n = 2$ and 5 .*

Acharya and Hegde [10] conjectured that 'No path P_m is set-sequential for any integer $m > 2$ ', which was later on disproved in [88].

Theorem 78. [14] *A star $K_{1,p}$ is set-sequential if and only if $p = 2^{n-1} - 1$ for some integer $n \geq 2$.*

Theorem 79. [60] *Binary trees are not set-sequential.*

The following problems are open.

Problem 80. [1] *Characterize set-sequential graphs, in particular set-sequential trees.*

Problem 81. [1] *Given a set-graceful graph G , does there exist a connected set-sequential graph H such that G is an induced subgraph of H ?*

An immediate observation from the very definition is that a necessary condition for a (p, q) -graph $G = (V, E)$ to admit a set-sequential labeling is $p + q + 1 = 2^m$ for some positive integer m [10]. But the following embedding regarding set sequential graph gives the NP-completeness of determining the clique number and the chromatic number of a connected set sequential graph.

Theorem 82. [15] *Every (p, q) -graph of order $p \geq 5$ can be embedded into a connected set-sequential graph.*

Let G be the 'host' graph and H the embedded graph and, if the chromatic number and the clique number of G are ≥ 3 , then the chromatic number $\chi(H) = \chi(G) + 1$ and the clique number $\omega(H) = \omega(G) + 1$. Therefore, the problems of determining the chromatic number and the clique number of a connected set sequential graph are NP-complete [15].

Following result describes a method of constructing an ascending chain of set-sequential graphs for an arbitrarily given graph G containing it as an induced subgraph.

Theorem 83. [15] *Let G be any graph. Then, there exists an infinite sequence $\mathcal{H} = (H = G_1, G_2, \dots)$ of set-sequential graphs where H contains G as an induced subgraph and G_i contains G_{i-1} as an induced subgraph for all integers $i \geq 2$.*

For more results on set-sequential graphs the reader to refer to [1, 5, 10, 13, 14, 15, 16, 18, 77, 86, 94].

4.1. Set-sequential topogenic graphs

Let $G=(V, E)$ be any graph and X be any set. A set-indexer f of G is called a segregation of X on G if the sets $\{f(u): u \in V(G)\}$ and $\{f^\ominus(e): e \in E(G)\}$ are disjoint and if, in addition their union is the set $Y(X) = \tau-\emptyset$ for some topology τ on X , then f is called a *sequential topogenic labeling* of G [59]. A graph is called *sequential topogenic* if it admits a sequential topogenic labeling with respect to some set X . The sequential topogenic index $\gamma(G)$ of a graph G is the least cardinality of a set X with respect to which G has a sequential topogenic labeling. Further, if $f : V \cup E \rightarrow 2^X$ is a sequential topogenic labeling of G with $|X| = \gamma(G)$ we call f an optimal sequential topogenic labeling of G . Germina *et al.* [59] proved many classes of topogenic graphs such as the complete bipartite graph $K_{m,n}$ is sequential topogenic for every non-negative integers m,n .

In [59], it is proved that any arbitrary graph G can be embedded as an induced subgraph of a set-graceful (set-sequential, topologically set-graceful, topogenic, sequentially topogenic) graph which is set-graceful (set-sequential, topologically set-graceful, topogenic, sequentially topogenic) and studied the complexity in determining the various parameters like chromatic number, clique number, independence number, domination number etc. of set-graceful (set-sequential, topologically set-graceful, topogenic, sequentially topogenic, bitopological) graphs.

5. Set-magic graphs

A graph G is said to be *set-magic* if its edges can be assigned distinct subsets of a set X such that for every vertex u of G union of the subsets assigned to the edges incident at u is X , such a set assignment to the edges of G being called a *set-magic labeling* of $G[1]$.

Following are some interesting results on set-magic graphs.

Theorem 84. [3] *For any finite graph G having a set-magic labeling $f:E(G) \rightarrow 2^X$ we must have $|E(G)| \leq 2^X$ which gives $\lceil \log_2 |E(G)| \rceil \leq |X|$. Hence, if $m=m(G)$ denotes the least cardinality of a set with respect to which G has a set-magic labeling then, $\lceil \log_2 |E(G)| \rceil \leq m(G)$.*

Lemma 85. [102] *Consider any integer $m \geq 2$ and let $\underline{m}=\{1,2,\dots,m\}$. Order the set $2^{\underline{m}}$ by putting $A < B$ for distinct $A, B \in 2^{\underline{m}}$ if and only if either $|A| < |B|$ or $|A|=|B|$ and $\min(A-B) < \min(B-A)$. Let A_1, A_2, \dots, A_{2^m} be the increasing sequence obtained in accordance with the ordering relating $<$ defined above. Then for each $i, 1 \leq i \leq 2^{m-1}$, one has $A_i = \underline{m} - A_p$ where $p = 2^m - i + 1$. That is, A_i and A_p are complements of each other in \underline{m} .*

Problem 86. [10] *Determine the graphs, finite and infinite which admit set-magic labeling f such that $|f(e)| = |f(e')|$ for any two edges e, e' in the component of G .*

Theorem 87. [102] For every integer $m \geq 3$, $m(W_n) = 1 + \lceil \log_2 n \rceil$.

Theorem 88. [102] There exists a connected infinite graph $G = (V, E)$ with a set-magic labeling g with $|g(e)| < \infty$ for each $e \in E(G)$.

Theorem 89. [104] An infinite graph G has a set-magic labeling f such that $|f(e)| < \infty$ for every $e \in E(G)$ if and only if $d(u) = |V(G)| \forall u \in V(G)$.

Another interesting class of set-magic labeling [10], of a (finite or infinite) graph G are those $f : E(G) \rightarrow 2^X$ with the property that $\{f(e) : e \in E_u\}$ is a partition of X for each $u \in V(G)$. Such a set-magic labeling may be called *partitioning set-magic labeling* of G , and a graph G which admits such a set-magic labeling may be called *partition set-magic*.

The class of partition set-magic graphs is a subclass of multicolorable graphs. A graph $G = (V, E)$ is said to be *multicolorable* if it admits a multicoloring, which is essentially a set-assignment $f : E(G) \rightarrow 2^X$ such that $\{f(e) : e \in E_u\}$ is a partition of X for each $v \in V(G)$. Thus, an injective multicoloring is same as a partitioning set-magic labeling and vice versa.

Alternatively, we may regard an n -multicoloring of a graph G as assignment of one or more colors from the color set $X = \{X_1, X_2, \dots, X_n\}$ to the edges of G so that at each vertex u of G every color appears on exactly one edge incident at u . Equivalently, an n -multicoloring of a graph $G = (V, E)$ may be thought of as assignment λ of n -tuples from the involutory group M_n to the edges of G and for each $i \in n$, exactly one of the n -tuples as assigned to the edges in E_u has a -1 entry at the i th coordinate. Thus, the following holds, $\prod_{x \in E_u} \lambda(x) = \bar{I} \forall u \in V(G)$ ($\bar{I} = -1$).

In general, an n -assigning $\lambda : E(G) \rightarrow M_n$ is called an *odd n -signing* if it satisfies $\prod_{x \in E_u} \lambda(x) = \bar{I} \forall u \in V(G)$ ($\bar{I} = -1$), [1] that is an odd number of n -tuples assigned by λ to the edges in E_u have a -1 entry in their i -th coordinate for each $i \in n$ and for each $u \in V(G)$. In terms of the n -tuple representation ψ of set of assignments $\prod_{x \in E_u} \lambda(x) = \bar{I} \forall u \in V(G)$ ($\bar{I} = -1$) is equivalent to the set theoretic condition $\sum_{x \in E_u} \psi^{-1}(\lambda(x)) = X \forall u \in V(G)$.

Theorem 90. [34] Let G be a simple graph. Then a regular multigraph H can be obtained from G by edge multiplication (i.e., replacement of some edges by several parallel edges) if and only if for every independent set S of vertices in G we have,

$$(i) |N(S)| \geq |S|$$

$$(ii) |N(S)| = |S| \Rightarrow N(N(S)) = S.$$

Theorem 91. [34] A simple graph $G = (V, E)$ has a multicoloring if and only if some regular multigraph $H = (V, F)$ obtained from G by edge multiplication satisfies:

$m_H(S, V - S) \geq \Delta(H)$ for every $S \subseteq V(G)$ with $|S|$ odd where for a graph K , and for any two sets A, B of vertices of K , $m_K(A, B)$ denotes the number of edges in A and the other in B .

Definition 92. [1] Let H be a multigraph and $\delta(H)$ denotes the minimum of the vertex degree $d_H(u)$ in H . Acharya called H a uniform degree parity (or u.d.p) multigraph if the degrees of the vertices of H are of the same parity. H is said to be odd (or even) if the number of vertices in H is odd (even). A spanning subgraph of H in which the degrees of the vertices are all odd is called an odd degree factor of H .

Lemma 93. [10] If H is a multigraph having an odd degree factor then, H is an even graph.

Acharya [1] defined multigraph H is odd degree factorable if it can be written as the edge disjoint union degree factors H_1, H_2, \dots and the collection $\{H_i\}$ is then called an odd degree factorization. He [1] also defined the multiplication index of G denoted by $\kappa(G)$ as the least integer n for which G has an n -multicoloring. If G is not multicolorable then $\kappa(G) = \infty$. He proved a necessary condition for a graph G to have a generalized multicoloring (see [1]) and conjectured the following.

Conjecture 94. [10] The conditions (i) All the vertex degree in H_g have the parity of n , and (ii) H_g has an odd degree factorization, are sufficient for a graph G to have a generalized multicoloring.

$\kappa(G) \geq \Delta^*(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the least integer t for which a t -regular multigraph can be obtained from G by edge multiplication. If G is a graph having a generalized multicoloring then an odd degree factorable multigraph H may be obtained by the edge multiplication. Hence, let $\delta^*(G)$ denote the least of the minimum vertex degrees $\delta(H)$ of the degree factorable multigraph H obtainable from G by edge multiplication, and let $\kappa_g(G)$ denote the least integer n for which G has a generalized n -multicoloring; and he called $\kappa_g(G)$ the generalized multicoloring index of G and $\kappa_g(G) = \infty$ if G has no generalized multicoloring; $\kappa_g(G) \geq \delta^*(G) \geq \delta(G)$.

Definition 95. [10] Given a set-assignment $h: V(G) \cup E(G) \rightarrow 2^X$ to the elements of a graph $G = (V, E)$ its norm $\|h\|$ is defined as the number $\|h\| = \min_{x \in V \cup E} |h(x)|$

Remark 96. [10] Put $\|h\|_V = \min_{u \in E_u} |h(u)|$, $\|h\|_E = \min_{e \in E_u} |h(e)|$,

then h is a set-magic labeling of G whenever

(i) the restriction map h/E is injective, and

(ii) $\bigcap_{e \in E_u} h(e) = h(u) = X$ for each $u \in V(G)$.

Set-Valuations of Graphs and their Applications—A Survey

If $OSM_X(G)$ denotes the set of all optimal set-magic labelings with respect to the set X of a set-magic graph G then the set-magic number $|G|_m$ defined by $|G|_m = \max_{h \in OSM_X(G)} |h|_E$. Also, an optimal set-magic labeling h with $|h|_E = |G|_m$ is said to be extremal [1]. Any two $g_1, g_2 \in OSM_X(G)$ are said to be equivalent [89], written $g_1 \sim g_2$ if there is a permutation π of X , and an automorphism ψ of G such that $g_1(uv) = \pi g_2(\psi(u)\psi(v)) \forall uv \in E(G)$. A set-magic graph G is said to be uniquely set-magic [10] if $g_1 \sim g_2$ for each pair $g_1, g_2 \in OSM_X(G)$.

Theorem 97. [10] *For every integer $n \geq 2$, there are only a finite number of graphs G for which $m(G) = n$.*

Sedlacek [102] claimed that a graph G is set-magic if and only if G has at most one vertex of degree one, which was easily disproved by the argument that K_2 has two vertices of degree one; but it is set-magic. Vijayakumar [105] proved the following result.

Theorem 98. [105] *A graph G is set-magic if and only if it has at most one pendant vertex.*

Problem 99. [105] *For a set-magic graph G , find the best possible upper bound $m(G)$.*

Theorem 100. [105] *For a infinite graph G the following are equivalent.*

1. G has set-magic labeling f such that $|f(e)| < \infty$.
2. G has set-magic labeling f such that $|f(e)| < \infty$ for all $e \in E$ and $|f(e)| = |f(e')|$ whenever e and e' are edges in the same connected component of G .
3. For $k \in \mathbb{N}$, G has a k -magic labeling.
4. G has 2-magic labeling.
5. For all $v \in V(G)$, $\deg(v) = |V|$.

Theorem 101. [105] *If an infinite graph G has a set-magic labeling f which satisfies the condition $|f(e)| < \infty$ for all $e \in E$ then $\deg(v) = |V|$ for all $v \in V(G)$.*

Theorem 102. [105] *Any infinite graph G has a set-magic labeling f which satisfies the condition $\deg v = |V|$ for all $v \in V(G)$, has a set-magic labeling satisfying $|f(e)| = 2$ for all $e \in E(G)$.*

Theorem 103. [105] *For an infinite graph $G = (V, E)$ the following are equivalent*

1. G has a set-magic labeling f such that $|f(e)| < \infty$ for all $e \in E$.
2. G has a set-magic labeling f such that $|f(e)| < \infty$ for all $e \in E$ and $|f(e)| = |f(e')|$ whenever, e, e' are edges in the same connected component of G .
3. G has a set-magic labeling satisfying $|f(e)| = \eta$ for all $e \in E(G)$, where η is a positive integer.
4. G has a set-magic labeling satisfying $|f(e)| = 2$ for all $e \in E(G)$.
5. $\text{Deg}(v) = |V|$ for all $v \in V(G)$.

6. Distance-patterns of vertices in a graph

Let $G = (V, E)$ be a given connected simple (p, q) -graph, $\emptyset \neq M \subseteq V(G)$ and $u \in V(G)$.

Then, the M -distance-pattern of u is the set $f_M(u) = \{d(u, v) : v \in M\}$. If f_M is injective then the set M is a distance pattern distinguishing set (DPD-set) of G and G is a DPD-graph [6,48]. By Acharya [6, 48] while defining this new concept, following were the problems identified.

Problem 104. [48] For what structural properties of the graph G , the function f_M is injective (or respectively uniform)?

Problem 105. [48] Characterize DPD-graphs having the given DPD-number.

Problem 106. [48] Which graphs G have the property that every k -subset of $V(G)$ is a DPD-set of G . Solve this problem in particular when $k = \rho(G)$?

Problem 107. [48] Which graphs G have exactly on $\rho(G)$ -set ?

Problem 108. [48] For which values of n it is possible to extract a proper n -distance coloring of a given graph G using a distance-pattern function as a listing of colors for the vertices?

Problem 109. [48] Given any positive integer k , does there exist a graph G with $\rho(G) = k$?

Acharya [6, 48], while sharing his many incisive thoughts during our discussion in June 2008, introduced a new approach, namely, distance neighborhood pattern matrices (dnp-matrices), to study dpd-graphs, as follows.

For an arbitrarily fixed vertex u in G and for any nonnegative integer j , we let $N_j[u] = \{v \in V(G) : d(u, v) = j\}$. Clearly, $N_0[u] = \{u\}$ for all $u \in V(G)$ and $N_j[u] = V(G) - V(C_u)$ whenever j exceeds the eccentricity $\varepsilon(u)$ of u in the component C_u to which u belongs. Thus, if G is connected then, $N_j[u] = \emptyset$ if and only if $j > \varepsilon(u)$. If G is a connected graph then the vectors

$$\overline{u} = (|N_0[u]|, |N_1[u]|, |N_2[u]|, \dots, |N_{\varepsilon(u)}[u]|)$$

associated with $u \in V(G)$ can be arranged as a $p \times (d_G + 1)$ nonnegative integer matrix D_G given by

$$\begin{pmatrix} 1 & |N_1[v_1]| & |N_2[v_1]| & |N_3[v_1]| & \cdots & 0 & 0 & 0 \\ 1 & |N_1[v_2]| & |N_2[v_2]| & |N_3[v_2]| & \cdots & |N_e[v_2]| & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & |N_1[v_p]| & |N_2[v_p]| & |N_3[v_p]| & \cdots & 0 & 0 & |N_{e(p)}[v_p]| \end{pmatrix}$$

where d_G denotes the diameter of G : we call D_G the *distance neighborhood pattern* (or, *dnp*-) *matrix* of G .

In general, for an arbitrarily given nonempty subset M of vertices in G , the M -*dnp* matrix D_G^M of G is defined by replacing $N_j[v_i]$ by $N_j^M[v_i]$ for all indices j and for all vertices $v_i \in M$ in the above *dnp*-matrix. It is important to note here that all the parameters, like the eccentricities $\varepsilon(v_i)$ and the diameter, are then to be with respect to the *marker set* M . Thus, D_G^M would be a nonnegative $p \times (d_G + 1)$ matrix. We will denote by D_G^{*M} the $(0,1)$ -matrix obtained from D_G^M by replacing all its nonzero entries by 1.

In an attempt to solve these problems many researchers [25, 26, 28, 29, 31, 32, 33, 48, 50, 54, 55, 56, 61, 62, 65, 66] studied different concepts of distance pattern of vertices in a graph.

6.1. Distance pattern distinguishing (DPD-set) of a graph

Let $G = (V, E)$ be a given connected simple (p, q) -graph with diameter d_G , $\emptyset \neq M \subseteq V(G)$ and for each $u \in V(G)$, let $f_M(u) = \{d(u, v) : v \in M\}$ be the distance-pattern of u with respect to the marker set M . If f_M is injective then the set M is a distance pattern distinguishing set of G and G is a DPD-graph [48, 6]. For a given connected simple (p, q) -graph, $G = (V, E)$ and an arbitrary nonempty subset $M \subseteq V(G)$ of G and for each $v \in V(G)$, define $N_j^M[v] = \{v \in M : d(u, v) = j\}$. The $p \times (d_G + 1)$ nonnegative integer matrix $D_G^M = (|N_j^M[v_i]|)$ is called the M -distance neighborhood pattern (or, M -*dnp*) matrix of G . Many interesting results are established in [25, 48, 50, 55, 56, 62]. Some of them are listed below.

1. For any (p, q) -graph G , $V(G)$ is a DPD-set if and only if G is isomorphic to K_1 .
2. Let G be any graph having a DPD-set M . Then, any vertex of G is adjacent to at most two pendent vertices. Further, if G has a vertex with exactly two pendent vertices adjacent to it then, exactly one of them belongs to M .
3. If a block G of order $p \geq 3$ has a DPD-set M then, G is not complete and $3 \leq |M| \leq p-1$.
4. There is no DPD-graph of diameter two, except P_4 .
5. Let G be a (p, q) -graph. Then for any positive integer k , $1 \leq k \leq p-1$, G is k -DPD-set uniform if and only if $G \cong K_1$ or $G \cong K_2$.
6. For any graph G , $q(G) = 1$ if and only if G is a path.
7. A tree T of order $p \geq 2$ has a DPD-set of cardinality $p-1$ ($\neq 2$) if and only if T is isomorphic to a path or to the tree consisting of the path $P_6 := (v_1, v_2, v_3, v_4, v_5, v_6)$ with one other vertex w that is adjacent to v_3 or v_4 .
8. Every path $P_n := (v_1, v_2, v_3, \dots, v_n)$, $n \geq 4$ has a DPD-set $M = (v_i, v_{i+1}, v_{i+3})$ for every fixed i , $1 \leq i \leq n-3$.

9. Let T be any caterpillar with distance between any two pendant vertices greater than two. Then T has a DPD-set.
10. In any graph G , a nonempty $M \subseteq V(G)$ is a DPD-set if and only if no two rows of $D_G^*{}^M$ are identical.
11. For any DPD-graph G possessing a nontrivial DPD-set, all the nonzero entries in the first column of $D_G^*{}^M$ are unity and their number is less than the number of rows.
12. Let G be a graph with DPD-set M and the M -DNP matrix $D_G^*{}^M$ is such that the rows of $D_G^*{}^M$ are the elements of a basis of the Euclidean space \mathbb{R}^n . Then $G \cong P_n$, a path on n vertices.

6.2. Open distance pattern uniform (ODPU)-sets of graph

We [48] can also associate with each vertex u of a graph $G = (V, E)$ its *open A -distance pattern* (or, ‘ODP’ in short) $f_A^0(u) = \{d(u, v) : v \in A, u \neq v\}$ and intend to study graphs in which every vertex has the same open distance pattern; we call such graphs *ODP-uniform graphs* (or, simply, ‘ODPU-graphs’), where the *set-valued function* (or, *set-valuation*) f_A^0 is called the *open distance pattern uniform* (or, a *ODPU-*) *function* and A is called an *ODPU-set* of G . *ODPU-number* of a graph G , denoted $\zeta(G)$, is the minimum cardinality of an ODPUs-set in G ; if G does not possess an ODPUs-set then we postulate that $\zeta(G)=0$. Following are some interesting results we could establish on ODPUs graphs (see [48]).

1. A tree T has an ODPUs-set M if and only if $T \cong P_2$.
2. In any graph G , if there exists an ODPUs-set M , then $M \subseteq C(G)$.
3. If G has an ODPUs-set M then $\max \{f_M^0(v)\} = |f_M^0(v)| = r(G) \quad \forall v \in V(G)$.
4. Let $G = (V, E)$ be any graph and $M \subseteq V$. Then, M is an ODPUs-set if and only if $\max \{f_M^0(v)\} = |f_M^0(v)| = r(G)$ for all $v \in V(G)$.
5. There is no graph with ODPUs-number three.
6. A connected graph G is an ODPUs-graph if and only if the center $C(G)$ of G is an ODPUs-set.
7. Every ODPUs-graph G satisfies, $r(G) \leq d(G) \leq r(G)+1$.
8. A graph with radius 1 and diameter 2 is an ODPUs-graph if and only if there exists an $M \subset V(G)$ with $|M| \geq 2$ such that the induced subgraph $\langle M \rangle$ is complete and any vertex in $\langle G-M \rangle$ is adjacent to all the vertices of M .
9. For any ODPUs-graph G , every ODPUs-set in G is a total dominating-set of G .

10. For every integer $n \geq 3$ there is a graph G with ODPU-number $n + 2$. We have proved that 3 cannot be the ODPU number of any graph. Hence, for an ODPU-graph, the number three is forbidden as the ODPU-number. Thus, 1 and 3 are the only two numbers forbidden as ODPU-numbers of any graph. Any graph G (may or may not be connected) with every vertex having positive degree and no vertex has full-degree can be embedded into an ODPU-graph H with G as an induced subgraph of H of order $|V(G)| + 2$ such that $V(G)$ is an ODPU-set of the graph H .
11. A bipartite ODPU-graph $G = (XUY, E)$ with the bipartition $\{X, Y\}$ of its vertex set V has ODPU-number 4 if and only if the set X has at least two vertices of degree $|Y|$ and the set Y has at least two vertices of degree $|X|$.
12. Let H be a connected graph with radius $r \geq 2$. Then the new graph $K = H[G_1, G_2, \dots, G_n]$ has the same radius and diameter as that of H .
13. Given a finite integer $n \neq 1, 3$, any graph G can be embedded in an ODPU-graph H with ODPU-number n and G as an induced subgraph of H .

Some of these results may be found in [33]. Following problems are open [48].

Problem 110. *In an ODPU-graph G , what is the maximum order of a collection of pairwise disjoint ODPU-sets?*

Problem 111. *Characterize a graph that possess an ODPU-set M such that $\langle M \rangle$ is a block.*

Problem 112. *Characterize graphs in which every total dominating set is an ODPU set.*

6.3. Distance-pattern segregated (dps) graphs

Let $G = (V, E)$ be a (p, q) graph. Given an arbitrary nonempty subset M of vertices in G , each vertex u in G is associated with the set $f'_M(u) = \{d(u, v) : v \in M\}$, where $d(x, y)$ denotes the usual distance between the vertices x and y in G , called the M -distance pattern of u . G is called a *distance-pattern segregated* (or, in short, *dps*) graph if f'_M is independent of the choice of $u \in M$ and injective set-valued function when restricted to the set $V - M$. The set M is called *distance-pattern segregating (dps) set* for G . The graph G itself is a *dps-graph* if G admits a dps-set. The least cardinality of dps-set in G is called *dps-number* denoted by $\sigma(G)$. We have proved many results on dps-graphs and dps-number of a graph (See [26, 28, 32, 48]). Some of them are listed below:

- (1) Compute graph K_n is a dps-graph having a dps-set of cardinality $n-1$.
- (2) If T is isomorphic to either star $K_{1,k}$ or $K_{1,k+1}$ or a graph obtained by subdividing at most two legs of $K_{1,k+1}$ or one leg of $K_{1,k}$, then $\sigma(T) \leq k$.

6.4. Complementary distance pattern uniform (CDPU) graph

Consider M be any non-empty subset of $V(G)$. For each vertex u in G if the distance pattern $f_M(u)$ is independent of the choice of $u \in V-M$, then G is called a complementary Distance Pattern Uniform (CDPU) Graph, the set M is called the CDPU set. The least cardinality of CDPU set in G is called the *CDPU number* of G , denoted $\sigma(G)$. Listed below are some results under CDPU-graphs (See [26, 28, 29, 32, 33, 48]).

- (1) Every self centered graph of order p has a CDPU set M with $|M| \leq p-2$.
- (2) If G is a self-median graph of order n ($2n-13$), $n \geq 8$, then $\sigma(G) \leq 2n(n-7)$.

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- (3) Let G be a graph with n vertices. If G is a self centered graph, then $1 \leq \sigma(G) \leq n-2$.
 If G is not a self centered graph, then $1 \leq \sigma(G) \leq n-r$, where r is the number of vertices with maximum eccentricity.
- (4) A graph G has $\sigma(G) = 1$ if and only if G has atleast one vertex of full degree.
- (5) For all integers $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 2$, $\sigma(K_{\alpha_1, \alpha_2, \dots, \alpha_n}) = n$.
- (6) $\sigma(C_n) = n - 2$, if n is odd and $\sigma(C_n) = \frac{n}{2}$, if $n \geq 8$ is even.
- (7) $\sigma(G + \overline{K}_m) \leq m$, if G has no vertex of full degree.

Following problems are open.

Problem 113. Characterize graphs G in which every minimal CDPU-set is independent.

Problem 114. What is the maximum cardinality of a minimal CDPU set in G ?

Problem 115. Determine whether every graph has an independent CDPU-set.

Problem 116. Characterize minimal CDPU-sets.

Problem 117. For any graph G find good bounds for $\sigma(G)$.

We [48,54] also studied Independent CDPU graph (if there exists an independent CDPU set) for G . We identified many classes of independent CDPU graphs and calculated the independent CDPU (ICDPU) number of various classes of graphs.

6.5. Distance-compatible set-labeling (dcsl) graphs

A distance-compatible set-labeling (dcsl) is an injective set-assignment $f: V(G) \rightarrow 2^X$, X a nonempty ground set, such that the corresponding induced function $f^\oplus : V(G) \times V(G) \rightarrow 2^X - \{\emptyset\}$ defined by $f^\oplus(uv) = f(u) + f(v)$ satisfies $|f^\oplus(u, v)| = k_{(u,v)}d(u, v)$ for all distinct $u, v \in V(G)$, where $d(u, v)$ is the distance between u and v and $k_{(u,v)}$ is a constant; G is a dcsl-graph if it admits a dcsl. Further, G is integrally dcsl if all the proportionality constants $k_{(x,y)}$ are integers and such a dcsl of G is referred to as an integral dcsl of G . A dcsl f of a graph G is k -uniform dcsl if the constants of proportionality are all equal to k ; G itself is a k -uniform dcsl graph if it admits a k -uniform dcsl. The minimum cardinality of a ground set X such that G admits a 1-uniform dcsl graph is called the 1-uniform dcsl index δ_d of graph G . We have identified many classes of k -uniform dcsl graphs for $k \geq 1$ and also studied (k, r) -arithmetic dcsl graphs and characterized (k, r) -arithmetic complete dcsl graphs and also proved that all trees admit 1-uniform dcsl. We also established the relationship between k -uniform graphs and l_1 graphs and found that k -uniform graphs are generalization of l_1 graphs. The topic is of special interest for further investigation because of its applications in cryptography and signalling problems. We also studied the hypergraph connection of dcsl graphs. (See [29, 31, 33, 48, 50, 54]. Many conjectures and open problems have been identified for further investigation.

Conjecture 118. *If a graph contains an odd cycle as an induced subgraph then it is not 1-uniform dcsl.*

Problem 119. *Prove or disprove: For $n \geq 4$, no graph C_n has a 1-uniform dcsl f such that $f(u) = \emptyset$ for some $u \in C_n$.*

Problem 120. *Prove or disprove: There exists a unicyclic graph G with its unique cycle having odd length such that $f(u) = \emptyset$ for some $u \in V(G)$, where f is a 1-uniform dcsl.*

Problem 121. *Prove or disprove: If a graph G has an odd cycle as an induced subgraph then G does not admit a 1-uniform dcsl.*

Problem 122. *For any even integer $n \geq 4$, consider any 1-uniform dcsl $f:V(G) \rightarrow 2^X$. It defines a hypergraph $\mathcal{M}_f = (X, \mathcal{E}_n)$ where $\mathcal{E}_n = \{f(v_i): 1 \leq i \leq n\}$. What properties of hypergraphs can be identified in \mathcal{H}_f with the characteristics of the cycle C_n ? What is its cyclomatic number?*

6.6. Bi-distance pattern uniform graphs

A graph $G=(V, E)$ is *Bi-Distance Pattern Uniform* (Bi-DPU) if there exists $M \subseteq V(G)$ such that the M -distance pattern $f_M(u) = \{d(u,v): v \in M\}$ is identical for all u in M and $f_M(v)$ is identical for all v in $V \setminus M$. The set M is called a *Bi-DPU set*. More details are found in [66] and [67].

6.7. Open distance pattern coloring of a graph

Given a connected (p, q) -graph $G=(V, E)$ of diameter $d(G)$, $\emptyset \neq M \subseteq V(G)$ and a nonempty set $X = \{1, 2, 3, \dots, d(G)\}$ of colors of cardinality $d(G)$, let f_M^0 be an assignment of subsets of X to the vertices of G , such that $f_M^\oplus(u) = \{d(u,v): u \in M, u \neq v\}$ where $d(u,v)$ is the usual distance between u and v . Given such a function f_M^0 for all vertices in G , an induced edge function f_M^\oplus of an edge $uv \in E(G)$, $f_M^\oplus(uv) = f_M^0(u) \oplus f_M^0(v)$. We call f_M^0 an M -open distance pattern coloring of G , if no two adjacent vertices have same f_M^0 and if such an M exists for a graph G , then G is called an M -open distance pattern colorable graph. The M -open distance pattern edge coloring number of a graph G is the cardinality of $f_M^\oplus(G)$, taken over all $M \subseteq V(G)$, denoted by $\eta(G)$ (see [65]).

7. Set-valuations of digraphs

Given a simple digraph $D = (V, \mathcal{A})$ and a set-valuation $f:V \rightarrow 2^X$, to each arc (u,v) in D we assign the set $f(u) - f(v)$. A set-valuation f of a given digraph $D=(V, \mathcal{A})$ is a *set-indexer* of D if both f and its ‘arc-induced function’ g_f defined by letting $g_f(u, v) = f(u) - f(v)$ for each arc (u,v) of D , are injective. Further, f is *arc-bounded* [22] if $|g_f(u, v)| < |f(v)|$, for each $(u, v) \in \mathcal{A}$.

Lemma 123. [22] *The out-degree $od(v)$ of any vertex v in a set-indexed digraph (D, f) is at most $2^{|f(v)|}$.*

Theorem 124. [22] *Every digraph admits a set-indexer.*

The following problem is open [22].

Problem 125. *What is the least cardinality of a ground set X with respect to which a given digraph D admits a set-indexer?*

If $\omega'(D)$ denotes the least cardinality of a ground set X with respect to which the given digraph D admits a set-indexer then from the proof of Theorem one can infer that

$$\omega'(D) \leq |V(D) \cup \mathcal{A}(D)|. \quad (2)$$

It would be interesting to determine digraphs D for which equality is attained in (2). The question whether the bound in (2) could be improved in general is open.

Note that any set-indexer f of a digraph $D=(V, \mathcal{A})$ has the property that $|g_f(u, v)| = |f(u) - f(v)| \leq |f(v)|$ for any arc (u, v) , but $|g_f(u, v)|$ could be equal to or larger than, $|f(v)|$. Thus, for any given digraph $D=(V, \mathcal{A})$ and for any arc-binding set-indexer f we have

$$|g_f(u, v)| \leq \min\{|f(u)|, |f(v)| - 1\} \text{ for every } (u, v) \in \mathcal{A} \quad (3)$$

The following problems arise [22].

Problem 126. *Characterize digraphs $D=(V, \mathcal{A})$ that admit arc-binding set-indexers f such that*

$$|g_f(u, v)| = \min\{|f(u)|, |f(v)| - 1\} \text{ for every } (u, v) \in \mathcal{A} \quad (4)$$

Problem 127. *What is the least cardinality of a ground set X with respect to which a given digraph D admits an arc-binding set-indexer?*

Definition 128. *A set-indexer f of a given digraph $D=(V, \mathcal{A})$ is arc-binding if $|g_f(u, v)| < |f(v)|$, for each $(u, v) \in \mathcal{A}$.*

Theorem 129. [22] *Every digraph admits an arc-binding set-indexer.*

If $\omega^{ab}(D)$ denotes the least cardinality of a ground set X with respect to which the given digraph D admits an arc-binding set-indexer then from the proof of Theorem one can infer that

$$w^{ab}(D) \leq 2n, \text{ where } n = |V(D)| \quad (5)$$

It would be interesting to determine digraphs D for which equality is attained in (5). Again, the question whether the bound in (5) could be improved in general is open. In any case, we have

$$w'(D) \leq w^{ab}(D) \text{ for any digraph } D \quad (6)$$

Problem 130. [22] *Characterize digraphs D for which equality holds in (6).*

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For digraphs D of order n , the inequalities (5) and (6) can be put together as

$$w'(D) \leq w^{ab}(D) \leq 2n \quad (7)$$

Another problem that naturally arises is the following.

Problem 131. *Find a 'good' lower bound for $w'(D)$.*

8. Set-valuations of digraphs

Given a simple signed digraph $S=(V, \mathcal{A}, \sigma)$ and a set-valued function, in general called a *set-valuation*, assigns a subset of a nonempty 'ground set' X to each element (i.e., a vertex and/or an arc) of S . In particular, a set-valuation $f:V \rightarrow 2^X$ is called a *vertex set-valuation* of S , a set-valuation $g:\mathcal{A} \rightarrow 2^X$ is called an *arc set-valuation* of S and a set-valuation $h:V \cup \mathcal{A} \rightarrow 2^X$ is called a *total set-valuation* of S . Further, an injective vertex set-valuation $f:V \rightarrow 2^X$ is called a *vertex set-labeling* of S if the induced arc set-valuation $g_f:\mathcal{A} \rightarrow 2^X$ defined by letting $g_f(u, v) = f(u) - f(v)$ for each $(u, v) \in \mathcal{A}$ satisfies for each $(u, v) \in \mathcal{A}$, $\sigma(u, v) = (-1)^{|f(u)-f(v)|}$. Furthermore, a vertex set-labeling f of S is called a *vertex set-indexer* if g_f is also injective. This note attempts to answer the question whether every signed digraph admits a vertex set-labeling (set-indexer). Several open problems are posed and new directions of study of the notion and its applications are suggested in [8]. We list here some of the open problems cited in [8].

Problem 132. [8] *Characterize graphs G that satisfy equality in (3).*

Problem 133. [8] *Characterize canonical signing of the vertex set-labelings (VSVC) of signed digraphs.*

Problem 134. [8] *Characterize finite sequences of integers that are degree sequences of signed digraphs.*

Marking of a signed digraph $S=(V, E, \sigma)$ is simply a function $\mu:V \rightarrow \{-1, +1\}$. It is *degree-compatible* if it satisfies

$$\mu(v) = -1 \Leftrightarrow d(v) < 0, \quad v \in V \quad (10)$$

and *canonical* if

$$\mu(v) = -1 \Leftrightarrow \partial^-(v) \equiv 1 \pmod{2}, v \in V \quad (11)$$

Definitions (10) and (11) motivate one to regard S as *degree-canonical* if

$$d(v) < 0 \Leftrightarrow \partial^-(v) \equiv 1 \pmod{2}, v \in V \quad (12)$$

Problem 135. [8] *Does every degree-canonical signed digraph admit a vertex set-labeling?*

Acharya in his paper [8] comments that if a signed digraph does not admit a vertex set-labeling then it must be an unbalanced signed digraph. It also implies that no connected unbalanced signed graph admits a vertex set-labeling; note, however, that it does not preclude the possibility of having weakly connected signed digraphs, balanced as well unbalanced ones, that admit vertex set-labelings. Thus, he [8] raises the following problems.

Problem 136. [8] *Characterize signed digraphs that admit vertex set-labelings.*

The following conjecture, if true, will pose serious obstacles towards solving Example 5.

Conjecture 137. [8] *Every signed digraph can be embedded as an induced subgraph in a signed digraph that admits a vertex set-labeling.*

9. Linear hypergraph set-indexer (LHSI)

For a graph $G = (V, E)$ and a non-empty set X , a *linear hypergraph set-indexer* (LHSI) is a function $f : V(G) \rightarrow 2^X$ satisfying the following conditions: (i) f is injective (ii) the ordered pair $H_f(G) = (X, f(V))$, where $f(V) = \{f(v) : v \in V(G)\}$, is a linear hypergraph, (iii) the induced set-valued function $f^\oplus : E(G) \rightarrow 2^X$, defined by $f^\oplus(uv) = f(u) \oplus f(v), \forall uv \in E$ is injective, and (iv) $H_{f^\oplus}(G) = (X, f^\oplus(E))$, where $f^\oplus(E) = \{f^\oplus(e) : e \in E\}$, is a linear hypergraph. Recently these hypergraphs are being studied (e.g., see [20,107,109, 110]).

Theorem 138. [20] *Let $G = (V, E)$ be a (p, q) -graph and let the $f : V(G) \rightarrow 2^X$ be an LHSI of G . Let u be any vertex of G with its vertex degree $d(u) \geq 2$. Then, $|f(u)| \leq 3$.*

Theorem 139. [109] *Let $G = (V, E)$ be a graph and let $f : V(G) \rightarrow 2^X$ be an LHSI of G . Let u be any vertex of G with $d(u) \geq 4$. Then, $|f(u)| \leq 2$.*

Theorem 140. [20] *For a simple graph G admitting an LHSI $f : V(G) \rightarrow 2^X$, $|X|$ can be any arbitrary positive integer greater than $I_L(G)$ if and only if G contains a pendent vertex.*

Proposition 141. [110] *If $G = (V, E)$ is a (p, q) -graph without pendent vertices and isolated vertices, then $I^{UL}(G) \leq 2p$.*

Theorem 142. [110] *For a (p, q) -graph G with $\delta(G) \geq 3$, $I^{UL}(G) \leq \frac{3p}{2}$.*

Theorem 143. [109] *If a graph G admits a 3-uniform LHSI, then G contains no cycles of length ≤ 4 .*

Theorem 144. [110] *If G is a (p, q) -graph with $2 \leq \delta(G) \leq \Delta(G) \leq 3$, then $I^{UL}(G) \leq 3p - q$.*

Theorem 145. [20] *A graph G without isolated points admits a 3-uniform LHSI if and only if (1) $\Delta(G) \leq 3$ and (2) girth $g(G) \geq 5$.*

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Theorem 146. [109] *If G is a (p, q) -graph with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ and $\text{girth}(G) \geq 5$, then $I^{UL}(G) = 3p - q$.*

Theorem 147. [110] *If G is a $\text{conn}(p, q)$ -graph with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ and $\text{girth } g(G) \geq 5$, there exists a 3-uniform LHSI f of G satisfying the following.*

1. $\mu(H_f(G)) = \mu(G)$
2. $\mu(H_{f \oplus}(G)) = \mu(L(G)) + q$, where $L(G)$ represents the line graph of G .

10. Conclusions and scope

This survey paper on set-valuations of graphs which includes a various type of set indexers, we hope that it will pave the way for any researcher for studying the topic. The conjectures and open problems identified in various sections appear would be quite interesting, for further investigation. In this perspective, the authors wish a general study on set-valuations of graphs would be a long term goal.

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