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On Glivenko Congruence of a 0-Distributive Nearlattice

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Abstract. In this paper the authors have studied the Glivenko congruence R in a 0-distributive nearlattice S defined by "$a \equiv b(R)$ if and only if $a \wedge x = 0$ is equivalent to $b \wedge x = 0$ for each $x \in S$". They have shown that the quotient nearlattice $\frac{S}{R}$ is weakly complemented. Moreover, $\frac{S}{R}$ is distributive if and only if S is 0-distributive. They also proved that every Sectionally complemented nearlattice S in which every interval $[0,x]$ is unicomplemented is SemiBoolean if and only if S is 0-distributive.

Keywords: 0-distributive nearlattice, Glivenko Congruence, Weakly complemented nearlattice, Semi Boolean nearlattice.

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1. Introduction

J.C. Varlet [5] has given the definition of a 0-distributive lattice to generalize the notion of pseudocomplemented lattice. Then many authors including [1] and [3] studied the 0-distributive properties in lattices and meet semilattices. Recently, [6] have studied the 0-distributive nearlattices.

A *nearlattice* is a meet semilattice together with the property that any two elements possessing a common upper bound have a supremum. This property is known as the *upper bound property*.

A nearlattice S is called *distributive* if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, provided $y \vee z$ exists in S. For detailed literature on nearlattices, we refer the reader to consult [2] and [4].

A nearlattice S *with 0* is called *0-distributive* if for all $x, y, z \in S$ with $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists imply $x \wedge (y \vee z) = 0$.

For $A \subseteq S$, we denote $A^{\perp} = \{x \in S \mid x \wedge a = 0 \text{ for all } a \in A\}$. Clearly $A^{\perp}$ is a down set . If S is 0- distributive then by[6], $A^{\perp}$ is an ideal of S. Moreover, $A^{\perp} = \bigcap_{a \in A} \{\{a\}^{\perp}\}$. *If A is an ideal, then obviously* $A^{\perp}$ *is the pseudocomplement* of *A* in $I(S)$.

Define *a relation* R on a nearlattice with *0* by aRb if and only if $a \wedge x = 0$ is equivalent to $b \wedge x = 0$. In otherwords, $a \equiv b(R)$ if and only if $(a]^{\perp} = (b]^{\perp}$.

In this paper, we will study this binary relation in a 0-distributive nearlattice. We will show that R is a nearlattice congruence when S is 0-distributive. We call it as Glivenko congruence.

2. Some Results

Theorem 1. *Let S be a nearlattice with 0. then the above relation* R *is a meet congruence. Moreover, when S is 0- distributive, then it is a nearlattice congruence*.

Proof. Clearly R is an equivalence relation. Let $a \equiv b(R)$ and $t \in S$. Then $a \wedge x = 0$ if and only if $b \wedge x = 0$. Let $(a \wedge t) \wedge x = 0$ for some $x \in S$. Then $a \wedge (t \wedge x) = 0$ implies $b \wedge (t \wedge x) = 0$, and so $(b \wedge t) \wedge x = 0$. Similarly $(b \wedge t) \wedge x = 0$ implies $(a \wedge t) \wedge x = 0$. Therefore, $a \wedge t \equiv b \wedge t(R)$ and so R is a meet congruence.
Now let $a \vee t, b \vee t$ exists and $a \equiv b(R)$. Then for any $x \in S$, $a \wedge x = 0$ if and only if $b \wedge x = 0$. Let $(a \vee t) \wedge x = 0$ for some $x \in S$. Then $a \wedge x = 0$ and $t \wedge x = 0$ which implies $b \wedge x = 0$ and $t \wedge x = 0$, and so $(b \vee t) \wedge x = 0$ as S is 0-distributive. Similarly $(b \vee t) \wedge x = 0$ for some $x \in S$ implies $(a \vee t) \wedge x = 0$. Therefore, R is a nearlattice congruence.●

By [6], we know that *a* nearlattice S with 0 is 0-distributive if and only if the lattice of *ideals* $I(S)$ is pseudocomplemented. Thus we have the following result.

Corollary 2. *If I(S) is pseudocomplemented, then R is a nearlattice congruence* . ●

A nearlattice S with 0 is called *Weakly complemented* if for any pair of distinct elements a, b of S, there exists an element c disjoint from one of these elements but not from the other.

Theorem 3. *S is weakly complemented if and only if R is an equality relation and hence is a nearlattice congruence* .

Proof. Suppose S is weakly complemented. Let suppose $a \neq b$. Then there exists c such that $a \wedge c = 0$ but $b \wedge c \neq 0$ (or vice versa). This implies $a \not\equiv b(R)$, a contradiction . Hence $a = b$. Therefore, R is an equality relation and so R is a nearlattice congruence.

Suppose R is an equality relation. We need to prove S is weakly complemented. Let $a, b \in S$ and $a \neq b$. Then $a \not\equiv b(R)$. This implies there exists $c \in S$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$ (or vice versa). Hence S is weakly complemented. ●

In the following nearlattice S, R is a nearlattice congruence. Here the classes are {0},{a},{b},{1}{c,d,e}. But S is neither 0-distributive nor weakly complemented.

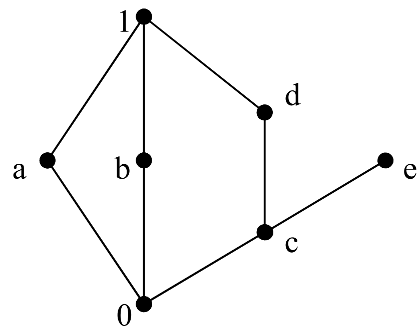

Figure 1

Theorem 4. *For any nearlattice S, the quotient lattice* $\frac{S}{R}$ *is weakly complemented . Furthermore, a nearlattice S with 0 is 0-distributive if and only if* $\frac{S}{R}$ *is a distributive nearlattice and R is a nearlattice congruence.*

Proof. Let A and B be two classes in $\frac{S}{R}$ such that A< B. Then there exists $a \in A$ and $b \in B$ such that a< b in S . So, by the definition of R there is an element $c \in S$, such that $a \wedge c = 0$ but $b \wedge c \neq 0$. Suppose $x \in [0]$.Then $x \equiv 0(R)$.Then $0 \wedge x = 0$ implies $x \wedge x = x = 0$. So $[0] = \{0\}$. This implies $A \wedge C = [a] \wedge [c] = \{0\}$ but $B \wedge C \neq \{0\}$. Hence $\frac{S}{R}$ is weakly complemented.

Now let S be a nearlattice for which R is a nearlattice congruence and $\frac{S}{R}$ is distributive .
Let $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ such that $b \vee c$ exists.
Then $[a] \wedge ([b] \vee [c]) = ([a] \wedge [b]) \vee ([a] \wedge [c]) = [0] \vee [0] = [0]$.
This implies $[a \wedge (b \vee c)] = [0]$. Since $[0] = \{0\}$, so $a \wedge (b \vee c) = 0$. Hence S is 0-distributive .

Conversely, let S be 0-distributive. Then by Theorem1, R is a nearlattice congruence. Let $[a],[b],[c]\in\frac{S}{R}$. We need to prove $[a]\wedge([b]\vee[c])=([a]\wedge[b])\vee([a]\wedge[c])$ provided $[b]\vee[c]$ exists. Suppose $[b]\vee[c]=[d]$. Then $[b]=[b]\wedge[d]=[b\wedge d]$, $c=[c]\wedge[d]=[c\wedge d]$, and so $[b]\vee[c]=[(b\wedge d)\vee(c\wedge d)]$. So we need to prove that $[a\wedge((b\wedge d)\vee(c\wedge d))]=[(a\wedge b\wedge d)\vee(a\wedge c\wedge d)]$.

Let $a\wedge((b\wedge d)\vee(c\wedge d))\wedge x=0$. Since

$(a\wedge b\wedge d)\vee(a\wedge c\wedge d)\leq a\wedge((b\wedge d)\vee(c\wedge d))$

so, $((a\wedge b\wedge d)\vee(a\wedge c\wedge d))\wedge x=0$.

On the other hand, if $((a\wedge b\wedge d)\vee(a\wedge c\wedge d))\wedge x=0$, then

$a\wedge b\wedge d\wedge x=0=a\wedge c\wedge d\wedge x$ and by 0-distributivity of S,

$a\wedge((b\wedge d)\vee(c\wedge d))\wedge x=0$.

Thus $a\wedge((b\wedge d)\vee(c\wedge d))\equiv(a\wedge(b\wedge d))\vee(a\wedge(c\wedge d))(R)$ and hence

$[a]\wedge([b]\vee[c])=([a]\wedge[b])\vee([a]\wedge[c])$. ●

Corollary 5. *If a 0-distributive nearlattice S is weakly complemented then S is distributive.*

Proof. If S is weakly complemented . Then by Theorem3, R is an equality relation and so by above theorem $S\cong\frac{S}{R}$ implies S is distributive. ●

A nearlattice S with 0 is called *Sectionally complemented* if the intervals [0,x] are complemented for each $x\in S$. A nearlattice which is sectionally complemented and distributive is called a *Semi Boolean nearlattice.*

Corollary 6. *If a 0-distributive nearlattice S is sectionally complemented and weakly complemented , then S is semi Boolean.* ●

Corollary 7 *Suppose S is sectionally complemented and in every interval [0,x] , every element has a unique relative complement . Then S is semi Boolean if and only if it is 0-distributive.*

Proof. Let S be 0-distributive and for every $x\in S$, the interval [0,x] is unicomplemented. Let $x,y\in S$ with $x\neq y$. If they are comparable, without loss of generality, suppose x< y. Then $0\leq x<y$. Then there exists a unique $t\in[0,y]$ such that $t\wedge x=0$ and $t\vee x=y$. Thus $t\wedge x=0$ but $t\wedge y=t\neq 0$. If x, y are not comparable, then $0\leq x\wedge y<x$ and $0\leq x\wedge y<y$. Then there exist $s,t\in S$ such that $x\wedge y\wedge s=0$, $(x\wedge y)\vee s=x$, $x\wedge y\wedge t=0$ and $(x\wedge y)\vee t=y$. Now $s\wedge t\leq x\wedge y$ implies $s\wedge t\leq x\wedge y\wedge s=0$, which implies $s\wedge t=0$. Now $s\wedge t=0$

and $s \wedge x \wedge y = 0$ implies $0 = s \wedge ((x \wedge y) \vee t) = s \wedge y$ as S is 0-distributive, but $s \wedge x \neq 0$. Therefore, S is weakly complemented and so by above corollary S is semi Boolean. Since the reverse implication always holds in a Semi-Boolean nearlattice, this completes the proof. ●

We conclude the paper with the following result

Theorem 8. *Let S be a nearlattice with 0. Then the following conditions are equivalent.*

(i) *S is 0-distributive.*

(ii) *$(0]$ is the kernel of some homomorphism of S onto a distributive nearlattice with 0.*

(iii) *$(0]$ is the kernel of a homomorphism of S onto a 0-distributive nearlattice.*

Proof: $(i) \Rightarrow (ii)$. Suppose S is 0-distributive. Then by Theorem 1, the binary relation R define by $x \equiv y(R)$ if and only if $(x]^{\perp} = (y]^{\perp}$ is a congruence on S. Moreover by Theorem 4, $\frac{S}{R}$ is a distributive nearlattice. Clearly the map $a \to [a]R$ is a homomorphism. Now let $a \equiv 0(R)$. Then $0 \wedge a = 0$ implies $a = a \wedge a = 0$. Here $[0]R$ contains only 0 of S.

$(ii) \Rightarrow (iii)$ is obvious as every distributive nearlattice with 0 is 0-distributive.

$(iii) \Rightarrow (i)$ Let θ be a congruence on S for which $(0]$ is the zero element of the 0-distributive nearlattice $\frac{S}{\theta}$. Suppose $x, y, z \in S$ with $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists. Then $[x]\theta \wedge [y]\theta = [x \wedge y]\theta = [0]\theta = [x \wedge z]\theta = [x]\theta \wedge [z]\theta$. Hence by 0-distributivity of $\frac{S}{\theta}$ $[x]\theta \wedge ([y]\theta \vee [z]\theta) = [0]\theta$, and so $[x \wedge (y \vee z)]\theta = [0]\theta$. This implies $x \wedge (y \vee z) \in (0]$ and so $x \wedge (y \vee z) = 0$. Therefore, S is 0-distributive. ●

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