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O-Modular Nearlattice

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Abstract. J.C. Varlet introduced the concept of 0-distributive and 0-modular lattices. Recently, Zaidur Rahman et al. [6] have introduced the concept of 0-distributivity in a nearlattice. In this paper, we discuss 0-modularity in a nearlattice. Here, we include several characterizations of 0-modular nearlattices. We prove that a section complemented 0-modular and 0-distributive nearlattice is semi Boolean. We also show that for two filters F and G of a 0-modular nearlattice if $F \vee G = [0)$ and $F \cap G = [x)$; $x \in S$, then both F and G are principal. Finally we show that a nearlattice S is semi Boolean if and only if S is 0-modular, every $[0, x]$ is semi complemented and 0 is the meet of a finite number of meet primes.

Keywords. 0-distributive nearlattice, 0-modular nearlattice, Prime filter, Semi complemented nearlattice, Section complemented nearlattice, Weakly complemented nearlattice, Semi Boolean nearlattice.

AMS Mathematics Subject Classification (2010): 06A12, 06A99, 06B10

1. Introduction

J.C Varlet [5] introduced the concept of 0-distributive and 0-modular lattices to study a larger class of non-distributive lattices. A lattice L with 0 is called 0-distributive if for all $a, b, c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. L is called 0-modular if for all $a, b, c \in L$ with $c \le a$ and $a \wedge b = 0$ imply $a \wedge (b \vee c) = c$. Of course, every distributive lattice is both 0-distributive and 0-modular. Every pseudocomplemented lattice is 0-distributive but not necessarily 0-modular.[1, 3, 4, 5] have studied different properties of 0-distributivity and 0-modularity in lattices and in semilattices. Recently, Zaidur Rahman et al. [6] have studied 0-distributive nearlattices. In this paper, we study some properties of 0-modular nearlattices.

O-Modular Nearlattice

A nearlattice S is a meet semi-lattice together with the property that any two elements possessing a common upper bound, have a supremum. This property is known as the upper bound property. S is called distributive if for all $x, y, z \in S$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $y \vee z$ exists. Observe that the right hand expression exists by the upper bound property of S. S is called a modular nearlattice if for all $x, y, z \in S$ with $z \leq x$ and $y \vee z$ exists imply $x \wedge (y \vee z) = (x \wedge y) \vee z$. By [2], we know that a nearlattice is modular if it does not contain a sublattice isomorphic to a pentagonal lattice $R_5 = \{d, a, b, c, e \mid a < b, a \wedge b = a \wedge c = d, a \vee c = b \vee c = e\}$. Moreover, S is distributive if it does not contain any sublattice isomorphic to R_5 or $M_5 = \{d, a, b, c, e \mid a \wedge b = a \wedge c = b \wedge c = d, a \vee b = a \vee c = b \vee c = e\}$.

A nearlattice S with 0 is called 0-distributive if for all $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists imply $a \wedge (b \vee c) = 0$. Thus every distributive nearlattice with 0 is 0-distributive, Moreover, if S is section pseudocomplemented then it is 0-distributive.

A nearlattice S with 0 is called a 0-modular nearlattice if for all $a, b, c \in S$ with $c \leq a$, $a \wedge b = 0$ imply $a \wedge (b \vee c) = c$ provided $b \vee c$ exists. It is easy to see that this definition is equivalent to "for all $t, a, b, c \in S$ with $c \leq a$ $a \wedge b = 0$ imply $a \wedge [(t \wedge b) \vee (t \wedge c)] = t \wedge c$". Moreover, it is easy to show that the definition of 0-modular nearlattice coincides with the definition of 0-modular lattice when S is a lattice. Of course every modular nearlattice with 0 is 0-modular. By [5] we know that S with 0 is 0-modular if it contains no non-modular five element pentagonal sublattice including 0. Also S with 0 is 0-distributive if it contains no five element modular but non distributive sublattice including 0. Now we include some examples:

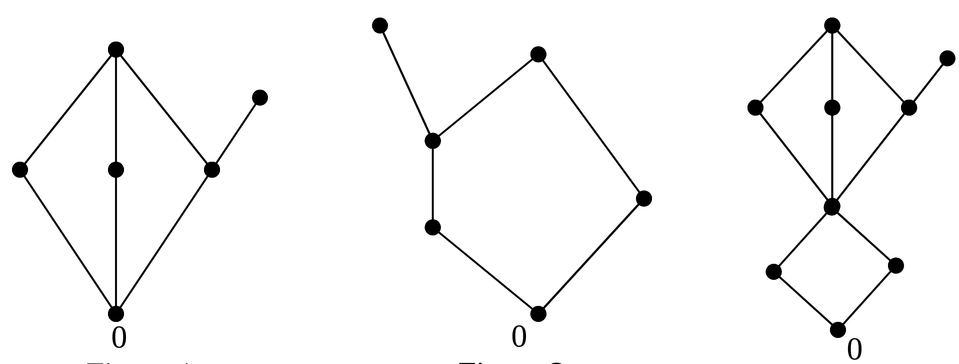

Figure 1 Figure 2 Figure 3

Figure 1 is 0-modular but not 0-distributive, Figure 2 is 0-distributive but not 0-modular, Figure 3 is both 0-modular and 0-distributive, figure 4 is 0-distributive but not 0-modular, Figure 5 is 0-modular but not 0-distributive, Figure 6 is both 0-modular and 0-distributive.

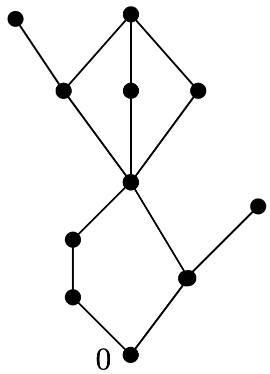

Figure 4

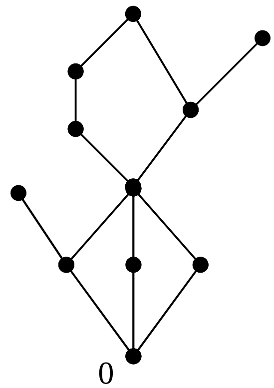

Figure 5

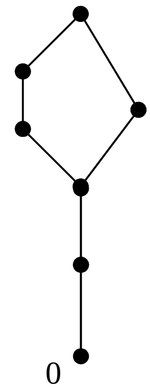

Figure 6

A lattice L with 1 is called 1-distributive if for all $a, b, c \in L$ with $a \vee b = a \vee c = 1$ imply $a \vee (b \wedge c) = 1$. A lattice L with 1 is called 1-modular if for all $a, b, c \in L$ with $c \geq a$ and $a \vee b = 1$ imply $a \vee (b \wedge c) = c$.

A lattice *L* with 0 is semi complemented if for any $a \in L$, $(a \neq 1)$ there exists $b \in L,\ b \neq 0$ such that $a \wedge b = 0$. Dually a lattice *L* with 1 is called dual semi complemented if for any $a \in L$, $(a \neq 0)$ there exists $b \in L,\ b \neq 1$, such that $a \vee b = 1$.

A lattice *L* with 0 and 1 is called complemented if for any $a \in L$ there exist $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$.

A nearlattice *S* with 0 is called weakly complemented if for any distinct elements $a, b \in S$, there exists $c \in S$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$ (or vice versa).

An element a of a nearlattice *S* is called meet prime if $b \wedge c \leq a$ implies either $b \leq a$ or $c \leq a$. A non-zero element *x* of a nearlattice *S* with 0 is an atom if for any $y \in S$, with $0 \leq y \leq x$ implies either $0 = y$ or $y = x$. Dually in a lattice *L* with 1, an element *x* is called a dual atom if for any $y \in L$, $x \leq y \leq 1$ implies $x = y$ or $y = 1$.

A non-empty subset *F* of a nearlattice *S* is called a filter if for $x, y \in S$, $x \wedge y \in F$ if and only if $x \in F$ and $y \in F$.

The set of all filters of a nearlattice is just a join semi-lattice. But in case of a lattice, the set of filters is again a lattice.

2. Some Results

Theorem 1. A nearlattice *S* with 0 is 0-modular if and only if for all $a, b, c \in S$ with $c \leq a$, $a \wedge b = 0$, $a \vee b = c \vee b$ imply $a = c$, provided $a \vee b$ exist.

Proof: Suppose *S* is 0-modular and $a, b, c \in S$ with $c \leq a$, $a \wedge b = 0$ and $a \vee b = c \vee b$. If $a \vee b$ exists then $c \vee b$ exists by the upper bound property. Then $a = a \wedge (a \vee b) = a \wedge (b \vee c) = c$.

Conversely, let the stated conditions are satisfied in *S*. Let $a, b, c \in S$ with $c \leq a$, $a \wedge b = 0$ and $b \vee c$ exists. Here $c \leq a \wedge (b \vee c)$ and $b \wedge [a \wedge (b \vee c)] = b \wedge a = 0$.

Now $a \wedge (b \vee c) \le b \vee c$, so $b \vee [a \wedge (b \vee c)] \le b \vee c$. Also $c \le a \wedge (b \vee c)$ implies $b \vee [a \wedge (b \vee c)] \ge b \vee c$ and so $b \vee c = b \vee [a \wedge (b \vee c)]$, so by the given conditions $c = a \wedge (b \vee c)$, which implies *S* is 0-modular. ●

Theorem 2. A nearlattice *S* with 0 is 0-modular if and only if the interval $[0, x]$ for each $x \in S$ is 0-modular.

Proof: If *S* is 0-modular then trivially $[0, x]$ is 0-modular for each $x \in S$.

Conversely, let $[0, x]$ is 0-modular for each $x \in S$. Let $a, b, c \in S$ with $a \wedge b = 0$, $c \le a$ and $b \vee c$ exists.

Choose $t = b \vee c$.

Then $a \wedge (b \vee c) = a \wedge [(t \wedge b) \vee (t \wedge c)] = (t \wedge a) \wedge [(t \wedge b) \vee (t \wedge c)] = t \wedge c = c$ as the interval $[0, t]$ is 0-modular. ●

In a similar way we can easily prove the following result.

Corollary 3. A nearlattice *S* with 0 is 0-distributive if and only if the interval $[0, x]$ for each $x \in S$ is 0-distributive. ●

Theorem 4. For a nearlattice *S* with 0, if $I(S)$ is 0-modular, then *S* is 0-modular, but the converse need not be true.

Proof: Suppose $I(S)$ is 0-modular. Let $a, b, c \in S$ with $a \wedge b = 0$, $c \le a$ and $b \vee c$ exist. Then $(a] \wedge ((b] \vee (c]) = (c]$ as $I(S)$ is 0-modular. Thus $(a \wedge (b \vee c)] = (c]$ and so $a \wedge (b \vee c) = c$, which implies *S* is 0-modular.

For the converse, we consider the nearlattice *S* given below which is due to [2].

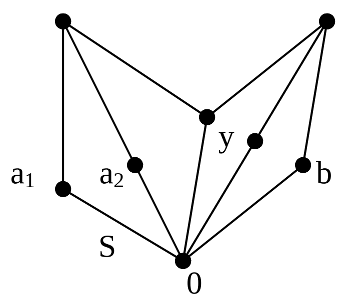

Figure 7

Here S is 0-modular. But in $I(S)$, $\{(0], (a_1], (a_1, y], (a_2, b], S\}$ is a pentagonal sublattice including 0. So $I(S)$ is not 0-modular. ●

Theorem 5. A nearlattice *S* with 0 is 0-modular if and only if the lattice of filters of the interval $[0, x]$ for each $x \in S$ is 1-modular.

Proof: Let S be 0-modular. Choose any $x \in S$. Then $[0, x]$ is also 0-modular. Let *F*, *G*, *H* be filters of the lattice $[0, x]$ such that $H \supseteq F$, $F \vee G = [0)$.

Then $F \vee (G \cap H) \subseteq H$ is obvious. Let $h \in H$. Now $F \vee G = [0)$ implies $0 = f \wedge g$ for some $f \in F$ and $g \in G$. Thus $h \wedge f \leq f$ and $f \wedge g = 0$ implies $f \wedge [g \vee (h \wedge f)] = h \wedge f$ as S is 0-modular. So $h \wedge f \in F \vee (G \cap H)$ and hence $h \in F \vee (G \cap H)$. Therefore, $F \vee (G \cap H) = H$ and so the lattice of filters of $[0, x]$ is 1-modular.

Conversely, suppose the lattice of filters of $[0, x]$ is 1-modular. Let $a, b, c \in [0, x]$, $(x \in S)$ such that $c \leq a$, $a \wedge b = 0$. Then $[a) \subseteq [c)$ and $[a) \vee [b) = [0)$. So by 1-modular property, $[a) \vee ([b) \wedge [c)) = [c)$. Thus $[a \wedge (b \vee c)) = [c)$ and hence $a \wedge (b \vee c) = c$. This implies $[0, x]$ is 0-modular. Therefore by Theorem 2, *S* is 0-modular. ●

Theorem 6.

a) If a nearlattice *S* is 0-distributive and the interval $[0, x]$ for each $x \in S$ is semi complemented, then the interval $[0, x]$ is 1-distributive for all $x \in S$.

b) If a dual nearlattice *S* with 1 is 1-distributive and $[x, 1]$ is dual semi complemented for each $x \in S$, then the interval $[x, 1]$ is 0-distributive for each $x \in S$.

Proof: a) Let $a, b, c \in [0, x]$ with $a \vee b = x = a \vee c$. Suppose $a \vee (b \wedge c) \neq x$. Then there exists $p \neq 0$ in $[0, x]$ such that $p \wedge (a \vee (b \wedge c)) = 0$. Then $a \wedge p = 0 = (b \wedge c) \wedge p$. Thus $p \wedge b \wedge a = 0 = (p \wedge b) \wedge c$ which implies $(p \wedge b) \wedge (a \vee c) = 0$ as *S* is 0-distributive. This implies $0 = p \wedge b \wedge x = p \wedge b$. Then using the 0-distributivity of *S* again, $p \wedge (a \vee b) = 0$. That is, $0 = p \wedge x = p$, which gives a contradiction. Therefore, $a \vee (b \wedge c) = x$ and so $[0, x]$ is 1-distributive.

b) This is trivial by a dual proof of (a). ●

A nearlattice *S* with 0 is called a semi Boolean lattice if it is distributive and the interval $[0, x]$ for each $x \in S$ is complemented.

Theorem 7. If a section complemented 0-modular nearlattice *S* is 0-distributive, then it is semi Boolean.

Proof: Let $a < b$ for some $a, b \in S$. Then $0 \leq a < b$. Since $[0, b]$ is complemented, so there exists $c \in [0, b]$ such that $c \wedge a = 0$, $c \vee a = b$. Now if $b \wedge c = 0$, then by the 0-modularity of *S*, $b = b \wedge (c \vee a) = a$, which is a contradiction. Therefore, $b \wedge c \neq 0$. This implies *S* is weakly complemented. Since *S* is also 0-distributive. Therefore, by

Corollary 3 and [5, Corollary2.2] $[0, x]$ is Boolean for each $x \in S$ and so *S* is semi Boolean. ●

Theorem 8. Let *S* be a 0-modular nearlattice and *F*, *G* are two filters such that $F \vee G = [0)$ and $F \cap G = [x)$ for some $x \in S$. Then both *F* and *G* are principal filters.

Proof: Suppose $F \vee G = [0)$ and $F \cap G = [x)$. Then $0 \geq f \wedge g$ for some $f \in F$ and $g \in G$. That is, $f \wedge g = 0$. Let $b = x \wedge f$ and $c = x \wedge g$. Then $b \in F$ and $c \in G$. We claim that $F = [b)$ and $G = [c)$. Indeed if for instance $G \neq [c)$, then there exists $a \in G$ such that $a < c$. Then $\{0, a, c, b, x\}$ is a pentagonal sublattice of *S*. This implies *S* is not 0-modular and this gives a contradiction.

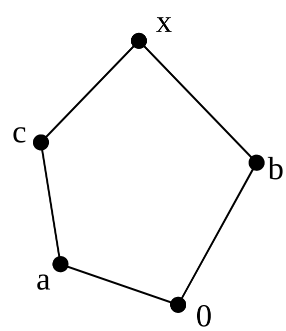

Figure 8

Therefore, $G = [c)$. Similarly $F = [b)$ and so both *F* and *G* are principal. ●

Lemma 9. In a bounded semi complemented lattice *L*, every meet prime element is a dual atom.

Proof: Suppose *x* is a meet prime element. Let $x \leq y < 1$. Then $0 \leq y < 1$. Since *L* is semi complemented, so there exists $t \neq 0 \in L$ such that $t \wedge y = 0$. Since $x \leq y$, so $t \wedge x = 0$. Since x is meet prime so this implies either $t \leq x$ or $y \leq x$. Now $t \leq x$ implies $t = t \wedge x = 0$, which is a contradiction. Thus $y \leq x$ and so $x = y$. Therefore *x* is a dual atom. ●

Lemma 10. Let *L* be a bounded semi complemented lattice. If 0 is the meet of a finite number of meet prime elements of *L*, then *L* is dual semi complemented and 0-distributive.

Proof: Let *x* be a non-zero element of *L*. Then by hypothesis, there is a meet prime element p in L such that $x \not\leq p$. Since *L* is semi complemented, so by Lemma 9 is a dual atom and $x \vee p = 1$. Therefore, *L* is dual semi complemented. Now suppose $a \wedge b = 0 = a \wedge c$ for some $a, b, c \in L$. Let us assume that $0 = \overset{n}{\underset{i=1}{\wedge}} p_i$ where p_i are meet prime elements in *L*. Observe that for each i, $p_i \geq a \wedge b$ and $p_i \geq a \wedge c$. Then for each *i*, $p_i \in [a)$ or $p_i \in [b) \cap [c)$. Therefore for each *i*, $p_i \in [a) \vee ([b) \cap [c))$. This implies $[a) \vee ([b) \cap [c)) = [0)$, consequently, $a \wedge (b \vee c) = 0$, and so *L* is 0-distributive.

Lemma 11. Let L be a bounded 0-modular lattice. If $b \in L$ is a dual atom and $a \wedge b = 0$ for some $a \neq 0$, $(a \in L)$, then *a* is an atom.

Proof: Suppose $0 < c \leq a$ for some $c \in L$. As $c \leq a$ and $a \wedge b = 0$, so by 0-modularity, $a \wedge (b \vee c) = c$. Since $0 < c$, it follows that $b < b \vee c$ and so $b \vee c = 1$ as *b* is a dual atom. Consequently, $a = a \wedge 1 = a \wedge (b \vee c) = c$ by 0-modular. Therefore, *a* is an atom.●

Lemma 12. Let *S* be a 0-modular nearlattice and $[0, x]$ is semi complemented for each $x \in S$. If for each $x \in S$, 0 is the meet of a finite number of meet prime elements in $[0, x]$. Then *x* is the join of finite number of atoms in $[0, x]$.

Proof: Let $0 = \overset{n}{\underset{i=1}{\wedge}} p_i$, where p_i's are meet prime elements in $[0, x]$. Observe that by Lemma 9, each p_i is a dual atom in $[0, x]$. Since each $p_i \neq x$, and $[0, x]$ is semi complemented, so there exists $q_i \in [0, x]$ such that $p_i \wedge q_i = 0$, *i=1,2,…, n*. Also by Lemma 11, each q_i is an atom in $[0, x]$. Now let $c = \overset{n}{\underset{i=1}{\vee}} q_i$. Then $c \vee p_i = x$ as p_i is a dual atom for each *i*. As $[0, x]$ is bounded semi complemented and 0 is the meet of finite number of meet primes, by Lemma 10, $[0, x]$ is 0-distributive and so by theorem5, $[0, x]$ is 1-distributive. Therefore, $c \vee \left(\overset{n}{\underset{i=1}{\wedge}} p_i \right) = x$. That is, $c = c \vee 0 = x$. Hence $\overset{n}{\underset{i=1}{\vee}} q_i = x$.●

Theorem 13. A nearlattice *S* with 0 is a semi Boolean lattice if and only if the following conditions are satisfied

(i) $[0, x]$ for each $x \in S$ is 1-distributive.

(ii) S is 0-distributive.

(iii) $F([0, x])$ is semi complemented for each $x \in S$.

Proof: By [3, Theorem 3], every $[0, x]$, $x \in S$ is a finite Boolean algebra. Therefore, *S* is semi Boolean.●

We conclude the paper with the following result which also trivially follows from [3, Theorem 4].

Theorem 14. For a nearlattice S with 0, S is semi-Boolean if and only if the following conditions are satisfied.

(i) $[0,x]$ is semi complemented for each $x \in S$.

(ii) S is 0-modular.

(iii) 0 is the meet of a finite number of meet primes. ●

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