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An Analysis of Friedmann Robertson Walker Metric with the Variation of G and Λ

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Abstract. We study the general Friedmann-Robertson–Walker (FRW) metric ($k = +1, 0, -1$) into explicit "Schwarzschild" or "Curvature" [1] form, which is important from the view point of cosmology. By the form of the FRW model we find out some new solutions of Einstein's field equation for a perfect fluid sphere. From new solution we see that when t increases, R increases but ρ and p decrease, which is true for the concept of expanding universe. This paper also presents cosmological model in which the gravitational and cosmological constants G and Λ respectively are time–dependent.

Keywords: Newtonian Gravitational constant, Cosmological constant, FRW Metric, Energy-momentum Tensor, Ricci Tensor, Ricci Scalar.

AMS Mathematics Subject Classification (2010): 85A40, 83F05

1. Introduction

The discovery of Friedmann solutions within the framework of homogeneous and isotropic universe models allowed the cosmological considerations to be treated in a mathematical manner, which was a subject so far dominated by largely speculative ideas about the overall structure of the universe. The major assumption used in arriving at the Robertson- Walker geometry is the large scale homogeneity and isotropy of the universe. The homogeneity in space means that the universe is roughly the same at all spatial points and that the matter is uniformly distributed all over the space. Even through the universe is clearly in homogeneous at the local scales of stars and star dusters. It is generally argued that an overall homogeneity will be achieved only at a large enough scale in a statistical sense only. It is possible to have observational tests on the assumption of isotropy that is the universe must be in the same in all directions.

In the standard model of cosmology , the expanding universe of galaxies is described by a Friedmann - Robertson–Walker (FRW) metric, which clarifies that spherical coordinates have a line element given by [2,3]:

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2[d\theta^2 + \sin^2\theta d\phi^2\right\} \quad (1)$$

In this model, which accounts for things on the largest length scale, the universe is approximated by a space of uniform density and pressure at each fixed time, and the expansion rate is determined by the cosmological factor $R(t)$ that evolves according to the Einstein equations. Astronomical observations show that the galaxies are uniform on a scale of about one billion light-years, and the expansion is critical that is, k = 0 in (1) and so, according to (1) , on the largest scale , the universe is infinite flat Euclidian space R^3 at each fixed time. Matching the Hubble constant to its observed values, and invoking the Einstein equations, the FRW model implies that the entire infinite universe R^3 emerged all at once from a singularity, ($R = 0$).

In general relativity the cosmological constant Λ may be regarded as the measure of energy density of the vacuum and can, in principle, lead to the avoidance of the big-bang singularity, which is a characteristic of other Friedmann-Robertson–Walker (FRW) models. The cosmological constant problem has a long history and while there are many possible resolutions, none have gained widespread acceptance. In classical general relativity, the energy density and pressure of the vacuum obey the relation $\rho c^2 = -p = \Lambda c^4 / 8\pi G$, where c is the speed of light and G is the gravitational constant.

However, to obtain the static model of the universe, Einstein might have been able to predict the expansion of the universe or the universe is in a dynamic state (sec Islam [4,5], second edition, 2002). Actually Linde [6] has argued the cosmological term arises from spontaneous breaking and has suggested that the term is not a constant but a function of temperature. In cosmology the term may be understood by incorporation with Mach's [7] principle, which suggests the acceptance of Bran's- Dicke Lagrangian as a realistic case[8,9] and stimulates us to study the term with a modified Bran's –Dicke Lagrangian from cosmology and elementary particle physics. After that, Pradhan et al. [10] studied FRW universe with variable G and Λ terms. A new class of exact solutions of Einstein's field equation with a perfect fluid source, variable gravitational couple G and cosmological term Λ for FRW space time is obtained by considering variable deceleration parameter models of the universe.

2. Mathematical Analysis

We consider homogeneous and isotropic universe containing a single perfect fluid in which the matter is at rest in the local frames while the Friedmann Robertson Walker metric incorporates the symmetry properties and the kinematics of space-time. The Einstein equations provide the dynamics, that is the manner in which the matter and space-time intern, are affected by the forces present in the universe. The idealization is infecting a realist approximation in many situations. For example, if the mean free path between particles collision is much less than scales of physical interests, the fluid may be treated as perfect.

We consider a spatially flat FRW universe

$$ds^2 = dt^2 - R^2(t)\left[dr^2 + r^2\left(d\theta^2 + Sin^2\theta\, d\varphi^2\right)\right] \tag{2}$$

with a perfect fluid energy momentum tensor

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \tag{3}$$

and time dependent G and Λ.

The Einstein modified field equation as

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = -8\pi G T^{\mu\nu} \tag{4}$$

where, G is the Newtonian Gravitational Constant and $G^{\mu\nu}$ is called the Einstein tensor. Now putting equation (3) in (4) we get

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} + \Lambda g^{\mu\nu} = -8\pi G\left[(\rho + p)u^\mu u^\nu - pg^{\mu\nu}\right] \tag{5}$$

where $G^{\mu\theta} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ (6)

3. A New Solution of FRW Universe

If $\mu = \nu = 0$, we get from equation (5)

$$R^{00} - \frac{1}{2}g^{00}R + \Lambda g^{00} = -8\pi G\left[(\rho + p)u^0 u^0 - pg^{00}\right] \tag{7}$$

Now from equation (7) by Ricci tensor we have

$$\frac{3\ddot{R}}{R} - \frac{1}{2}\times 1\times\frac{\left(2\dot{R}^2 + R\ddot{R}\right)}{R^2} + \Lambda\times 1 = -8\pi G[\rho + p - p]$$

$$\text{or, } \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} \tag{8}$$

From equation (5), if we put $\mu = \nu = 1$ we get

$$\left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R} = -8\pi G\rho + \Lambda \tag{9}$$

If we put $\mu = \nu = 2$ or $\mu = \nu = 3$ we will get the same equation as (9)

Now the law of energy momentum conservation is

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{10}$$

where $T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\sigma\nu}T^{\sigma\nu} + \Gamma^{\nu}_{\sigma\nu}T^{\mu\sigma}$

If we put $\mu = 0$, we get

$$T^{0\nu}{}_{;\nu} = T^{0\nu}{}_{,\nu} + \Gamma^{0}_{\sigma\nu} T^{\sigma\nu} + \Gamma^{\nu}_{\sigma\nu} T^{0\sigma} = \dot{\rho} + 3\rho\frac{\dot{R}}{R} + 3p\frac{\dot{R}}{R}$$

For other values of μ we get nothing.

From equation (10) we can write

$$T^{\mu\nu}{}_{;\nu} = 0$$

or, $T^{0\nu}{}_{;\nu} = 0$

or, $\dot{\rho} + 3\rho\frac{\dot{R}}{R} + 3\rho\omega\frac{\dot{R}}{R} = 0 \qquad [p = \rho\omega]$

or, $\frac{\dot{\rho}}{\rho} + 3\frac{\dot{R}}{R}(1+\omega) = 0$ (11)

$$\rho = \frac{a}{R^{3(1+\omega)}} \qquad (a = \text{Integrating Constant}) \tag{12}$$

From equations (8) and (12) we have

$$R = \left[\sqrt{\frac{3C}{\Lambda}}\sinh\left\{\frac{3}{2}\sqrt{\frac{\Lambda}{3}}(1+\omega)t + m\right\}\right]^{\frac{2}{3(1+\omega)}} \quad \text{where } C = \frac{8\pi G a}{3}$$

and $p = \omega\rho = a\omega\left[\sqrt{\frac{3C}{\Lambda}}\sinh\left\{\frac{3}{2}\sqrt{\frac{\Lambda}{3}}(1+\omega)t + m\right\}\right]^{-2}$ (13)

The above solution is a new solution.

4. Solutions with variation of G and Λ and its physical significance

From the Bianchi identities we know

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\right)_{;\nu} = 0$$

or, $8\pi G T^{\mu\nu}{}_{;\nu} + 8\pi\dot{G}T^{\mu\nu} + \dot{\Lambda}g^{\mu\nu} + \Lambda g^{\mu\nu}{}_{,\nu} = 0$

Consider the energy momentum tensor is divergent free. So we can write

$$8\pi G T^{\mu\nu}{}_{;\nu} = 0 \tag{14}$$

$$8\pi\dot{G}T^{\mu\nu} + \dot{\Lambda}g^{\mu\nu} = 0 \tag{15}$$

or, $\dot{\Lambda} = -8\pi\dot{G}\rho$ [If $\mu = \upsilon = 0$] (16)

From equations (8) and (11) we get

$$\left(\frac{\dot{\rho}}{\rho}\right)^2 = 9(1+\omega)^2\left(\frac{8\pi G\rho}{3} + \frac{\Lambda}{3}\right) \tag{17}$$

And by differing with respect to time t and using (16) we get

$$\ddot{\rho}\rho - \dot{\rho}^2 = 12(1+\omega)^2\pi G\rho^3 \tag{18}$$

Let us now assume that $G(t)$ is given by a power law

$$G = K\,t^n \ \ (K = \text{Constant}\ ) \tag{19}$$

From equation (18) we get

$$\frac{\ddot{\rho}}{\rho^2} - \frac{\dot{\rho}^2}{\rho^3} = At^n \qquad \text{where } A = 12(1+\omega)^2\pi K \tag{20}$$

Now assume $\rho = bt, \quad \dot{\rho} = brt^{r-1}, \ \ddot{\rho} = br(r-1)t^{r-2}$

From equation (20) we get

$$-rt^{-r-2} = bAt^n \tag{21}$$

From (21) we have, $bA = -r$ and $n = -(r+2)$

So $b = \dfrac{n+2}{A} = \dfrac{n+2}{12(1+\omega)^2\pi K}$

and $$\rho = \frac{n+2}{12(1+\omega)^2\pi K}\cdot\frac{1}{t^{n+2}} \qquad ,n \neq -2 \tag{22}$$

For the density to be positive definite we must have that $n > -2$. Next on combining (16), (19) and (22) we get

$$\dot{\Lambda} = -8\pi Kt^{n-1}n\frac{n+2}{12(1+\omega)^2\pi K}\frac{1}{t^{n+2}} = -\frac{2n(n+2)}{3(1+\omega)^2}\frac{1}{t^3} \tag{23}$$

Integrating equation (23) and setting a constant to zero, gives

$$\Lambda = \frac{2n(n+2)}{3(1+\omega)^2t^2} \qquad ,n \neq -2 \tag{24}$$

We see that Λ varies as t^{-2} [11,12] , which matches its natural dimensions and means there is no longer a dimensional constant associated with the cosmological term in the field equations. Form (24) , (19) , (22) and (8) we get

$$\frac{\dot{R}}{R} = \frac{(n+2)}{3(1+\omega)t} = Kt^{-1}, \quad \text{where } K = \frac{n+2}{3(1+\omega)} \tag{25}$$

Integrating equation (25) we have

$$\log R = K \log t + \log K_1$$

$$\therefore R = K_1 t^K = K_1 t^{(n+2)/[3(1+\omega)]} \quad , n \neq -2$$

If $n = 0$, $R(t) =$ constant $\times t^{2/[3(1+\omega)]}$

$$\therefore R \propto t^{2/[3(1+\omega)]} \tag{26}$$

For the flat FRW Model with constant G.

Obviously, one solution of equation (18) is $\dot{\rho} = 0$. One way this can be compatible with equation (11) is the scale factor is a constant too. Equation (16) then yields

$$8\pi G\rho = -\Lambda \tag{27}$$

The functions $G(t)$ and $\Lambda(t)$ in equation (27) are arbitrary. The vacuum in this solution is, $\rho_{vac} = -\frac{\Lambda}{8\pi G} = -\rho$ (28)

and the total energy be denoted by

$$\rho_{tot} = \rho + \rho_{vac} = \rho - \rho = 0 \tag{29}$$

Therefore, we have a spatially flat static universe with varying G and Λ and total energy zero. If $n = -2$ and $\omega \neq -1$, the above solution is necessary. For that value of n, however, we can see from equation (27) that Λ still falls as $\frac{1}{t^2}$ and therefore, this law holds for every $n \geq -2$.

An alternative way of satisfying equation (11) with a constant energy density is to have $\omega = -1$, so R can be nonzero. Using $\dot{\rho} = 0$ and equation (16) the time derivative equation (8) yields

$$R = e^{Kt} = \exp\,(\pm const \times t) \tag{30}$$

The only constraint on G and Λ comes form equation (16) and as in the case of the static solutions their functional form is otherwise free. This means that for any $G(t)$ and $\Lambda(t)$ satisfying (16), we obtain

inflationary solutions from the assumptions of a vacuum equation of state.

In particular, we consider

$$G(t) = Bt^{s-2} \exp(-ft^{s}) \quad ; B > 0 \tag{31}$$

and the energy density

$$\rho(t) = A\exp(qt^{s}) \tag{32}$$

$$\dot{\rho}(t) = Aqst^{s-1} \exp(qt^{s}) \tag{33}$$

$$\ddot{\rho}(t)Aqs(s-1)t^{s-2} \exp(qt^{s}) + q^{2}s^{2}t^{2(s-1)}A\exp(qt^{s}) \tag{34}$$

From equation (1.22) we get

$$A = \frac{s(s-1)qt^{s-2} \exp(2qt^{s})}{12(1+\omega)^{2} \pi Bt^{s-2} \exp[(3q-f)t^{s}]} \tag{35}$$

Now let $q = f$, so $A = \dfrac{s(s-1)f}{12(1+\omega)^{2} \pi B}$ (36)

$$\rho(t) = \frac{s(s-1)f}{12(1+\omega)^{2} \pi B}\left[\exp(ft^{s})\right] \tag{37}$$

For the energy density to be positive we must have

$$fs(s-1) > 0 \tag{38}$$

Then the cosmological constant be

$$\Lambda = -\frac{2s(s-1)}{3(1+\omega)^{2}}\left[t^{s-2} - \frac{fs}{2(s-1)}t^{2(s-1)}\right] \tag{39}$$

Putting the value of G, ρ and Λ in to equation (8), we get

$$\frac{\dot{R}}{R} = Kt^{(s-1)} \quad \text{Where } K = \frac{fs}{3(1+\omega)} \tag{40}$$

$$\therefore R = K_{1} \times \exp\left[-\frac{fst^{s}}{3s(1+\omega)}\right] = K_{1} \times \exp\left[-\frac{ft^{s}}{3(1+\omega)}\right] \tag{41}$$

In order to have expansion , f has to be negative. These are exponential solutions that do not arise from an equation of state $p = -\rho$ as in inflation, because $\omega \neq -1$ is not allowed in the above expression.

$$\rho_{vac} = \frac{\Lambda}{8\pi G}$$

$$= -\frac{s(s-1)f\exp\left(ft^s\right)}{12(1+\omega)^2\pi B}\left[1-\frac{fst^s}{2(s-1)}\right] \tag{42}$$

Now $$\frac{\rho}{\rho_{vac}} = \frac{\dfrac{s(s-1)f}{12\pi B(1+\omega)^2}\exp\left(ft^s\right)}{-\dfrac{s(s-1)f}{12\pi B(1+\omega)^2}\exp\left(ft^s\right)\left[1-\dfrac{fst^s}{2(s-1)}\right]}$$

$$= \quad -\frac{1}{1-\dfrac{fs}{2(s-1)}t^s} \tag{43}$$

So in the limit of $t \to 0$ we have

$$\rho = -\rho_{vac} \tag{44}$$

This corresponds to an initial state with zero total energy.

5. Conclusions

As we believe that the existence of wave at the leading edge of the expansion of the galaxies in the most likely possibility, so the alternatives are that either universe of expanding galaxies goes on infinity or else the universe is not simply connected. Besides for expanding universe FRW metric (k = +1, 0, -1) in the explicit "Schwarzschild" form is already expressed in cosmology. Using this FRW form with the variation of Gravitational constant G and cosmological constant Λ we have found and seen in (13) when t increases, R increases but ρ and p decrease, which indicates that the universe is expanding.. We have a spatially flat static universe with varying Λ and G and total energy is zero i.e. $\rho = -\rho_{vac}$ in equation (44) this corresponds to an initial state with zero total energy. The scale factor and the energy densities ρ and ρ_{vac} are finite in this limit, but both Λ and G diverge at time zero for the expanding solutions [13, 14]. So the solution is realistic.

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