

Gamma Rings of Gamma Endomorphisms

Md. Sabur Uddin¹ and Md. Sirajul Islam²

¹Department of Mathematics
Carmichael College, Rangpur-5400 Bangladesh

²Department of Mathematics
Bangabandhu Sheikh Mujibur Rahman Science & Technology University
Gopalganj, Bangladesh

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Abstract. In this paper we study Γ -rings of Γ -endomorphisms. Some properties of Irreducible Γ -rings are developed by the help of Γ -endomorphisms.

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1. Introduction

As a generalization of ring theories, the concept of Γ -rings was first introduced by N. Nobusawa [6]. After wards Barnes [1] generalized the notion of Nobusawa's Γ -rings and gave a new definition of a Γ -ring. Now a days, Γ -rings means the Γ -rings due to Barnes and other Γ -rings are known as Γ_N -rings i.e., gamma rings in the sense of Nobusawa. Many Mathematicians worked on Γ -rings and obtained some fruitful results which are the generalizations of that of the classical ring theories.

W.E. Barnes [1] introduced the notation of Γ -homomorphisms, Prime and Primary ideals, m -systems and the radical of an ideal for Γ -rings.

In classical ring theories N.H. McCoy [5] studied rings of endomorphisms.

In this paper we generalized the results of N.H. McCoy [5] into Γ -rings of Γ -endomorphisms. We also developed some characterizations of Γ -rings by the help of Γ -endomorphisms.

2. Preliminaries

Gamma Ring. Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \rightarrow M$ (sending (x, α, y) into $x\alpha y$) such that

- (i) $(x + y)\alpha z = x\alpha z + y\alpha z$
 $x(\alpha + \beta)z = x\alpha z + x\beta z$
 $x\alpha(y + z) = x\alpha y + x\alpha z$
- (ii) $(x\alpha y)\beta z = x\alpha(y\beta z),$

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where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then M is called a Γ -ring in the sense of Barnes [1].

Sub- Γ -ring. Let M be a Γ -ring. A non-empty subset S of a Γ -ring M is a sub- Γ -ring of M if $a, b \in S$, then $a - b \in S$ and $a\gamma b \in S$ for all $\gamma \in \Gamma$.

Ideal of Γ -rings. A subset A of the Γ -ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c \in M, \alpha \in \Gamma, a \in A\}$ ($A\Gamma M$) is contained in A . If A is both a left and a right ideal of M , then we say that A is an ideal or two-sided ideal of M .

If A and B are both left (respectively right or two-sided) ideals of M , then $A + B = \{a + b \mid a \in A, b \in B\}$ is clearly a left (respectively right or two-sided) ideal, called the sum of A and B . We can say every finite sum of left (respectively right or two-sided) ideal of a Γ -ring is also a left (respectively right or two-sided) ideal.

It is clear that the intersection of any number of left (respectively right or two-sided) ideal of M is also a left (respectively right or two-sided) ideal of M .

If A is a left ideal of M , B is a right ideal of M and S is any non-empty subset of M , then the set, $A\Gamma S = \{\sum_{i=1}^n a_i\gamma s_i \mid a_i \in A, \gamma \in \Gamma, s_i \in S, n \text{ is a positive integer}\}$ is a left ideal of M and $S\Gamma B$ is a right ideal of M . $A\Gamma B$ is a two-sided ideal of M .

If $a \in M$, then the principal ideal generated by a denoted by $\langle a \rangle$ is the intersection of all ideals containing a and is the set of all finite sum of elements of the form $na + x\alpha a + a\beta y + u\gamma a\mu v$, where n is an integer, x, y, u, v are elements of M and $\alpha, \beta, \gamma, \mu$ are elements of Γ . This is the smallest ideal generated by a . Let $a \in M$. The smallest left (right) ideal generated by a is called the principal left (right) ideal $\langle a \mid (\mid a) \rangle$.

Unity element of a Γ -ring. Let M be a Γ -ring. M is called a Γ -ring with unity if there exists an element $e \in M$ such that $a\gamma e = e\gamma a = a$ for all $a \in M$ and some $\gamma \in \Gamma$.

We shall frequently denote e by 1 and when M is a Γ -ring with unity, we shall often write $1 \in M$. Note that not all Γ -rings have an unity. When a Γ -ring has an unity, then the unity is unique.

Γ -homomorphism. Let M and S be two Γ -rings. A mapping θ of a Γ -ring M into a Γ -ring S is said to be a Γ -homomorphism of M into S if addition and multiplication are preserved under this mapping, that is, if $a, b \in M, \alpha \in \Gamma$

(i) $(a + b)\theta = a\theta + b\theta$

(ii) $(a\alpha b)\theta = (a\theta)\alpha(b\theta)$.

If θ is one-one and onto then θ is called a Γ -isomorphism from M into S .

Kernel of Γ -homomorphism. If θ is a homomorphism of a Γ -ring M into a Γ -ring S , then $(0)\theta^{-1}$, that is, the set of all elements a of M such that $a\theta = 0$ (the zero of S), is called the kernel of the Γ -homomorphism θ .

3. Gamma Rings of Gamma Endomorphisms

Definition 3.1. Mapping $a: M \rightarrow M$ of the Γ -ring M into itself is called a Γ -endomorphism of M if for $x, y \in M, \alpha \in \Gamma$, then

$$(x + y)a = xa + ya \quad (1)$$

$$(x\alpha y)a = xa\alpha ya. \quad (2)$$

If a is a Γ -endomorphism of M , then $0a = 0, 0 \in M, (-x)a = -(xa)$ and $(x\alpha y)0 = 0, x, y \in M, \alpha \in \Gamma$. It is to be understood that equality of mappings is the usual equality of mappings, in other words $a = b$ for Γ -endomorphisms a and b of M means that $xa = xb$ for every $x \in M$.

Let us denote by Δ , the set of all Γ -endomorphism of the Γ -ring M . We now define multiplication and addition on the set Δ as follows: where it is understood that a and b are elements of Δ :

$$x(a\alpha b) = (xa)\alpha b, x \in M, \alpha \in \Gamma \quad (3)$$

$$x(a + b) = xa + xb, x \in M. \quad (4)$$

Ofcourse (3) is just the usual definition of multiplication of mappings of any set into itself, whereas (4) has meaning only because we already have an operation of addition defined on M . The fact that $a\alpha b$ and $a+b$ are indeed Γ -endomorphism of M and therefore elements of Δ follows from the following simple calculations in which x and y are arbitrary elements in M :

$$\begin{aligned} (x + y)(a\alpha b) &= ((x + y)a)\alpha b \quad \text{by (3)} \\ &= (xa + ya)\alpha b \quad \text{by (1)} \\ &= (xa)\alpha b + (ya)\alpha b \quad \text{by (1)} \\ &= x(a\alpha b) + y(a\alpha b) \quad \text{by (3);} \end{aligned}$$

$$\begin{aligned} \text{and } (x + y)(a + b) &= (x + y)a + (x + y)b \quad \text{by (1)} \\ &= xa + ya + xb + yb \quad \text{by (1)} \\ &= xa + xb + ya + yb \\ &= x(a + b) + y(a + b) \quad \text{by (4)}. \end{aligned}$$

Now that we have addition and multiplication defined on the set Δ , we may state the following theorem.

Theorem 3.2. With respect to the operations (4) and (3) of addition and multiplications the set Δ of all Γ -endomorphisms of the Γ -ring M is a Γ -ring with unity.

Proof. (i) Let $a, b, c \in \Delta, \alpha \in \Gamma$ and $x \in M$,

$$\begin{aligned} \text{then } x((a + b)\alpha c) &= x(a + b)\alpha c \\ &= (xa + xb)\alpha c \\ &= (xa)\alpha c + (xb)\alpha c \\ &= x(a\alpha c) + x(b\alpha c) \\ &= x(a\alpha c + b\alpha c). \end{aligned}$$

Hence $(a + b)\alpha c = a\alpha c + b\alpha c$.

Now $x(a(\alpha + \beta)c) = (xa)(\alpha + \beta)c, a, c \in \Delta, \alpha, \beta \in \Gamma, x \in M$

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$$\begin{aligned}
 &= (xa)\alpha c + (xa)\beta c \\
 &= x(a\alpha c) + x(a\beta c) \\
 &= x(a\alpha c) + a\beta c.
 \end{aligned}$$

Thus $a(\alpha + \beta)c = a\alpha c + a\beta c$.

Again $x(a\alpha(b+c)) = (xa)\alpha(b+c)$, $a, b, c \in \Delta, \alpha \in \Gamma, x \in M$

$$\begin{aligned}
 &= (xa)\alpha b + (xa)\alpha c \\
 &= x(a\alpha b) + x(a\alpha c) \\
 &= x(a\alpha b + a\alpha c).
 \end{aligned}$$

Hence $a\alpha(b+c) = a\alpha b + a\alpha c$.

(ii) $x((a\alpha b)\beta c) = (x(a\alpha b))\beta c$, $a, b, c \in \Delta, \alpha, \beta \in \Gamma$

$$\begin{aligned}
 &= ((xa)\alpha b)\beta c \\
 &= (xa)\alpha(b\beta c) \\
 &= x(a\alpha(b\beta c)).
 \end{aligned}$$

Hence $(a\alpha b)\beta c = a\alpha(b\beta c)$.

(iii) For all $a \in \Delta$, then exists unity element $1 \in \Delta$ such that

$$\begin{aligned}
 x(1\alpha a) &= ((x1)\alpha)a = xa, \alpha \in \Gamma, x \in M \text{ and } x(a\alpha 1) = ((xa)\alpha)1 = xa. \text{ Thus} \\
 x(a\alpha 1) &= x(1\alpha a) = xa.
 \end{aligned}$$

Hence $a\alpha 1 = 1\alpha a = a$. Thus Δ satisfies all the conditions of a Γ -ring. Hence Δ is a Γ -ring with unity.

Theorem 3.3. Let Δ be the Γ -ring of all Γ -endomorphisms of the Γ -ring M . If $a \in \Delta$, then a has an inverse in Δ if and only if a is a one-one mapping of M onto M .

Proof. Suppose first that a has an inverse b in Δ , so that $a\alpha b = b\alpha a = 1, \alpha \in \Gamma$. Then for each $x \in M$, we have $(xb)\alpha a = x(b\alpha a) = x1 = x$ and a is clearly a mapping onto M . Moreover, if $x_1, x_2 \in M$ such that $x_1 a = x_2 a$, then

$$x_1 = x_1 1 = x_1 (a\alpha b) = (x_1 a)\alpha b = (x_2 a)\alpha b = x_2 (a\alpha b) = x_2 1 = x_2.$$

This shows that the mapping is a one-one mapping.

Conversely, let us assume that the Γ -endomorphism a is a one-one mapping of M onto M , so that every element of M is uniquely expressible in the form $xa, x \in M$. We may therefore define a mapping b of M into M as follows: $((xa)\alpha)b = x, x \in M, \alpha \in \Gamma$. Since, if $x, y \in M$, then

$$\begin{aligned}
 ((xa + ya)\alpha)b &= (((x+y)a)\alpha)b = x + y = ((xa)\alpha)b + ((ya)\alpha)b \text{ and} \\
 ((xa\alpha y)\alpha)b &= ((x\alpha y)\alpha)b = (x\alpha y) = ((xa)\alpha)b\alpha((ya)\alpha)b.
 \end{aligned}$$

We see that b is a Γ -endomorphism of M . Moreover $(xa)\alpha b = x(a\alpha b) = x1 = x$ for every x in M and hence $a\alpha b = 1$. Finally if $x \in M$, then

$$((xa)\alpha)(b\alpha a) = (x(a\alpha b))\alpha a = (x1)\alpha a = x(1\alpha a) = xa.$$

Thus is equivalent to the statement that $y(b\alpha a) = y$ for every $y \in M$. Hence $b\alpha a = 1$ and b is the inverse of a in Δ .

Suppose now that we start with a given Γ -ring M and let Δ be the Γ -ring of all Γ -endomorphisms of M . If a is a fixed element of M , then the mapping $x \rightarrow x\alpha a, \alpha \in \Gamma, x \in M$ of M into M is called the right multiplication by a . It will be convenient to denote this right multiplication by a , that is, the mapping a_m is defined by $x\alpha a_m = x\alpha a, x \in M, \alpha \in \Gamma$.

It is clear that a_m is a Γ -endomorphism of M and therefore $a_m \in \Delta$ for each $a \in M$. If $a, b \in M$, then $a_m + b_m$ and $a_m \alpha b_m, \alpha \in \Gamma$ are defined in Δ . Moreover we observe that for each x in M ,

$$x\alpha(a+b)_m = x\alpha(a+b) = x\alpha a + x\alpha b = x\alpha a_m + x\alpha b_m = x\alpha(a_m + b_m)$$

$$\text{and } x\alpha(a\alpha b)_m = x\alpha(a\alpha b) = (x\alpha a)\alpha b = (x\alpha a)\alpha b_m = (x\alpha a_m)\alpha b_m = x\alpha(a_m \alpha b_m).$$

Thus we

$$\left. \begin{aligned} (a+b)_m &= a_m + b_m \\ (a\alpha b)_m &= a_m \alpha b_m \end{aligned} \right\} \quad (5)$$

Hence $a_m + b_m$ and $a_m \alpha b_m, \alpha \in \Gamma$ we themselves right multiplications and the set S of all right multiplications is a sub- Γ -ring of Δ . Moreover, the relations (5) shows that the mapping $a \rightarrow a_m, a \in M$, is a Γ -homomorphims of M onto S . If a is in the kernel of this Γ -homomorphism, then $x\alpha a = 0$ for every element $x \in M$ and $\alpha \in \Gamma$. If M happens to have a unity, this implies that $a = 0$ and in this case the kernel is certainly zero and the Γ -homomorphism is a Γ -isomorphism. We have therefore established the following result.

Lemma 3.4. If the Γ -ring has a unity, then M is Γ -isomorphic to the Γ -ring of all its right mutiplications. For our purposes, the importance of this Lemma is that it leads atmost immediately to the following result.

Theorem 3.5. Every Γ -ring M is Γ -isomorphic to a Γ -ring of Γ -endomorphisms of M .

If Δ is a Γ -ring of Γ -endomorphisms of a Γ -ring M and U is a sub- Γ -ring of M . We naturally define $U\Delta$ as follows: $U\Delta = \{x\alpha | x \in U, \alpha \in \Delta\}$. If U consists of a single element y of M , we write $y\Delta$ instead of $\{y\}\Delta$, that is $y\Delta = \{y\alpha | \alpha \in \Delta\}$.

Definition 3.6. If Δ is a Γ -ring of Γ -endomorphisms of a Γ -ring M and W is a sub- Γ -ring of M such that $W\Delta \subseteq W$, then W may be called a Δ -sub- Γ -ring of M .

Clearly, the zero sub- Γ ring denote simply by '0' and the entire Γ -ring M are always Δ -sub- Γ -rings. We shall be particularly interested in the case in which there are no Δ -sub- Γ -rings except these two trivial ones.

Definition 3.7. Let Δ be a nonzero Γ -ring of Γ -endomorphisms of the Γ -ring M . If the only Δ -sub- Γ -rings W of M are $W = 0$ and $W = M$. We say that Δ is an irreducible Γ -ring of Γ -endomorphisms of M .

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We may remark that if M is the zero Γ -ring (having only the zero element) the only Γ -endomorphism of M is the zero Γ -endomorphism. Since in the preceding definition Δ is required to be a non-zero Γ -ring of Γ -endomorphisms of M , when we speak of an irreducible Γ -ring of Γ -endomorphisms of M it is implicit that M must have nonzero elements.

We shall next prove the following useful result.

Lemma 3.8. If Δ is a nonzero Γ -ring of Γ -endomorphisms of the Γ -ring M , then Δ is an irreducible Γ -ring of Γ -endomorphisms of M if and only if $x\Delta = M$ for every nonzero element x of M .

Proof. One part is essentially trivial. For if $x\Delta = M$ for every nonzero element x of M , it is clear that M is the only nonzero Δ -sub- Γ -ring of M and Δ is therefore irreducible.

Conversely, let us assume that Δ is an irreducible Γ -ring of Γ -endomorphisms of M and that x is an arbitrary nonzero element of M . It is easily verified that $x\Delta$ is a sub- Γ -ring of M and since $(x\Delta)\Gamma\Delta \subseteq x\Delta$, it is a Δ -sub- Γ -ring of M . It follows that $x\Delta = 0$ or $x\Delta = M$. Suppose that $x\Delta = 0$ and let $\langle x \rangle$ be the sub- Γ -ring of M generated by x . Then $\langle x \rangle\Delta = 0$ and therefore $\langle x \rangle$ is an Δ -sub- Γ -ring of M . Since $x \in \langle x \rangle$ and $x \neq 0$, we must have $\langle x \rangle = M$ and therefore $M\Delta = 0$. However, this is impossible since M has nonzero elements and the assumption that $x\Delta = 0$ has led to a contradiction. Hence $x\Delta = M$ and the proof is completed.

Theorem 3.9. Let Δ be an irreducible Γ -ring of Γ -endomorphisms of the Γ -ring M . If A is a nonzero ideal in Δ , then A also is an irreducible Γ -ring of Γ -endomorphisms of M .

Proof. Let x be an arbitrary nonzero element of M . Since $A\Gamma\Delta \subseteq A$. We have that $(xA)\Gamma\Delta = x(A\Gamma\Delta) \subseteq xA$, so that xA is a Δ -sub- Γ -ring of M . Since Δ is irreducible, it follows that $xA = 0$ or $xA = M$. Suppose that $xA = 0$. Then using Lemma 3.8, we have $MA = (xA)\Gamma\Delta \subseteq xA = 0$, which is impossible since A is assumed to have nonzero elements. Accordingly we must have $xA = M$. The same lemma, now applied to the Γ -ring A , shows that A is indeed an irreducible Γ -ring of Γ -endomorphism of M and the proof is completed.

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