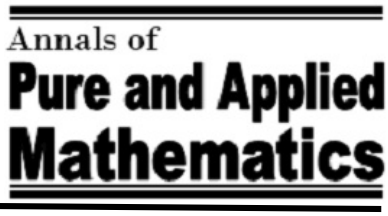
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The Connected Domination Number and Some Hamiltonian Properties of Graphs

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Abstract. Let G = (V, E) be a graph. A connected dominating set of G is a subset of V that is a dominating set and induces a connected subgraph of G. The connected domination number of G is the minimum size of a connected dominating set of G. In this note, we present sufficient conditions based on the connected domination number for some Hamiltonian properties of graphs.

Keywords: Hamiltonian graph; traceable graph; connected domination number

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. For a graph $G = (V(G), E(G))$, we use n and e to denote its order $|V(G)|$ and size $|E(G)|$, respectively. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The neighborhood of a vertex u in G is denoted by $N_G(u)$. If a vertex $u \in V - V_1$, where V_1 is a subset of the vertex set V, we define $d_{G[V_1]}(u)$ as $|N_G(u) \cap V_1|$, where $G[V_1]$ is a subgraph of G which is induced by V_1. A subset V_1 of the vertex set $V(G)$ of G is called an independent set if no two vertices in V_1 are adjacent in G. A maximum independent set in a graph G is an independent set of largest possible size. The independence number, denoted $\alpha(G)$, of a graph G is the cardinality of a maximum independent set in G. A subset D of the vertex set $V(G)$ of G is called a dominating set if every vertex v in G is either in the set D or is adjacent to a vertex in D. A connected dominating set of G is a subset of V that is a dominating set and induces a connected subgraph of G. The connected domination number of G, denoted $\gamma_c(G)$, is the minimum size of a connected dominating set of G. Let $A(G)$ be the adjacency matrix of G. The Laplacian spectral radius of G, denoted $\mu_1(G)$, is defined as the largest eigenvalue of $L(G) := D(G) - A(G)$, where $D(G)$ is a diagonal matrix whose entries are the degrees of vertices in G. The signless Laplacian spectral radius of G,

denoted $q_1(G)$, is defined as the largest eigenvalue of $Q(G) := D(G) + A(G)$, where $D(G)$ is a diagonal matrix whose entries are the degrees of vertices in G. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. In this note, we present sufficient conditions based on the connected domination number for some Hamiltonian properties of graphs. The main results are as follows.

Theorem 1. Let G be a k-connected graph ($k \geq 2$) of order $n \geq 3$ and size e. If $\gamma_c + \frac{n\delta}{2(n-k-1)} \geq n$, then G is Hamiltonian.

Theorem 2. Let G be a k-connected graph ($k \geq 1$) of order $n \geq 9$ and size e. If $\gamma_c + \frac{n\delta}{2(n-k-2)} \geq n$, then G is traceable.

2. Lemmas

We need the following results as lemmas to prove Theorem 1 and Theorem 2. Lemma 1 and Lemma 2 below are from [2].

Lemma 1. Let G be a k-connected graph of order $n \geq 3$. If $\alpha \leq k$, then G is Hamiltonian.

Lemma 2. Let G be a k-connected graph of order n. If $\alpha \leq k + 1$, then G is traceable.

Lemma 3 below is from [5]. See also Proposition 10.2 on Page 272 in [4].

Lemma 3. Let G be a connected graph of order n. Then $\gamma_c + \Delta \leq n$.

Lemma 4 below is from Theorem 4.6 on Page 163 in [3].

Lemma 4. Let G be a connected graph. Then $q_1 \leq 2\Delta$ with equality if and only if G is regular.

Lemma 5 below is from Lemma 2.1 on Page 34 in [8].

Lemma 5. Let G be a connected graph. Then $\mu_1 \leq q_1$ with equality if and only if G is a bipartite graph.

Lemma 6 below follows from the proof of [4] in Theorem 1.1 on Page 584 in [6].

Lemma 6. Let G be a connected graph and I be an independent set with $|I| = \alpha$. Then $\mu_1 \geq \left(\sum_{y \in I} d(y)\right) \frac{n}{\alpha(n-\alpha)}$.

Lemma 7 below is from [7].

Lemma 7. Let G be a balanced bipartite graph of order $2n$ with bipartition (A, B). If $d(x) + d(y) \geq n + 1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then G is Hamiltonian.

3. Proofs

Proof of Theorem 1. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges satisfying the conditions in Theorem 1. Suppose G is not Hamiltonian. Lemma 1

implies that $\alpha \geq k+1$. Let I be any maximum independent set in G. Then $|I| = \alpha < n$. From Lemmas 3, 4, 5, and 6, we have that

$$\gamma_c + \frac{n\delta}{2(n-k-1)} \geq n \geq \gamma_c + \Delta \geq \gamma_c + \frac{q_1}{2}$$

$$\geq \gamma_c + \frac{\mu_1}{2} \geq \gamma_c + \left(\sum_{y \in I} d(y)\right) \frac{n}{2\alpha(n-\alpha)}$$

$$\geq \gamma_c + \frac{\alpha\delta n}{2\alpha(n-\alpha)} = \gamma_c + \frac{n\delta}{2(n-\alpha)} \geq \gamma_c + \frac{n\delta}{2(n-k-1)}.$$

Thus $2\Delta = q_1$, $q_1 = \mu_1$, and $\alpha = k+1$. Therefore G is a k-connected ($k \geq 2$) regular bipartite graph. Suppose the partition sets of G are P and Q. Then $\delta|P| = |E(I, V-I)| = \delta|Q|$. Thus $|P| = |Q|$. Notice that both P and Q are maximum independent sets in G. Therefore $|P| = |Q| = \alpha = k+1$. From Lemma 7, we have that G is Hamiltonian, a contradiction. □

Proof of Theorem 2. Let G be a k-connected ($k \geq 1$) graph with $n \geq 9$ vertices and e edges satisfying the conditions in Theorem 2. Suppose G is not traceable. Lemma 2 implies that $\alpha \geq k+2$. Let I be any maximum independent set in G. Then $|I| = \alpha < n$. From Lemmas 3, 4, 5, and 6, we have that

$$\gamma_c + \frac{n\delta}{2(n-k-2)} \geq n \geq \gamma_c + \Delta \geq \gamma_c + \frac{q_1}{2}$$

$$\geq \gamma_c + \frac{\mu_1}{2} \geq \gamma_c + \left(\sum_{y \in I} d(y)\right) \frac{n}{2\alpha(n-\alpha)}$$

$$\geq \gamma_c + \frac{\alpha\delta n}{2\alpha(n-\alpha)} = \gamma_c + \frac{n\delta}{2(n-\alpha)} \geq \gamma_c + \frac{n\delta}{2(n-k-2)}.$$

Thus $2\Delta = q_1$, $q_1 = \mu_1$, and $\alpha = k+2$. Therefore G is a k-connected ($k \geq 1$) regular bipartite graph. Suppose the partition sets of G are P and Q. Then $\delta|P| = |E(I, V-I)| = \delta|Q|$. Thus $|P| = |Q|$. Notice that both P and Q are maximum independent sets in G. Therefore $|P| = |Q| = \alpha = k+2$. Since $n = 2k+4 \geq 9$, we have that $k \geq 3$. From Lemma 7, we have that G is traceable, a contradiction. □

Remark. In the proofs of Theorems 1 and 2, we establish an inequality on α and γ_c for a connected graph. Namely, $\gamma_c + \frac{n\delta}{2(n-\alpha)} \leq n$. Notice further that for some connected graphs the equality in the less-than-or-equal-to relationship is achievable. For instance, if G is an even cycle C_{2s} with $s \geq 2$, then $n = 2s$, $\gamma_c = 2s-2$, $\delta = 2$, $\alpha = \frac{n}{2}$, and $\gamma_c + \frac{n\delta}{2(n-\alpha)} = n$.

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REFERENCES

1. J.A.Bondy and U.S.R.Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York, 1976.
2. V.Chvátal and P.Erdős, A note on Hamiltonian circuits, *Discrete Mathematics*, 2 (1972) 111-113.
3. D.Cvetković, P.Rowlinson and S.K.Simić, Signless Laplacians of finite graphs, *Linear Algebra and its Applications*, 423 (2007) 155-171.
4. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc., 1998.
5. S.T.Hedetniemi and R.Laskar, Connected domination in graphs, in: B.Bollobás (ed.), *Graph Theory and Combinatorics*, Academic Press, London, 1984, 209-218.
6. R.Li, The Narumi-Katayama index and some Hamiltonian properties of graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 131 (2026) 577-591.
7. J.Moon and L.Moser, On Hamiltonian bipartite graphs, *Israel Journal of Mathematics*, 1 (1963) 163-165.
8. X.D.Zhang and R.Luo, The spectral radius of triangle-free graphs, *Australasian Journal of Combinatorics*, 26 (2002) 33-39.