

Picture Fuzzy Bondage Set and its Application in Indian Railways

Avishek Banerjee¹ and Sk Amanathulla^{2*}

¹Department of Mathematics, Sidho-Kanho-Birsha University, Purulia, 723104, India
 Email: 2avipr12@gmail.com

²Department of Mathematics, Raghunathpur College,
 Raghunathpur, Purulia, 723133, India.
 Email: samanathulla.math@raghunathpurcollege.ac.in

Received 27 August 2026; accepted 27 September 2026

Abstract. Many real-life situations can be formulated as a picture fuzzy graph (P_fG). Therefore, studying P_fG s is very important. In this paper, we introduce the bondage set and the non-bondage set in P_fG . We describe the union, intersection, and join of these sets. Some properties and theorems are discussed. We propose two algorithms to compute the P_f strong edge and P_f bondage number. Finally, we present an application to the Indian railway network using the bondage set in P_fG .

Keywords: Picture Fuzzy Graph; Bondage set; Bondage number; Non-bondage set; Domination Number

AMS Mathematics Subject Classification (2020): 05C69, 03E72, 90C70

Table of Abbreviations: The following symbols have been used throughout the paper.

$\zeta(\wp)$	P_f domination number of the P_fG $\wp$
$\alpha(\wp)$	Set of all P_fBS s of the P_fG $\wp$
$\beta(\wp)$	Set of all P_fNBS s of the P_fG $\wp$
$\gamma(\wp)$	Set of all P_fm_iBS s of the P_fG $\wp$
$\sigma(\wp)$	Set of all P_fm_xNBS s of the P_fG $\wp$
$\varrho(\wp)$	P_fBN of the P_fG $\wp$
$\varsigma(\wp)$	P_fNBN of the P_fG $\wp$
$CO_\wp(v_i, v_j)$	Strength of connectedness between P_fvs v_i and v_j .

1. Introduction

The crisp graph is converted to a fuzzy graph because the vertices and edges are uncertain.

In a fuzzy graph, every member has a degree of membership. Then the concept of non-membership of the vertices and edges reduces it to an intuitionistic fuzzy graph (I_fG). But this expression was not exact. There is one more term, neutrality, which is applicable to every member of a graph, and so the concept of P_fG is very essential. It has a wide range of applications in Mathematics, Engineering, Computer Science, etc. The structure of a P_fG is $\wp=(\Gamma, \Delta)$ on the underlying crisp graph (γ, δ) . Here Γ represents a picture fuzzy set (P_fS) on the crisp vertex set γ , and Δ is a PfS on the crisp edge set δ . Almost any real-life problem can be constructed as a P_fG . But the most challenging task is to keep the whole system functional. Bondage set (BS) indicates the vulnerability of the whole network. Link failure is one such example. The topics *BSS and non – bondage sets (NBSSs)* were introduced in crisp graphs and fuzzy graphs. Now we have been introducing these concepts in the Pf environment, as these have a lot of applications in real life.

1.1. Literature review

Euler introduced the crisp graph in 1736. Then, because of the uncertainty of objects, Zadeh [1] introduced fuzzy sets in 1965. A non-negative rational number less than or equal to 1 is attached to each member in a fuzzy set, and the relationship among them is converted into a fuzzy graph, proposed by Rosenfeld [2]. Atanassov [3] extended fuzzy sets to I_fS by introducing the concept of non-membership of the objects. Cuong *et al.* [4] extended the idea into a P_fS , introducing the neutrality value of membership for each vertex and edge. In 1990, Fink *et al.* [5] introduced the bondage number (BN) in graph theory. In 1994, Hartnell *et al.* [6] discussed bounds for BN . After a few years BN and NBN concepts were introduced in fuzzy graphs by Nagoor Gani *et al.* [7] in 2015. Zehui Shao *et al.* [8] presented the same concept in an I_fG . Cen Zuo *et al.* [9] gave the definition of P_fG in 2019. Recently, Amanathulla *et al.* used $PfGs$ [10], balanced $PfGs$ [11] in a decision-making problem and introduced PfG to select the best way in an airline network [12]. Xiao *et al.* [13] presented an application regarding regular P_fGs . In 2022, Das *et al.* investigated threshold [14], tolerance P_fGs with application in the medical sector [15]. Amanathulla *et al.* discussed some applications of discrete mathematics in [16]. Somasundaram *et al.* [17] introduced the domination concept with an effective edge in fuzzy graphs, and Parvathi *et al.* [18] in I_fG . Sivasankar *et al.* [19] introduced various types of domination parameters in I_fG . In P_fG , Ismayil *et al.* [20] studied the domination concept in a seminar paper. Recently, in 2024, Banerjee *et al.* [21] introduced split domination in PfG to control the disaster situation and further developed the domination concept in the papers [22], [23]. Adhikari *et al.* [24] developed some operations in P_fGs . Bera *et al.* [25] discussed P_f regular graphs and analysed social networks using the planar P_fG concept [26]. Khatun *et al.* [27] studied the decision-making problem in neutrosophic graphs. Mahapatra *et al.* also explored competition graphs under the m -polar interval-valued fuzzy graph ($mpl_{vf}G$) [28] and discussed planar $mpl_{vf}G$. Also, developed an mp fuzzy threshold graph [29] with an application in a power control system. Rashmanlou *et al.* [30] developed a cubic graph and applied it.

In this paper, the remaining portion is structured as follows: Section 2 presents preliminaries. Section 3 discusses $PfBS$, $PfNBS$, related theorems, and examples.

Picture Fuzzy Bondage Set and its Application in Indian Railways

Algorithms for P_fBN have been proposed. Section 4 presents an application in Indian Railways for P_fBS . Section 5 presents a comparative study with existing work. In Section 6, the conclusion and future work are given.

2. Preliminaries

To understand the main work of this paper, the following terms and ideas are helpful.

The form of a P_fS is $A = \{(b, \theta^+(b), \theta^0(b), \theta^-(b))\}$ satisfying $0 \leq \theta^+(b) + \theta^0(b) + \theta^-(b) \leq 1 \forall b \in Z$, Z being the universal set. Here, $\theta^+(b)$, $\theta^0(b)$, $\theta^-(b)$ represent the positive, neutral and negative degrees of membership for the picture fuzzy vertex ($P_f v$) b . The P_fG is $\wp = (\Gamma, \Delta)$ on the underlying crisp graph (γ, δ) . Here Γ represents a P_fS on crisp vertex set γ , and Δ is a P_fS on crisp edge set δ with the conditions $\theta^+(i, c) \leq \wedge \{\theta^+(i), \theta^+(c)\}$, $\theta^0(i, c) \leq \wedge \{\theta^0(i), \theta^0(c)\}$, $\theta^-(i, c) \geq \vee \{\theta^-(i), \theta^-(c)\}$. A $P_f v$ i dominates another $P_f v$ c if (i, c) is P_fSE . The $P_f e$ (i, c) is called P_fSE if $\theta^+(i, c) \geq \theta^{\infty+}(i, c)$, $\theta^0(i, c) \geq \theta^{\infty 0}(i, c)$ and $\theta^-(i, c) \leq \theta^{\infty-}(i, c)$. A P_fS η is called P_fDS if all $P_f v$ $c \in \Gamma - \eta$ is dominated by at least one $P_f v$ $i \in \eta$. If a P_fDS has no proper P_fD subset, then it is called a minimal P_fDS . The domination number $\zeta(\wp)$ is the cardinality of the minimum P_fDS .

3. Picture fuzzy bondage set

In this section, we have introduced the notion of P_fBS , P_fBN . We define various operations on P_fBS s, such as intersection, union, join, etc. We also investigate important new results on P_fBS .

Definition 1. A set $D \subseteq X$ is called a P_fBS or P_fNBS in the P_fG $\wp$ according as $\zeta(\wp - D) > \zeta(\wp)$ or $\zeta(\wp - D) = \zeta(\wp)$. $\alpha(\wp)$ and $\beta(\wp)$ respectively represents the set of all P_fBS s and P_fNBS s in a P_fG $\wp$.

Definition 2. A P_fBS is called P_fm_iBS if no proper subset is again a P_fBS . The collection of all such P_fm_iBS s is denoted by $\gamma(\wp)$. Also, a P_fNBS B_i becomes a P_fm_xNBS when no superset is again a P_fNBS . The collection of all P_fm_xNBS s is denoted by $\sigma(\wp)$.

Definition 3. A P_fm_mBS is defined as, $B = \wedge \{\partial(A_i) = \sum_{A_i} \frac{1+\theta^+(i,c)+\theta^0(i,c)-\theta^-(i,c)}{3} : A_i \in \gamma(\wp)\}$. $q(\wp) = |B|$ is called P_fBN . And $\varsigma(\wp) = |C|$ denotes P_fNBN . Where C is a $\sigma(\wp)$ -set.

Definition 4. A P_fuBS is a P_fm_iBS with maximum cardinality. P_fuBN is calculated by $q^u(\wp) = \vee \{|A_i| : A_i \in \gamma(\wp)\}$. A P_fsBS is a P_fBS with maximum cardinality. P_fsBN is calculated by $q^s(\wp) = \vee \{|A_i| : A_i \in \alpha(\wp)\}$.

Example 1. We have discussed P_fBS s and P_fNBS s of the P_fG given in Fig. 1. The $P_f v$ a and b has the following P_f -paths: $Q_1 = x_1$ with strength $(0.30, 0.10, 0.50)$, $Q_2 = x_5 - x_2$ with strength $(\wedge \{0.10, 0.10\}, \wedge \{0.10, 0.10\}, \vee \{0.50, 0.60\}) =$

$(0.10, 0.10, 0.60)$, $Q_3 = x_4 - x_3 - x_2$ with strength $(\wedge \{0.30, 0.10, 0.10\}, \wedge \{0.20, 0.10, 0.10\}, \vee \{0.50, 0.40, 0.60\}) = (0.10, 0.10, 0.60)$, $CO_{\wp}(a, b) = (\vee \{0.30, 0.10, 0.10\}, \vee \{0.10, 0.10, 0.10\}, \wedge \{0.50, 0.60, 0.60\}) = (0.30, 0.10, 0.50)$, $CO_{\wp}(a, c) = (0.10, 0.10, 0.50)$, $CO_{\wp}(a, d) = (0.30, 0.20, 0.50)$, $CO_{\wp}(b, c) = (0.10, 0.10, 0.50)$, $CO_{\wp}(c, d) = (0.10, 0.10, 0.40)$, $X = \{x_1, x_3, x_4, x_5\}$. Collection of all $P_f m_i DS$ s are $\epsilon(\wp) = \{\{a\}, \{b, c\}, \{b, d\}\}$. Therefore, $\zeta(\wp) = 1$. Now, $\zeta(\wp - \{x_4\}) = |\{a, c\}| = 2 > \zeta(\wp)$. So, $\{x_4\} \in \alpha(\wp)$. And $\zeta(\wp - \{x_1\}) = |\{c\}| = 1 = \zeta(\wp)$. So, $\{x_1\} \in \beta(\wp)$. Thus, $\alpha(\wp) = \{\{x_4\}, \{x_5\}, \{x_1, x_3\}, \{x_1, x_5\}, \{x_3, x_4\}, \{x_3, x_5\}, \{x_4, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}, \{x_1, x_4, x_5\}, \{x_3, x_4, x_5\}, \{x_1, x_3, x_4, x_5\}\}$; $\beta(\wp) = \{\{x_1\}, \{x_3\}, \{x_1, x_4\}\}$, $\gamma(\wp) = \{\{x_4\}, \{x_5\}, \{x_1, x_3\}\}$, $\sigma(\wp) = \{\{x_1, x_4\}\}$, $\partial(\{x_4\}) = \frac{1+0.30+0.20-0.50}{3} = 0.33$, $\partial(\{x_5\}) = \frac{1+0.10+0.10-0.50}{3} = 0.23$, $\partial(\{x_1, x_3\}) = \frac{1+0.30+0.10-0.50}{3} + \frac{1+0.10+0.10-0.40}{3} = 0.64$, $\varrho(\wp) = |\{x_4\}| = 1$ and $\varsigma(\wp) = |\{x_1, x_4\}| = 2$.

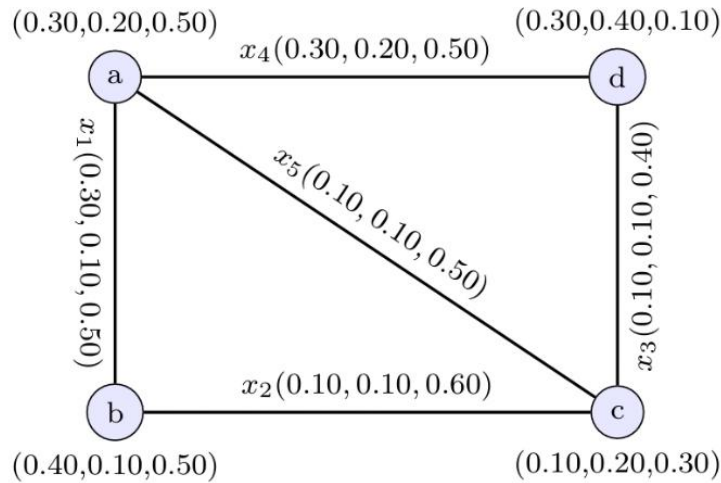


Figure 1: A $P_f G$

Definition 5. The union of two $P_f BS$ s A_1 and A_2 on the underlying crisp graph (γ_1, δ_1) and (γ_2, δ_2) in a $P_f G (\Gamma, \Delta)$ is defined as

$$A_1 \cup A_2 = \begin{cases} (\theta^+, \theta^0, \theta^-)(x_i) & \text{if } x_i \in \delta_1 - \delta_2 \\ (\theta^+, \theta^0, \theta^-)(x_j) & \text{if } x_j \in \delta_2 - \delta_1 \\ (\theta^+, \theta^0, \theta^-)(x_k) & \text{if } x_k \in \delta_1 \cap \delta_2 \end{cases}$$

Definition 6. $A_1 \cap A_2$ is denoted as the intersection of two $P_f BS$ s, which is defined by

$$A_1 \cap A_2 = \begin{cases} (\theta^+, \theta^0, \theta^-)(x) & \text{if } x \in \delta_1 \cap \delta_2 \\ \phi & \text{Otherwise} \end{cases}$$

Picture Fuzzy Bondage Set and its Application in Indian Railways

Definition 7. Let B and E be two P_fNBS s in P_fG $\wp$; then their union is defined as

$$B \cup E = \begin{cases} (\theta^+, \theta^0, \theta^-)(x_p) & \text{if } x_p \in \delta_1 - \delta_2 \\ (\theta^+, \theta^0, \theta^-)(x_q) & \text{if } x_q \in \delta_2 - \delta_1 \\ (\theta^+, \theta^0, \theta^-)(x_r) & \text{if } x_r \in \delta_1 \cap \delta_2 \end{cases}$$

Also, the intersection is defined as

$$B \cap E = \begin{cases} (\theta^+, \theta^0, \theta^-)(x) & \text{if } x \in \delta_1 \cap \delta_2 \\ \phi & \text{Otherwise} \end{cases}$$

Definition 8. Join of two P_fBS s, $A_1 = (\Gamma_1, \Delta_1)$ and $A_2 = (\Gamma_2, \Delta_2)$ on the underlying respective crisp graph (γ_1, δ_1) and (γ_2, δ_2) is $A_1 + A_2 = (\Gamma, \Delta)$ where,

$$\begin{aligned} (\theta_\Gamma^+, \theta_\Gamma^0, \theta_\Gamma^-)(u) &= \begin{cases} (\theta_{\Gamma_1}^+, \theta_{\Gamma_1}^0, \theta_{\Gamma_1}^-)(u_i) & \text{if } u_i \in \gamma_1 - \gamma_2 \\ (\theta_{\Gamma_2}^+, \theta_{\Gamma_2}^0, \theta_{\Gamma_2}^-)(u_i) & \text{if } u_i \in \gamma_2 - \gamma_1 \\ (\theta_{\Gamma_1}^+ \vee \theta_{\Gamma_2}^+, \theta_{\Gamma_1}^0 \vee \theta_{\Gamma_2}^0, \theta_{\Gamma_1}^- \wedge \theta_{\Gamma_2}^-)(u_i) & \text{if } u_i \in \gamma_1 \cap \gamma_2 \end{cases} \\ (\theta_\Delta^+, \theta_\Delta^0, \theta_\Delta^-)(u, v) &= \begin{cases} (\theta_{\Delta_1}^+, \theta_{\Delta_1}^0, \theta_{\Delta_1}^-)(u_i, u_j) & \text{if } (u_i, u_j) \in \delta_1 - \delta_2 \\ (\theta_{\Delta_2}^+, \theta_{\Delta_2}^0, \theta_{\Delta_2}^-)(u_i, u_j) & \text{if } (u_i, u_j) \in \delta_2 - \delta_1 \\ (\theta_{\Delta_1}^+ \vee \theta_{\Delta_2}^+, \theta_{\Delta_1}^0 \vee \theta_{\Delta_2}^0, \theta_{\Delta_1}^- \wedge \theta_{\Delta_2}^-)(u_i, u_j) & \text{if } (u_i, u_j) \in \delta_1 \cap \delta_2 \\ (\theta_{\Gamma_1}^+ \wedge \theta_{\Gamma_2}^+, \theta_{\Gamma_1}^0 \wedge \theta_{\Gamma_2}^0, \theta_{\Gamma_1}^- \wedge \theta_{\Gamma_2}^-)(u_i, u_j) & \text{if } (u_i, u_j) \in \delta' \end{cases} \end{aligned}$$

δ' contains all P_f es joining between P_f vs from Γ_1 and Γ_2 .

Theorem 1. The isolated P_fe of a P_fG construct the P_fBS of the P_fG .

Proof: $\wp$ be a P_fG with isolated P_fe $x=(i, y)$. Therefore, x is P_fSE , as there is no other P_f -path between the P_fv i and y . So, P_fDS of $\wp$ contains either P_fv i or P_fv y but not both. In the P_f -subgraph $\wp - x$, both the P_fv i and y become isolated. P_fDS contains both i and y in the P_f -subgraph $\wp - x$. Thus, $\zeta(\wp - x) > \zeta(\wp)$, i.e., isolated P_fe $x=(i, y)$ construct the P_fBS of the P_fG .

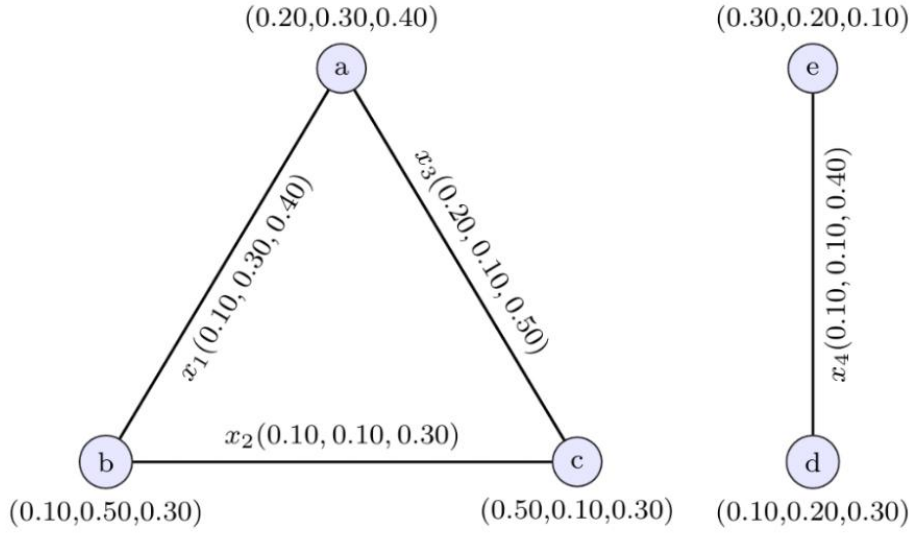


Figure 2: P_fG with isolated P_fe

Example 2. In the P_fG given in Fig. 2,

$CO_{\wp}(a, b) = (0.10, 0.30, 0.40)$, $CO_{\wp}(a, c) = (0.20, 0.10, 0.40)$, $CO_{\wp}(b, c) = (0.10, 0.10, 0.30)$, $CO_{\wp}(d, e) = (0.10, 0.10, 0.40)$, $X = \{x_1, x_2, x_4\}$, $\epsilon(\wp) = \{\{b, d\}, \{b, e\}\}$, $\zeta(\wp) = |\{b, d\}| = 2$, $\alpha(\wp) = \{\{x_4\}, \{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_1, x_2, x_4\}\}$, $\beta(\wp) = \{\{x_1\}\}$, $\varrho(\wp) = |\{x_4\}| = 1$, $\varsigma(\wp) = |\{x_1\}| = 1$. The isolated P_fe x_4 construct a P_fBS in Fig. 2.

Theorem 2. If $\wp^*$ is a P_f -star in a P_fG $\wp$ then $\varsigma(\wp^*) = 0$.

Proof: Considering $\wp^*$ as a P_f -star in the P_fG $\wp$. Its all P_fe are P_fSE . Therefore, all P_f -end vertices are dominated by the P_fv located at center of the P_f -star. So, $\zeta(\wp)=1$. Removing any P_fe from the P_fG $\wp$ increases the P_fDN of the resultant P_f -subgraph. Therefore, each P_fe forms a P_fBS . There is no P_fNBS . As a result $\varsigma(\wp)=0$.

Theorem 3. Let $\wp^*$ be a P_f -star in a P_fG $\wp$ then $\varrho(\wp)=1$.

Proof: In the P_fG $\wp$, $\wp^*$ be a P_f -star with $\{i, i_1, i_2, \dots, i_n\}$ as the P_fv -set and $\{x_1, x_2, \dots, x_n\}$ as the P_fe -set. The P_fv i be at the center that dominates the remaining P_fvs i_m , $m = 1, 2, \dots, n$. Therefore, $\{i\}$ is the only P_fm_iDS . Each P_fe of the P_f -star forms a P_fBS . Therefore, P_fm_iBS contains only one P_fe . Thus $\varrho(\wp)=1$.

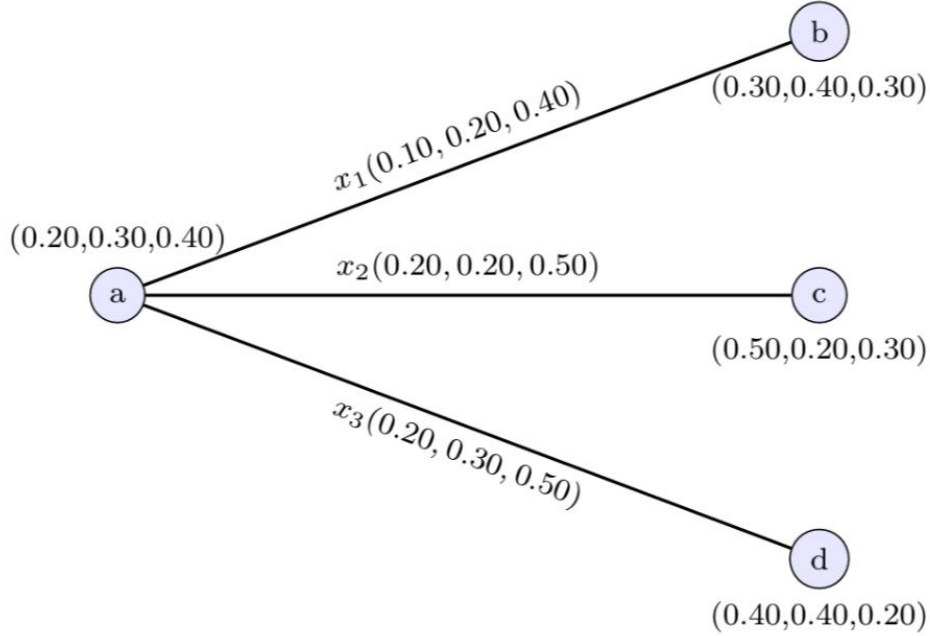


Figure 3: pf -star in a P_fG

Example 3. $CO_{\wp}(a, b) = (0.10, 0.20, 0.40)$, $CO_{\wp}(a, c) = (0.20, 0.20, 0.50)$, $CO_{\wp}(a, d) = (0.20, 0.30, 0.50)$, $X = \{x_1, x_2, x_3\}$, $\epsilon(\wp) = \{i_1\}$, $\zeta(\wp) = |\{i_1\}| = 1$, $\alpha(\wp) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}$, $\beta(\wp) = \phi$, $\varrho(\wp) = |\{x_1\}| = 1$, $\varsigma(\wp) = 0$. In the P_f -star of a P_fG given in Fig. 3, $\varrho(\wp)$ is 1.

Theorem 4. A P_fG $\wp$ with isolated P_fe has $\varrho(\wp)=1$.

Proof: Consider a P_fG with at least one isolated $P_f e$ $p = (p_1, p_2)$. Then by Theorem 1, $\{p\}$ is a P_fBS . This is also P_fm_iBS . Thus $\varrho(\wp)=1$.

Theorem 5. If $\wp^*$ is a P_f -star in a P_fG then all $\{e_i\}$ forms P_fm_iBS .

Proof: Let us consider the P_f -star $\wp^*$ containing $e_1, e_2, \dots, e_n$ $P_f e$. Elimination of each $P_f e$ create an isolated $P_f v$. This increases the P_fDN of the P_f -subgraph. Therefore, each $P_f e$ $\{e_i\}$ forms a P_fm_iBS among which one is $\varrho(\wp)$ -set. So, each singleton set of edges $\{e_i\}$ is the P_fm_iBS .

Theorem 6. $|\alpha(\wp^*)| = 2^r - 1$, here r denotes number of $P_f e$ in the P_f -star $\wp^*$ of P_fG $\wp$.

Proof: From Theorem 5 each $P_f e$ forms a P_fBS in the P_f -star $\wp^*$. There are r number of such P_fBS . Now, for two $P_f e$ $x_l, x_m, l \neq m; l, m = 1, 2, \dots, r$ implies greater P_fDN in the P_f -subgraph $\wp - \{x_l, x_m\}$. Therefore, number of P_fBS s having two elements are $\binom{n}{2}$. In such a way the number of P_fBS s having r elements are $\binom{n}{r}$. Thus, total number

of P_fBS s are $n + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = 2^n - 1$.

Example 4. In the P_f -star given in Fig. 3 has 3 P_f es.

Also, it has 7 P_fBS s namely $\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}$. Hence the result.

Theorem 7. If P_fG does not compress any P_fBS then $\zeta(\wp) = |X|$.

Proof: $\wp$ be a P_fG containing no P_fBS . Then for any $A \subseteq X$, $\zeta(\wp - A) \neq \zeta(\wp)$. For this type of P_fG deleting all P_fSE gives a P_f -subgraph with equal P_fDN with $\zeta(\wp)$. Therefore, X forms a P_fNBS with maximum cardinality. Hence, $\zeta(\wp) = |X|$.

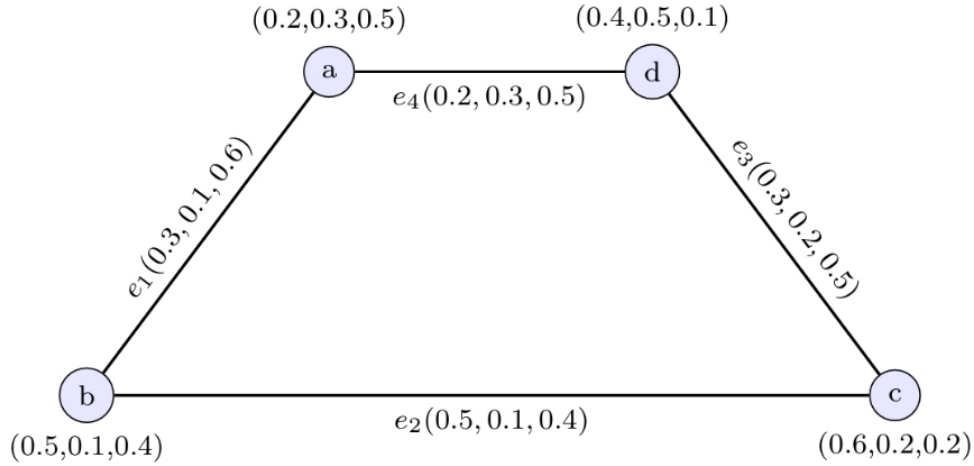


Figure 4: P_fG with no P_fBS

Example 5. From the P_fG shown in Fig. 4, we have,

$CO_{\wp}(a, b) = (0.30, 0.10, 0.50)$, $CO_{\wp}(a, d) = (0.30, 0.30, 0.50)$, $CO_{\wp}(b, c) = (0.50, 0.10, 0.40)$, $CO_{\wp}(c, d) = (0.30, 0.20, 0.50)$, $X = \{e_2, e_3\}$, $\epsilon(\wp) = \{a, c\}$, $\zeta(\wp) = |\{a, c\}| = 2$, $\alpha(\wp) = \phi$, $\beta(\wp) = \{e_2\}, \{e_2, e_3\}$, $\zeta(\wp) = |\{e_2, e_3\}| = 2$, Thus $\zeta(\wp) = |X|$.

Theorem 8. If $\wp$ is a P_fG with P_fBS s and X contains all P_fSE s, then X is a P_fBS .

Proof: Let the P_fG $\wp$ has at least one P_fBS . Then by Theorem 7, X never be a P_fNBS . Therefore, deleting all P_fSE s from the P_fG $\wp$ gives greater $\zeta(\wp)$. This implies X is a P_fBS .

Theorem 9. If a P_fG $\wp$ has a P_fBS then $q(\wp) < \zeta(\wp) + |P|$.

Proof: The P_fG $\wp$ contains P_fBS s. By Theorem 8, X is a P_fBS . Consider N be $\zeta(\wp)$ -set. So, $N \subset X$. Assume $P = X - N$, this indicates $N \cup P$ is a P_fBS . $q(\wp)$ is the

Picture Fuzzy Bondage Set and its Application in Indian Railways

cardinality of $P_f m_m BS$. This gives $\varrho(\wp) < |N \cup P| = \zeta(\wp) + |P|$.

Theorem 10. The union of two $P_f BS$ s is not necessarily a $P_f BS$.

Proof: Let A_1 and A_2 be two $P_f BS$ s in the $P_f G \wp$. Then $A_1 \subseteq X$ and $A_2 \subseteq X$, X being the set of all $P_f SE$. Then $A_1 \cup A_2$ is either equal to X or proper subset of X . If it is X , then by Theorem 8, $A_1 \cup A_2$ is a $P_f BS$. If it is proper subset of X , then it may be $P_f BS$ or $P_f NBS$. From Example 5, $A_1 \cup A_2 = \{(e_1, 0.16, 0.22, 0.50), (e_2, 0.16, 0.12, 0.50)\} = P_f BS$. But $A_8 \cup A_9 = \{(e_2, 0.16, 0.12, 0.50), (e_3, 0.44, 0.12, 0.44), (e_4, 0.36, 0.22, 0.42)\} = P_f NBS$. Hence, the result.

Theorem 11. If C_r , $r = 1, 2, \dots, n$ are $P_f BS$ s in a $P_f G$ then $\bigcup_{r=1}^n C_r$ is a $P_f BS$.

Proof: By Theorem 8, X is a $P_f BS$. All $P_f BS$ s of the $P_f G \wp$ are subsets of X . Thus union of all $P_f BS$ s is X itself. As a result it is always $P_f BS$.

Theorem 12. Let A_1 and A_2 be two $P_f BS$ s then $A_1 \cap A_2$ not necessarily a $P_f BS$.

Proof: We prove this result by considering an example. A_1, A_2 be two $P_f BS$ s, then $A_1 \cap A_2$ is ϕ or $\subseteq X$ or $\not\subseteq X$. From Example 4, $P_f BS$ s are- $A_1 = \{(e_4, 0.30, 0.20, 0.50)\}$, $A_2 = \{(e_5, 0.10, 0.10, 0.50)\}$, $A_3 = \{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40)\}$, $A_4 = \{(e_1, 0.30, 0.10, 0.50), (e_5, 0.10, 0.10, 0.50)\}$, $A_5 = \{(e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50)\}$, $A_6 = \{(e_3, 0.10, 0.10, 0.40), (e_5, 0.10, 0.10, 0.50)\}$, $A_7 = \{(e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$, $A_8 = \{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50)\}$, $A_9 = \{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40), (e_5, 0.10, 0.10, 0.50)\}$, $A_{10} = \{(e_1, 0.30, 0.10, 0.50), (e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$, $A_{11} = \{(e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$, $A_{12} = \{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$. Now $A_1 \cap A_2 = \phi$, $A_2 \cap A_4 = A_2$, a $P_f BS$. $A_5 \cap A_7 = A_5 \cap A_{10} = A_7 \cap A_8 = \{(e_4, 0.30, 0.20, 0.50)\} = A_1$, a $P_f BS$. $A_3 \cap A_4 = A_3 \cap A_{10} = A_4 \cap A_8 = \{(e_1, 0.30, 0.10, 0.50)\}$, a $P_f NBS$. Hence the Theorem.

Theorem 13. For two $P_f BS$ s P and Q , $P + Q$ is not always a $P_f BS$.

Proof: We establish this result by considering an example. Let P and Q be two $P_f BS$ s in the $P_f G \wp$, then the join of these two $P_f BS$ s $P + Q$ contains $P_f e$ other than $P_f e$ in P and Q . So, $P + Q$ may or may not be a $P_f BS$. From Example 4, we consider the $P_f BS$ s: $A_1 = \{(e_4, 0.30, 0.20, 0.50)\}$, $A_2 = \{(e_5, 0.10, 0.10, 0.50)\}$, $A_3 = \{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40)\}$. Then $A_1 + A_2 = \{(e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$. This is a $P_f BS$, shown in Fig. 5 to 7, but $A_1 + A_3 = \{(e_1, 0.30, 0.10, 0.50), (e_2, 0.10, 0.10, 0.60), (e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50), (e_6, 0.20, 0.10, 0.60)\}$. This is not a $P_f BS$, shown in Fig. 8 to 10.

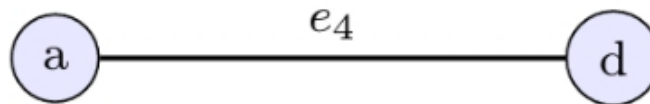


Figure 5: A_1

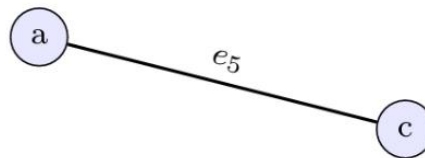


Figure 6: A_2

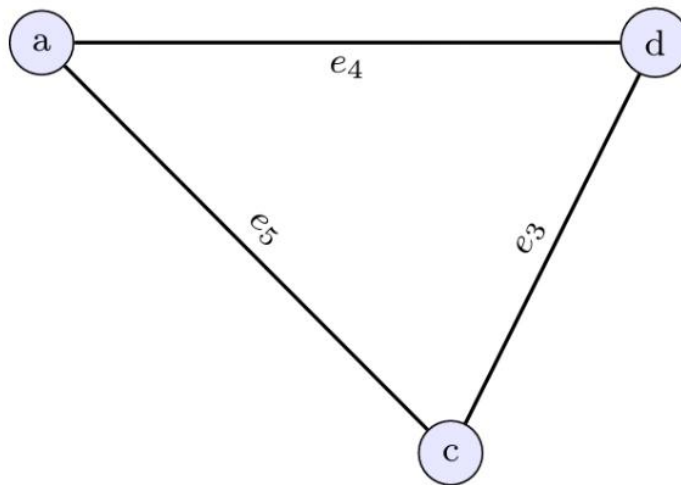


Figure 7: $A_1 + A_2$: Join of two P_fBS s is a P_fBS

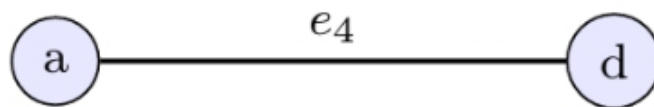


Figure 8: A_1

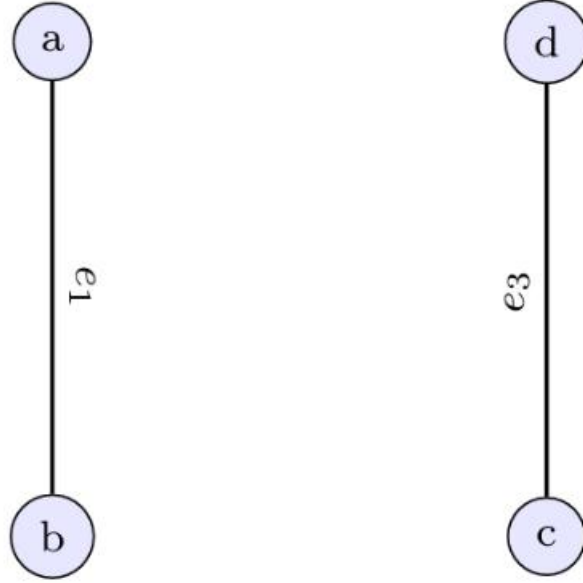


Figure 9: A_3

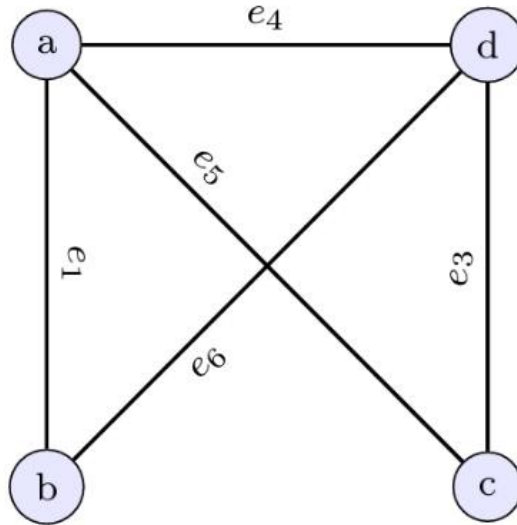


Figure 10: $A_1 + A_3$: Join of two P_fBS s is not a P_fBS

Theorem 14. The union of two P_fNBS is not always P_fNBS .

Proof: We illustrate this theorem by considering an example. In the P_fG given in Fig. 1, we have, $\alpha(\emptyset) = \{(e_4, 0.30, 0.20, 0.50)\}, \{(e_5, 0.10, 0.10, 0.50)\}, \{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40)\}, \{(e_1, 0.30, 0.10, 0.50), (e_5, 0.10, 0.10, 0.50)\}, \{(e_3, 0.10, 0.10,$

0.40), $(e_4, 0.30, 0.20, 0.50)$, $\{(e_3, 0.10, 0.10, 0.40), (e_5, 0.10, 0.10, 0.50)\}$, $\{(e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$, $\{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50)\}$, $\{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40), (e_5, 0.10, 0.10, 0.50)\}$, $\{(e_1, 0.30, 0.10, 0.50), (e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$, $\{(e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$, $\{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40), (e_4, 0.30, 0.20, 0.50), (e_5, 0.10, 0.10, 0.50)\}$. $\beta(\wp) = \{\{(e_1, 0.30, 0.10, 0.50)\}, \{(e_3, 0.10, 0.10, 0.40)\}, \{(e_1, 0.30, 0.10, 0.50), (e_4, 0.30, 0.20, 0.50)\}\}$. Taking P_fNBSs $B_1 = \{(e_1, 0.30, 0.10, 0.50)\}$, $B_2 = \{(e_3, 0.10, 0.10, 0.40)\}$ and $B_3 = \{(e_1, 0.30, 0.10, 0.50), \hat{A}(e_4, 0.30, 0.20, 0.50)\}$, then $B_1 \cup B_2 = \{(e_1, 0.30, 0.10, 0.50), (e_3, 0.10, 0.10, 0.40)\}$ is a P_fBS and $B_1 \cup B_3 = \{(e_1, 0.30, 0.10, 0.50), (e_4, 0.30, 0.20, 0.50)\}$ is a P_fNBS .

3.1. Algorithm

In this subsection we have proposed an algorithm to find P_fSE and another for $\varrho(\wp)$ of a P_fG $\wp$.

Algorithm 1: Computation of P_fSE in a P_fG $\wp$	
Input	A $P_f e$ in the P_fG $\wp$
Output	P_fSE of the P_fG $\wp$
Step-1 :	Choose one $P_f e$ $e=(i, j)$ from the $\wp$
Step-2 :	Find all P_f -paths between the $P_f v$ i and j other than e
Step-3 :	Compute $(\theta_p^+, \theta_p^0, \theta_p^-) = (\wedge \{\theta^+(e_i)\}, \wedge \{\theta^0(e_i)\}, \vee \{\theta^-(e_i)\})$, $i = 1, 2, \dots, k$
Step-4 :	Calculate $CO_{\wp}(i, j) = (\theta^{\infty+}(i, j), \theta^{\infty 0}(i, j), \theta^{\infty-}(i, j))$
Step-5 :	Check $\theta^+(i, j) \geq \theta^{\infty+}(i, j)$, $\theta^0(i, j) \geq \theta^{\infty 0}(i, j)$, $\theta^-(i, j) \leq \theta^{\infty-}(i, j)$. If yes then it is P_fSE .

Illustrations of Algorithm 1 :

We now illustrate this algorithm using the P_fG given in Fig. 11.

Step 1 : Taking a $P_f e$ $e_1 = (M, B)$.

Step 2 : Between the $P_f v$ M and B P_f -paths are $P_1 = e_4 - e_6$, $P_2 = e_4 - e_8 - e_9$, $P_3 = e_4 - e_3 - e_2$, $P_4 = e_4 - e_3 - e_{10} - e_9$, $P_5 = e_4 - e_8 - e_{10} - e_2$, $P_6 = e_5 - e_3 - e_6$, $P_7 = e_5 - e_3 - e_8 - e_9$, $P_8 = e_5 - e_{10} - e_8 - e_6$, $P_9 = e_5 - e_{10} - e_9$, $P_{10} = e_5 - e_2$, $P_{11} = e_7 - e_8 - e_6$, $P_{12} = e_7 - e_8 - e_3 - e_2$, $P_{13} = e_7 - e_{10} - e_2$, $P_{14} = e_7 - e_{10} - e_3 - e_6$, $P_{15} = e_7 - e_9$.

Step 3 : $(\theta_{P_1}^+, \theta_{P_1}^0, \theta_{P_1}^-) = (\wedge \{0.21, 0.24\}, \wedge \{0.26, 0.26\}, \vee \{0.38, 0.38\}) = (0.21, 0.26, 0.38)$.
 $(\theta_{P_2}^+, \theta_{P_2}^0, \theta_{P_2}^-) = (\wedge \{0.21, 0.24, 0.28\}, \wedge \{0.26, 0.26, 0.26\}, \vee \{0.38, 0.38, 0.34\}) = (0.21, 0.26, 0.38)$.
 $(\theta_{P_3}^+, \theta_{P_3}^0, \theta_{P_3}^-) = (\wedge \{0.21, 0.24, 0.42\}, \wedge \{0.26, 0.26, 0.26\}, \vee \{0.38, 0.38, 0.19\}) = (0.21, 0.26, 0.38)$.
 $(\theta_{P_4}^+, \theta_{P_4}^0, \theta_{P_4}^-) = (\wedge \{0.21, 0.24, 0.28, 0.28\}, \wedge \{0.26, 0.26, 0.32, 0.26\}, \vee \{0.38, 0.38, 0.34, 0.34\}) = (0.21, 0.26, 0.38)$.
 $(\theta_{P_5}^+, \theta_{P_5}^0, \theta_{P_5}^-) = (\wedge \{0.21, 0.24, 0.28, 0.42\}, \wedge \{0.26, 0.26, 0.32, 0.26\}, \vee \{0.38, 0.38, 0.34, 0.19\}) = (0.21, 0.26, 0.38)$.
 $(\theta_{P_6}^+, \theta_{P_6}^0, \theta_{P_6}^-) = (\wedge \{0.21, 0.24, 0.24\}, \wedge$

Picture Fuzzy Bondage Set and its Application in Indian Railways

$$\begin{aligned} & \{0.33, 0.26, 0.26\} \vee \{0.22, 0.38, 0.38\} = (0.21, 0.26, 0.38). (\theta_{P_7}^+, \theta_{P_7}^0, \theta_{P_7}^-) = (\wedge \\ & \{0.21, 0.24, 0.24, 0.28\}, \wedge \{0.33, 0.26, 0.26, 0.26\}, \vee \\ & \{0.22, 0.38, 0.38, 0.34\}) = (0.21, 0.26, 0.38). (\theta_{P_8}^+, \theta_{P_8}^0, \theta_{P_8}^-) = (\wedge \{0.21, 0.28, 0.24, 0.24\}, \wedge \\ & \{0.33, 0.32, 0.26, 0.26\}, \vee \{0.22, 0.34, 0.38, 0.38\}) = (0.21, 0.26, 0.38). (\theta_{P_9}^+, \theta_{P_9}^0, \theta_{P_9}^-) = (\wedge \\ & \{0.21, 0.28, 0.28\}, \wedge \{0.33, 0.32, 0.26\}, \vee \{0.22, 0.34, 0.34\}) = (0.21, 0.26, 0.34). (\theta_{P_{10}}^+, \theta_{P_{10}}^0, \\ & \theta_{P_{10}}^-) = (\wedge \{0.21, 0.42\}, \wedge \{0.33, 0.26\}, \vee \{0.22, 0.19\}) = (0.21, 0.26, 0.22). (\theta_{P_{11}}^+, \theta_{P_{11}}^0, \\ & \theta_{P_{11}}^-) = (\wedge \{0.21, 0.24, 0.24\}, \wedge \{0.32, 0.26, 0.26\}, \vee \{0.34, 0.38, 0.38\}) = (0.21, 0.26, 0.38). \\ & (\theta_{P_{12}}^+, \theta_{P_{12}}^0, \theta_{P_{12}}^-) = (\wedge \{0.21, 0.24, 0.24, 0.42\}, \wedge \{0.32, 0.26, 0.26, 0.26\}, \vee \\ & \{0.34, 0.38, 0.38, 0.19\}) = (0.21, 0.26, 0.38). (\theta_{P_{13}}^+, \theta_{P_{13}}^0, \theta_{P_{13}}^-) = (\wedge \{0.21, 0.28, 0.42\}, \wedge \\ & \{0.32, 0.32, 0.26\}, \vee \{0.34, 0.34, 0.19\}) = (0.21, 0.26, 0.34). (\theta_{P_{14}}^+, \theta_{P_{14}}^0, \theta_{P_{14}}^-) = (\wedge \\ & \{0.21, 0.28, 0.24, 0.24\}, \wedge \{0.32, 0.32, 0.26, 0.26\}, \vee \{0.34, 0.34, 0.38, 0.38\}) = \\ & (0.21, 0.26, 0.38). (\theta_{P_{15}}^+, \theta_{P_{15}}^0, \theta_{P_{15}}^-) = (\wedge \{0.21, 0.28\}, \wedge \{0.32, 0.26\}, \vee \\ & \{0.34, 0.34\}) = (0.21, 0.26, 0.34). \end{aligned}$$

Step 4 : $CO_G(M, B) = (\theta^{\infty+}(M, B), \theta^{\infty 0}(M, B), \theta^{\infty-}(M, B))$

$$\begin{aligned} \theta^{\infty+}(M, B) &= \vee \{0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, \\ & 0.21, 0.21\} = 0.21 \\ \theta^{\infty 0}(M, B) &= \vee \{0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, 0.21, \\ & 0.21, 0.26\} = 0.26 \\ \theta^{\infty-}(M, B) &= \wedge \{0.22, 0.22, 0.22, 0.22, 0.22, 0.22, 0.22, 0.22, 0.22, 0.22, 0.22, 0.22, 0.22, 0.22, \\ & 0.22, 0.34\} = 0.22 \end{aligned}$$

Step 5 : $(\theta^{\infty+}(M, B), \theta^{\infty 0}(M, B), \theta^{\infty-}(M, B)) = (0.21, 0.26, 0.22)$ and $(\theta^+(M, B), \theta^0(M, B), \theta^-(M, B)) = (0.21, 0.26, 0.22)$. Therefore, $\theta^{\infty+}(M, B) = \theta^+(M, B)$, $\theta^{\infty 0}(M, B) = \theta^0(M, B)$ and $\theta^{\infty-}(M, B) = \theta^-(M, B)$. So, (M, B) is a P_fSE .

Algorithm 2: Computation of P_fBN	
Input	A P_fG
Output	$\varrho(\wp)$ of the P_fG
Step-1 :	Compute the set of all P_fSE X of the P_fG $\wp$
Step-2 :	Identify all P_fDS s of $\wp$ i.e., $\epsilon(\wp)$
Step-3 :	Calculate $\zeta(\wp)$
Step-4 :	Check $\zeta(\wp - A) > \zeta(\wp)$ for $A \subseteq S$
Step-5 :	Calculate $\gamma(\wp)$
Step 6 :	Compute $\partial(A_i)$ of each $pfm_i b$ -sets A_i
Step 7 :	$\varrho(\wp)$ = cardinality of P_fm_mBS

Illustrations of Algorithm 2 :

Step 1 : $X = \{e_1, e_2, e_4, e_3, e_5, e_6, e_8, e_7, e_9, e_{10}\}$ in Fig. 11.

Step 2 : $\epsilon(\wp) = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$

Step 3 : $\zeta(\wp) = |v_4| = 1$

Step 4 : $\zeta(\emptyset - \{e_3\}) > \zeta(G)$, so, $\hat{A} \{e_3\}$ is a P_fBS .

Step 5 : $\gamma(\emptyset) = \{\{e_3\}, \{e_4\}, \{e_6\}, \{e_8\}\}$

Step 6 : $\partial(\{e_3\}) = \frac{1+0.24+0.26-0.38}{3} = 0.373$, $\partial(\{e_4\}) = \frac{1+0.21+0.26-0.38}{3} = 0.363$
 $\partial(\{e_6\}) = \frac{1+0.24+0.26-0.38}{3} = 0.373$, $\partial(\{e_8\}) = \frac{1+0.24+0.26-0.38}{3} = 0.373$

Step 7 : $\varrho(G) = |\{e_4\}| = 1$

4. An application

Indian Railway is a very essential medium for communication across India. Large number of people use this transport on their daily life work. Therefore, maintaining a good service facility is a very essential part of this system. Here, we consider 5 railway divisions M, S, B, N, P . Our main aim is to find the route between two divisions in which administration should focus to improve and maintain good services.

We have constructed a P_fG model given in Fig. 11 by taking divisions as P_fv and P_fe as railway connections between the divisions. Membership values of the P_fv represent the average amount of travelers who have entered into the division. This indicates positive membership of the P_fv . People who wait represent neutral membership and people who leave the place represent negative membership of the division. For example let during 8 a.m. to 9 a.m. x_1 number of people enters into Division M from outside the division for going to another station, x_2 is the number of people, coming by train from other stations. x_3 denotes people waiting before 8 a.m. and x_4 represents people leaving the division by transferring to another train, x_5 is the number of people exiting the station and x_6 people wait during this hour. Let, $x = x_1 + x_2 + x_3 + x_4 + x_5$. Here, $x_4 + x_5 \leq x_1 + x_2 + x_3$ and $x_6 = (x_1 + x_2 + x_3) - (x_4 + x_5)$. Therefore, $\theta^+(A) = \frac{x_1+x_2}{x}$, $\theta^0(A) = \frac{x_6}{x}$ and $\theta^-(A) = \frac{x_4+x_5}{x}$. Let 28% passengers enter, 32% wait and 34% leave from P division per mean hour. Therefore, (0.28, 0.32, 0.34) is the membership value of P_fv P . Also, degrees for P_fe (i, j) represent the average number of trains which appear late in division i and reach to j in the right time, run in scheduled time and arrive at time but reach late to the next division. For example, in M and S divisions we assume 42%, 26% and 19% represent respectively trains that appear late in M and reach right time in S , maintains scheduled time and arrive late in S while depart in right time at M . Thus (0.42, 0.26, 0.19) is the pf-membership value of the P_fe between M and S . Here, $pfDS$ represents divisions which connects the remaining divisions with strong rail routes. $pfBS$ computes the robustness of the rail network.

In this P_fG , $pfSE$ between two railway divisions indicates that this route is best among other possible routes i.e., time duration is minimum between the divisions. In our model it is a complete P_fG . So, all routes are $pfSE$. P_fDS s indicates the important divisions from which passengers can go to all the remaining destinations. In this model each division is represented as a P_fDS . Now the value $\partial(A_i)$ of each divisions A_i indicates crowd density. Number of P_fv in the P_fm_mDS denotes P_fDN . In our model P_fDN is 1. It has been seen that if route damage or not functional or problem in railway track, signal system or any type of accident occurs then number of P_fv in the P_fDS as a result P_fDN increases. Therefore, pressure on P_fm_iDS increases and passengers suffer from their basic needs. These types of railway routes represent $pfBS$. There are so many

Picture Fuzzy Bondage Set and its Application in Indian Railways

$pfBS$ s in the system. In this P_fG, P_fm_mBS contains only one pfe with a score 0.33. We have ranked them in Table 1. Route BN leads from the list. Thus among all routes first focus on the route BN .

Rank	Division	Score
1	BN	0.33
2	SN	0.37
3	MN	0.37
4	NP	0.37

Table 1: Ranking of railway divisions

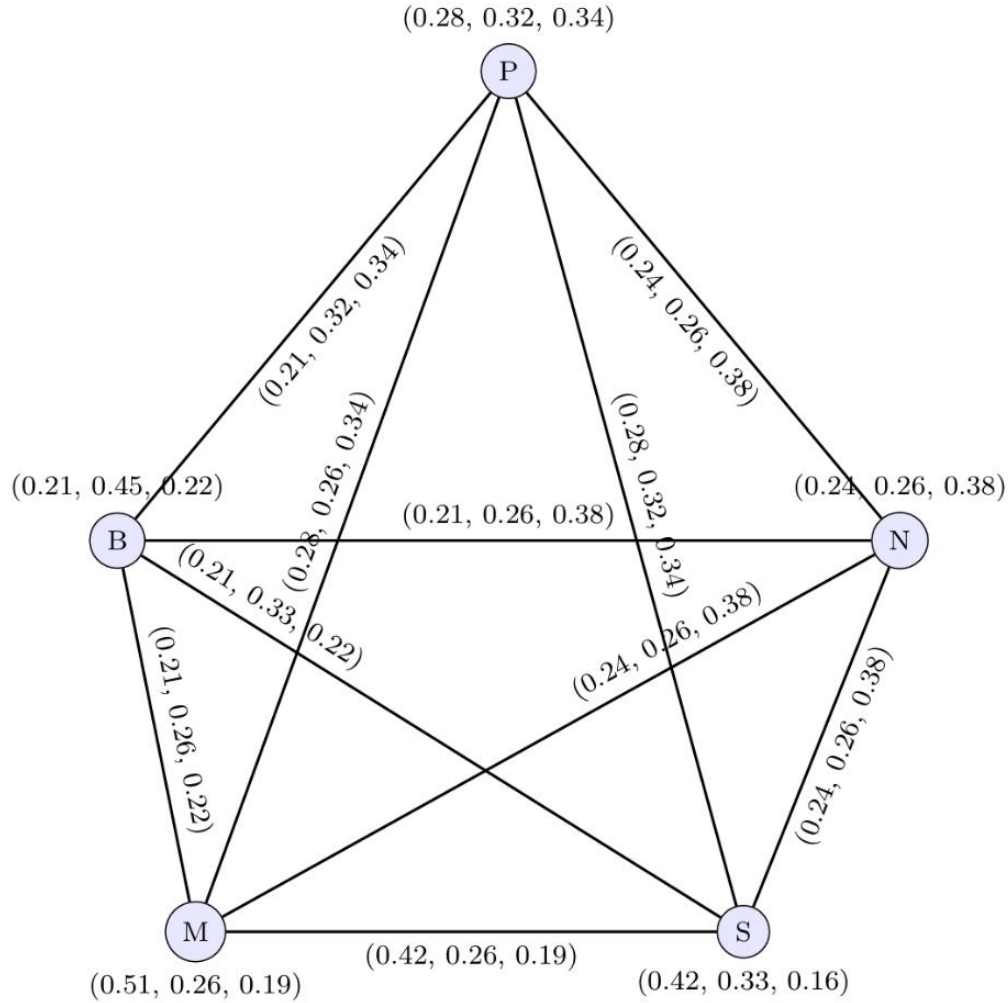


Figure 11: Graphical representation of railway divisions

4.1. Numerical calculations

We consider two P_f v B and M from Fig. 11. All P_f -paths with their strength between B and M are $P_1=e_4-e_6$ (0.21, 0.26, 0.38), $P_2=e_4-e_8-e_9$ (0.21, 0.26, 0.38), $P_3=e_4-e_3-e_2$ (0.21, 0.26, 0.38), $P_4=e_4-e_3-e_{10}-e_9$ (0.21, 0.26, 0.38), $P_5=e_4-e_8-e_{10}-e_2$ (0.21, 0.26, 0.38), $P_6=e_5-e_3-e_6$ (0.21, 0.26, 0.38), $P_7=e_5-e_3-e_8-e_9$ (0.21, 0.26, 0.38), $P_8=e_5-e_{10}-e_8-e_6$ (0.21, 0.26, 0.38), $P_9=e_5-e_{10}-e_9$ (0.21, 0.26, 0.34), $P_{10}=e_5-e_2$ (0.21, 0.26, 0.22), $P_{11}=e_7-e_8-e_6$ (0.21, 0.26, 0.38), $P_{12}=e_7-e_8-e_3-e_2$ (0.21, 0.26, 0.38), $P_{13}=e_7-e_{10}-e_2$ (0.21, 0.26, 0.34), $P_{14}=e_7-e_{10}-e_3-e_6$ (0.21, 0.26, 0.38), $P_{15}=e_7-e_9$ (0.21, 0.26, 0.34). Here, $\theta^+(M, B) \geq \theta^{\infty+}(M, B)$, $\theta^0(M, B) \geq \theta^{\infty 0}(M, B)$ and $\theta^-(M, B) \leq \theta^{\infty-}(M, B)$. So, $e_1=(M, B)$ is a P_f SE. Here, $X = \{ MB, BN, BP, SB, MN, MP, MS, NP, SN, SP \}$. $\epsilon(\wp) = \{\{M\}, \{S\}, \{B\}, \{N\}, \{P\}\}$.

$$\begin{aligned} \partial(M) &= \frac{1+0.51+0.26-0.19}{3} = 0.53, \quad \partial(S) = \frac{1+0.42+0.33-0.16}{3} = 0.53, \quad \partial(B) = \frac{1+0.21+0.45-0.22}{3} = 0.48, \\ \partial(N) &= \frac{1+0.24+0.26-0.38}{3} = 0.37, \quad \partial(P) = \frac{1+0.28+0.32-0.34}{3} = 0.42 \end{aligned}$$

So, $\zeta(G) = |N| = 1$. From the P_f -subgraph $\wp - e_4$, $\zeta(\wp - e_4) = 2 > \zeta(\wp)$. Therefore, $\{e_4\} \in \alpha(\wp)$. Thus $\gamma(\wp) = \{\{e_3\}, \{e_4\}, \{e_6\}, \{e_8\}\}$. Here, $\partial(SN) = \frac{1+0.24+0.26-0.38}{3} = 0.37$, $\partial(BN) = \frac{1+0.21+0.26-0.38}{3} = 0.33$, $\partial(MN) = \frac{1+0.24+0.26-0.38}{3} = 0.37$, $\partial(NP) = \frac{1+0.24+0.26-0.38}{3} = 0.37$. $\partial(BN)$ is minimum. So, $\varrho(\wp) = 1$.

Table 1 indicates the ranking of the pfb -edges. Therefore, $P_f e$ BN is our outcome. This route needs special attention to ensure smooth rail network operation.

4.2. Advantages and limitations of the application

The application proposed in Section 4 has the following advantages and limitations.

Advantages

- Identification of the most important links of the network.
- More accurate analysis for PFG model.

Limitations

- Collecting data with limited time.

5. Comparative study

All papers regarding BS s, NBS s on fuzzy graphs and I_fG s contain data in the form of fuzzy and I_f type. They fail to describe data in P_f type. Therefore, to upgrade the data, it is important to study BSS and $NBSS$ in P_fG . Somasundram et al. and Nagoorgani et al. discussed various domination and bondage parameters in fuzzy graphs. Parvathi et al. and Zehui et al. used the IfG model, in which vertices and edges have two degrees: membership and non-membership. But this model is not appropriate for picture fuzzy information. In this paper, the neutrality degree constructs the problems in a more accurate way. Thus, this activity is the extent of the work on BS , NBS in fuzzy graphs and I_fG . Table 2 presents the comparison.

Picture Fuzzy Bondage Set and its Application in Indian Railways

	Include membership value	Include non-membership value	Include neutral membership value
Bondage in fuzzy graph	√	×	×
Bondage in $I_f G$	√	√	×
Bondage in $P_f G$	√	√	√

Table 2: Comparison in fuzzy graph, $I_f G$, $P_f G$

6. Conclusion and future work

To understand and analyse the behaviour of everyday problems, PfG is an important tool. Therefore, it is an important topic to study. We analysed two important characteristics, PfBS and PfNBS. Computation of $P_f BN$ and $P_f NBN$ for various $P_f G$ s is presented here. In addition, we investigated an application to find a vulnerable edge in a railway network. This work will collaborate with researchers to develop $P_f G$ s. Our future work will focus on finding bounds for $\varrho(\varphi)$ and $\varsigma(\varphi)$, and exploring the proposed characteristics on the Pf-tree, Pf soft graph, IvPfG, tolerance PfG, threshold PfG, and mpIvPfG.

Acknowledgements. The authors sincerely thank the referee for valuable suggestions and constructive comments, which significantly improved the manuscript's quality, clarity, and overall presentation.

Conflict of Interest. The authors declare that there is no conflict of interest regarding the publication of this manuscript.

Authors' Contribution. All authors jointly contributed to the conceptualisation and design of the study. They participated in conducting the research, analyzing and interpreting the results, and preparing and revising the manuscript. All authors have read and approved the final version of the manuscript.

REFERENCES

1. L.A. Zadeh, Fuzzy sets, *Information and Control* 8 (1965) 338-353.
2. A. Rosenfield, Fuzzy sets and their applications, in: *Fuzzy Graphs*, Academic Press, 1975, pp. 77-95. <https://doi.org/10.1016/B978-0-12-775260-0.50008-6>.
3. K.T. Atanassov and S. Stoeva, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* 20 (1986) 87-96.
4. B.C. Cuong and V. Kreinovich, Picture fuzzy sets, *Journal of Computer Science and Cybernetics* 30 (2014) 409-420.
5. J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, The bondage number of a graph, *Discrete Mathematics* 86 (1990) 47-57.
6. B.L. Hartnell and D.F. Rall, Bounds on the bondage number of a graph, *Discrete Mathematics* 128 (1994) 173-177.
7. A.N. Gani, K.P. Devi and M. Akram, Bondage and non-bondage number of a fuzzy graph, *International Journal of Pure and Applied Mathematics* 103 (2015) 215-226.

8. Z. Shao, S. Kosari, H. Rashmanlou and M. Shoaib, New concepts in intuitionistic fuzzy graph with application in water supplier systems, *Mathematics* 8 (2020) 1241.
9. C. Zuo, A. Pal and A. Dey, New concepts of picture fuzzy graphs with application, *Mathematics* 7 (2019) 470.
10. S. Amanathulla, G. Muhiuddin, D. Al-Kadi and M. Pal, Multiple attribute decision-making problem using picture fuzzy graph, *Mathematical Problems in Engineering* (2021) 9937828. <https://doi.org/10.1155/2021/9937828>.
11. S. Amanathulla, B. Bera and M. Pal, Balanced picture fuzzy graph with application, *Artificial Intelligence Review* 54 (2021) 5255-5281.
12. S. Amanathulla and M. Pal, An introduction to picture fuzzy graph and its application to select best routes in an airlines network, in: *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, IGI Global, 2022, pp. 385-411. <https://doi.org/10.4018/978-1-7998-7979-4.ch018>.
13. W. Xiao, A. Dey and L.H. Son, A study on regular picture fuzzy graph with application in communication networks, *Journal of Intelligent and Fuzzy Systems* 39 (2020) 3633-3645.
14. S. Das, G. Ghorai and Q. Xin, Picture fuzzy threshold graphs with application in medicine replenishment, *Entropy* 24 (2022) 658.
15. S. Das, G. Ghorai and M. Pal, Picture fuzzy tolerance graphs with application, *Complex and Intelligent Systems* 8 (2022) 541-554.
16. S. Amanathulla, B. Bera and M. Pal, Real world applications of discrete mathematics, *Malaya Journal of Matematik* 9 (2021) 152-158.
17. A. Somasundaram and S. Somasundaram, Domination in fuzzy graphs-I, *Pattern Recognition Letters* 19 (1998) 787-791.
18. R. Parvathi and G. Thamizhendhi, Domination in intuitionistic fuzzy graphs, *Notes on Intuitionistic Fuzzy Sets* 16 (2010) 39-49.
19. M.G. Karunambigai, S. Sivasankar and K. Palanivel, Different types of domination in intuitionistic fuzzy graph, *Annals of Pure and Applied Mathematics* 14 (2017) 87-101.
20. A. Mohamedlsmayil and N. AshaBosely, Domination in picture fuzzy graphs, *American International Journal of Research in Science, Technology, Engineering and Mathematics* (2019) 205-210.
21. A. Banerjee and S. Amanathulla, Optimization of disaster management using split domination in picture fuzzy graphs, *Journal of Applied Mathematics and Computing* (2024) 1-25. <https://doi.org/10.1007/s12190-023-01965-6>.
22. A. Banerjee and S. Amanathulla, Picture fuzzy graphs with dominating energy: A new perspective for solving complex problem, *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences* (2026) 1-27.
23. A. Banerjee, J. Khatun, B. Bera, S. Amanathulla and M. Pal, A comparative analysis of domination parameters in picture fuzzy graphs for real-world application, *Journal of Uncertain Systems* (2026) 2650009.
24. B.D. Adhikari, A. Banerjee, S. Amanathulla and S. Mondal, Certain operations on interval-valued picture fuzzy graphs with application, *International Journal of Mathematics for Industry* (2023). <https://doi.org/10.1142/S2661335223500089>.

Picture Fuzzy Bondage Set and its Application in Indian Railways

25. B. Bera, S. Amanathulla and S.K. Mahato, A comprehensive study of picture fuzzy planar graphs with real-world applications, *Journal of Uncertain Systems* (2023). <https://doi.org/10.1142/S1752890932500095>.
26. B. Bera, A. Banerjee, J. Khatun, S. Amanathulla and M. Pal, Some properties of regular picture fuzzy graph with application in social network analysis, *Journal of Applied Mathematics and Computing* (2025) 1-24.
27. J. Khatun and S. Amanathulla, An application of neutrosophic graph in decision-making problem for alliances of companies, in: *Fuzzy Optimization, Decision-Making and Operations Research: Theory and Applications*, Springer International Publishing, 2023, pp. 241-255.
28. T. Mahapatra, G. Ghorai and M. Pal, Competition graphs under interval-valued m-polar fuzzy environment and its application, *Computational and Applied Mathematics* 41 (2022) 285-297.
29. T. Mahapatra and M. Pal, An investigation on m-polar fuzzy threshold graph and its application on resource power controlling system, *Journal of Ambient Intelligence and Humanized Computing* 13 (2022) 501-514.
30. H. Rashmanlou, G. Muhiuddin, S. Amanathulla, F. Mofidnakhaei and M. Pal, A study on cubic graphs with novel application, *Journal of Intelligent and Fuzzy Systems* 40 (2021) 89-101.