

## On the Diophantine Equation $31^x + n^y = z^3$

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**Abstract.** In this research, the solutions of the Diophantine equation  $31^x + n^y = z^3$  are investigated where  $x, y, z$  are non-negative integers and  $n$  is a positive integer such that  $n \equiv 5, 6, 25, 26 \pmod{31}$ . We show that if  $n \equiv 5, 6, 25 \pmod{31}$ , then the equation has no solutions. For  $n \equiv 26 \pmod{31}$ , the solution of the equation is  $(x, y, z) = (0, 1, (n+1)^{1/3})$ , where  $(n+1)^{1/3}$  is a positive integer.

**Keywords:** Diophantine equation; Integer solution; Congruence; Mihăilescu's Theorem

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### 1. Introduction

The exponential Diophantine equations of the form  $a^x + b^y = z^m$ , where  $a, b, m$  are fixed positive integers and  $x, y, z$  are non-negative integers, have received a lot of attention in recent decades. The Diophantine equations of this type have been studied for  $m = 2$ . Some of these can be seen in [1], [2], [3] and [4]. Meanwhile, the case of  $m = 3$  has also received attention. For example, in 2020, Burshtein [5] studied the Diophantine equation  $p^x + q^y = z^3$ , where  $p, q$  are distinct prime numbers and  $x, y, z$  are positive integers such that  $x, y \in \{1, 2\}$ . After that, in 2022, Mina and Bacani [6] found some sufficient conditions for the non-existence of integer solutions of the Diophantine equation  $p^x + (p+4)^y = z^3$ , where  $p$  and  $p+4$  are cousin prime numbers with  $p \equiv 1 \pmod{3}$ . Later, in 2023, Tadee [7] discovered all non-negative integer solutions of the Diophantine equation  $8^x + p^y = z^3$ , where  $p$  is a prime number. In 2024, Tadee [8] presented some conditions for the non-existence of non-negative integer solutions of the Diophantine equation  $3^x + n^y = z^3$ , where  $n$  is a positive integer. In the same year, Khanom et al. [9] solved the Diophantine equation  $3^x + a^y = z^3$ , where  $a$  is a positive integer.

Recently, in 2025, Tadee [10] showed that if  $a$  is a positive integer such that  $a \equiv 3, 9, 10 \pmod{13}$ , then the Diophantine equation  $13^x + a^y = z^3$  has no non-negative integer solutions. In 2026, Yiamras and Tadee [11] presented all non-negative integer solutions  $(x, y, z)$  of the Diophantine equation  $7^x + n^y = z^3$ , where  $n$  is a positive integer

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with  $n \equiv 2, 3, 4, 5 \pmod{7}$ . In the same year, Kaewsod and Tadee [12] proved that the Diophantine equation  $p^x + n^{3y} = z^3$ , when  $n$  is a positive integer and  $p \notin \{3, 7\}$  is a prime number with  $n \equiv 1 \pmod{p}$ , has no non-negative integer solutions. Moreover, Tadee [13] also showed that if  $n$  is a positive integer such that  $n \equiv 2 \pmod{91}$ , then the Diophantine equation  $13^x + n^y = z^3$  has no non-negative integer solutions.

In this paper, we investigate the Diophantine equation  $31^x + n^y = z^3$ , where  $x, y, z$  are non-negative integers and  $n$  is a positive integer with  $n \equiv 5, 6, 25, 26 \pmod{31}$ .

### 2. Preliminaries

In this section, we introduce certain a definition and theorems that will be applied throughout this paper.

**Definition 2.1.** [14] Let  $n > 1$  and  $\gcd(a, n) = 1$ . The order of  $a$  modulo  $n$  (in older terminology: the exponent to which  $a$  belongs modulo  $n$ ) is the smallest positive integer  $k$  such that  $a^k \equiv 1 \pmod{n}$ .

**Theorem 2.1.**[14] Let the integer  $a$  have order  $k$  modulo  $n$ . Then  $a^h \equiv 1 \pmod{n}$  if and only if  $k \mid h$ ; in particular,  $k \mid \phi(n)$ .

**Theorem 2.2.** [6] Let  $p$  be a prime number and  $a$  an integer such that  $\gcd(a, p) = 1$ . Then, the congruence  $x^3 \equiv a \pmod{p}$  has a solution if and only if  $a^{(p-1)/d} \equiv 1 \pmod{p}$ , where  $d = \gcd(3, p-1)$ . If it has a solution, then there are exactly  $d$  solutions modulo  $p$ .

**Theorem 2.3.** [13] Let  $p \neq 3$  be a prime number and let  $n$  be a positive integer such that  $\gcd(n, p) = 1$ . If the Diophantine equation  $p^x + n^y = z^3$  has a non-negative integer solution  $(x, y, z)$  and  $y \equiv 0 \pmod{3}$ , then  $1 + 3 \cdot n^{y/3} + 3 \cdot n^{2y/3} = p^x$ .

**Theorem 2.4.** (Catalan's conjecture or Mihăilescu's Theorem) [15] The Diophantine equation  $a^x - b^y = 1$ , where  $a, b, x$  and  $y$  are integers such that  $\min\{a, b, x, y\} > 1$ , has the unique integer solution  $(a, b, x, y) = (3, 2, 2, 3)$ .

By Theorem 2.4, we have the following corollary.

**Corollary 2.5.** [9] Let  $a$  be a positive integer. The Diophantine equation  $1 + a^y = z^3$  has at most one integer solution for fixed  $a$ ; namely,  $(y, z, a) = (1, \sqrt[3]{a+1}, a)$ .

**Lemma 2.6.** Let  $n$  be a positive integer and let  $h$  be a non-negative integer. Then:

(1) If  $n \equiv 5, 25 \pmod{31}$ , then  $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 7, 29 \pmod{31}$ .

(2) If  $n \equiv 6, 26 \pmod{31}$ , then  $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 1, 3, 7, 29, 30 \pmod{31}$ .

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**Proof:**

(1) Consider  $n \equiv 5 \pmod{31}$ . Notice that 3 is the order of 5 modulo 31. By the Division algorithm, there exist two integers  $q$  and  $r$  such that  $h = 3q + r$ , when  $r \in \{0, 1, 2\}$ . Now, we consider the following cases:

**Case 1.1.**  $r = 0$ . Then  $h = 3q$ . It implies that

$$1 + 3 \cdot 5^h + 3 \cdot 5^{2h} \equiv 1 + 3 \cdot 5^{3q} + 3 \cdot 5^{6q} \equiv 1 + 3 \cdot 1^q + 3 \cdot 1^{2q} \equiv 7 \pmod{31}.$$

**Case 1.2.**  $r = 1$ . Then  $h = 3q + 1$ . It implies that

$$1 + 3 \cdot 5^h + 3 \cdot 5^{2h} \equiv 1 + 3 \cdot 5^{3q+1} + 3 \cdot 5^{6q+2} \equiv 1 + 3 \cdot 5 \cdot 1^q + 3 \cdot 5^2 \cdot 1^{2q} \equiv 29 \pmod{31}.$$

**Case 1.3.**  $r = 2$ . Then  $h = 3q + 2$ . It implies that

$$1 + 3 \cdot 5^h + 3 \cdot 5^{2h} \equiv 1 + 3 \cdot 5^{3q+2} + 3 \cdot 5^{6q+4} \equiv 1 + 3 \cdot 5^2 \cdot 1^q + 3 \cdot 5 \cdot 1^{2q+1} \equiv 29 \pmod{31}.$$

From the above cases, we have  $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 7, 29 \pmod{31}$ . In the same way, we also prove the case  $n \equiv 25 \pmod{31}$ .

(2) Consider  $n \equiv 6 \pmod{31}$ . Notice that 6 is the order of 6 modulo 31. By the Division algorithm, there exist two integers  $q$  and  $r$  such that  $h = 6q + r$ , when  $r \in \{0, 1, 2, 3, 4, 5\}$ . Now, we consider the following cases:

**Case 2.1.**  $r = 0$ . Then  $h = 6q$ . It implies that

$$1 + 3 \cdot 6^h + 3 \cdot 6^{2h} \equiv 1 + 3 \cdot 6^{6q} + 3 \cdot 6^{12q} \equiv 1 + 3 \cdot 1^q + 3 \cdot 1^{2q} \equiv 7 \pmod{31}.$$

**Case 2.2.**  $r = 1$ . Then  $h = 6q + 1$ . It implies that

$$1 + 3 \cdot 6^h + 3 \cdot 6^{2h} \equiv 1 + 3 \cdot 6^{6q+1} + 3 \cdot 6^{12q+2} \equiv 1 + 3 \cdot 6 \cdot 1^q + 3 \cdot 6^2 \cdot 1^{2q} \equiv 3 \pmod{31}.$$

**Case 2.3.**  $r = 2$ . Then  $h = 6q + 2$ . It implies that

$$1 + 3 \cdot 6^h + 3 \cdot 6^{2h} \equiv 1 + 3 \cdot 6^{6q+2} + 3 \cdot 6^{12q+4} \equiv 1 + 3 \cdot 6^2 \cdot 1^q + 3 \cdot 6^4 \cdot 1^{2q} \equiv 29 \pmod{31}.$$

**Case 2.4.**  $r = 3$ . Then  $h = 6q + 3$ . It implies that

$$1 + 3 \cdot 6^h + 3 \cdot 6^{2h} \equiv 1 + 3 \cdot 6^{6q+3} + 3 \cdot 6^{12q+6} \equiv 1 + 3 \cdot 6^3 \cdot 1^q + 3 \cdot 1^{2q+1} \equiv 1 \pmod{31}.$$

**Case 2.5.**  $r = 4$ . Then  $h = 6q + 4$ . It implies that

$$1 + 3 \cdot 6^h + 3 \cdot 6^{2h} \equiv 1 + 3 \cdot 6^{6q+4} + 3 \cdot 6^{12q+8} \equiv 1 + 3 \cdot 6^4 \cdot 1^q + 3 \cdot 6^2 \cdot 1^{2q+1} \equiv 29 \pmod{31}.$$

**Case 2.6.**  $r = 5$ . Then  $h = 6q + 5$ . It implies that

$$1 + 3 \cdot 6^h + 3 \cdot 6^{2h} \equiv 1 + 3 \cdot 6^{6q+5} + 3 \cdot 6^{12q+10} \equiv 1 + 3 \cdot 6^5 \cdot 1^q + 3 \cdot 6^4 \cdot 1^{2q+1} \equiv 30 \pmod{31}.$$

Hence, we conclude that  $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 1, 3, 7, 29, 30 \pmod{31}$ . In the same way, we also prove the case  $n \equiv 26 \pmod{31}$ .

### 3. Main results

In this section, we will consider the Diophantine equation  $31^x + n^y = z^3$  for the cases  $n \equiv 5 \pmod{31}$ ,  $n \equiv 6 \pmod{31}$ ,  $n \equiv 25 \pmod{31}$  and  $n \equiv 26 \pmod{31}$ , respectively.

**Theorem 3.1.** Let  $n$  be a positive integer such that  $n \equiv 5 \pmod{31}$ . Then the Diophantine equation  $31^x + n^y = z^3$  has no solutions in non-negative integers  $x, y$  and  $z$ .

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**Proof:** Suppose, to get a contradiction, that there exist non-negative integers  $x, y$  and  $z$  such that  $31^x + n^y = z^3$ . If  $x = 0$ , then the equation becomes  $1 + n^y = z^3$ . By Corollary 2.5, we get  $z^3 = n + 1$ . Since  $n \equiv 5 \pmod{31}$ , it follows that  $z^3 \equiv 6 \pmod{31}$ . By Theorem 2.2, we obtain  $6^{10} \equiv 1 \pmod{31}$ . Since 6 is the order of 6 modulo 31, by Theorem 2.1, we have  $6 \nmid 10$ . This is impossible. Hence  $x > 0$ . Then  $31^x + n^y \equiv 5^y \pmod{31}$  and so  $z^3 \equiv 5^y \pmod{31}$ . Since  $\gcd(3, 30) = 3$ , by Theorem 2.2, we get  $5^{10y} \equiv 1 \pmod{31}$ . Notice that 3 is the order of 5 modulo 31. By Theorem 2.1, it implies that  $3 \mid 10y$ . Therefore, we have  $3 \mid y$ . So, there exists a non-negative integer  $h$  such that  $y = 3h$ . By Theorem 2.3, we obtain  $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 31^x$ . It follows that  $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 0 \pmod{31}$ . This contradicts Lemma 2.6(1).

**Example 3.1.** The Diophantine equation  $31^x + 5^y = z^3$  has no solutions in non-negative integers  $x, y$  and  $z$ .

**Theorem 3.2.** Let  $n$  be a positive integer such that  $n \equiv 6 \pmod{31}$ . Then the Diophantine equation  $31^x + n^y = z^3$  has no solutions in non-negative integers  $x, y$  and  $z$ .

**Proof:** Suppose, to get a contradiction, that there exist non-negative integers  $x, y$  and  $z$  such that  $31^x + n^y = z^3$ . If  $x = 0$ , then the equation becomes  $1 + n^y = z^3$ . By Corollary 2.5, we get  $z^3 = n + 1$ . Since  $n \equiv 6 \pmod{31}$ , it follows that  $z^3 \equiv 7 \pmod{31}$ . By Theorem 2.2, we obtain  $7^{10} \equiv 1 \pmod{31}$ . Since 15 is the order of 7 modulo 31, by Theorem 2.1, we have  $15 \nmid 10$ . This is impossible. Hence  $x > 0$ . Then  $31^x + n^y \equiv 6^y \pmod{31}$  and so  $z^3 \equiv 6^y \pmod{31}$ . Since  $\gcd(3, 30) = 3$ , by Theorem 2.2, we get  $6^{10y} \equiv 1 \pmod{31}$ . Notice that 6 is the order of 6 modulo 31. By Theorem 2.1, this implies that  $6 \mid 10y$ . Therefore, we have  $3 \mid y$ . So, there exists a non-negative integer  $h$  such that  $y = 3h$ . By Theorem 2.3, we obtain  $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 31^x$ . It follows that  $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 0 \pmod{31}$ . This contradicts Lemma 2.6(2).

**Example 3.2.** The Diophantine equation  $31^x + 6^y = z^3$  has no solutions in non-negative integers  $x, y$  and  $z$ .

**Theorem 3.3.** Let  $n$  be a positive integer with  $n \equiv 25 \pmod{31}$ . Then the Diophantine equation  $31^x + n^y = z^3$  has no solutions in non-negative integers  $x, y$  and  $z$ .

**Proof:** Suppose, to get a contradiction, that there exist non-negative integers  $x, y$  and  $z$  such that  $31^x + n^y = z^3$ . If  $x = 0$ , then the equation becomes  $1 + n^y = z^3$ . By Corollary 2.5, we obtain  $z^3 = n + 1$ . Since  $n \equiv 25 \pmod{31}$ , it follows that  $z^3 \equiv 26 \pmod{31}$ . By Theorem 2.2, we have  $26^{10} \equiv 1 \pmod{31}$ . Since 6 is the order of 26 modulo 31, by Theorem 2.1,

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it implies that  $6 \mid 10$ , which is impossible. Hence  $x > 0$ . Therefore, we get  $31^x + n^y \equiv 25^y \pmod{31}$  and so  $z^3 \equiv 25^y \pmod{31}$ . Since  $\gcd(3, 30) = 3$ , by Theorem 2.2, we obtain  $25^{10y} \equiv 1 \pmod{31}$ . Notice that 3 is the order of 25 modulo 31. This implies, by Theorem 2.1, that  $3 \mid 10y$ . Therefore, we have  $3 \mid y$ . So, there exists a non-negative integer  $h$  such that  $y = 3h$ . By using Theorem 2.3, we obtain  $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 31^x$ . Consequently,  $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 0 \pmod{31}$ . This contradicts Lemma 2.6(1).

**Example 3.3.** The Diophantine equation  $31^x + 25^y = z^3$  has no solutions in non-negative integers  $x, y$  and  $z$ .

**Theorem 3.4.** Let  $n$  be a positive integer with  $n \equiv 26 \pmod{31}$ . Then the non-negative integer solution of the Diophantine equation  $31^x + n^y = z^3$  is  $(x, y, z) = (0, 1, \sqrt[3]{n+1})$ , where  $\sqrt[3]{n+1}$  is a positive integer.

**Proof:** Assume that  $x > 0$ . Take the Diophantine equation  $31^x + n^y = z^3$  modulo 31 to get  $z^3 \equiv 26^y \pmod{31}$ . Since  $\gcd(3, 30) = 3$ , by Theorem 2.2, we have  $26^{10y} \equiv 1 \pmod{31}$ . Notice that 6 is the order of 26 modulo 31. This implies, by Theorem 2.1, that  $6 \mid 10y$ . Therefore, we have  $3 \mid y$ . So, there exists a non-negative integer  $h$  such that  $y = 3h$ . By Theorem 2.3, we get  $1 + 3 \cdot n^h + 3 \cdot n^{2h} = 31^x$ . It follows that  $1 + 3 \cdot n^h + 3 \cdot n^{2h} \equiv 0 \pmod{31}$ . This contradicts Lemma 2.6(2). Thus, we get  $x = 0$ . The equation becomes  $1 + n^y = z^3$ . By Corollary 2.5, it implies that  $(x, y, z) = (0, 1, \sqrt[3]{n+1})$ , where  $\sqrt[3]{n+1}$  is a positive integer.

**Example 3.4.** The Diophantine equation  $31^x + 26^y = z^3$  has the unique non-negative integer solution  $(x, y, z) = (0, 1, 3)$ .

**Example 3.5.** The Diophantine equation  $31^x + 57^y = z^3$  has no solutions in non-negative integers  $x, y$  and  $z$ .

### 3. Conclusion

In this article, we consider the Diophantine equation  $31^x + n^y = z^3$ , where  $x, y, z$  are non-negative integers and  $n$  is a positive integer such that  $n \equiv 5, 6, 25, 26 \pmod{31}$ . Using a combination of modular arithmetic and Mihăilescu's Theorem, we demonstrate that if  $n \equiv 5, 6, 25 \pmod{31}$ , then the equation is unsolvable. For the case  $n \equiv 26 \pmod{31}$ , the non-negative integer solution  $(x, y, z)$  of the equation is  $(0, 1, \sqrt[3]{n+1})$ , where  $\sqrt[3]{n+1}$  is a positive integer.

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