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Short Communication

A Generalization of the Erdős-Szekeres Theorem

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Abstract. Let S be an infinite set. Suppose B is a binary operator which can be applied to any two elements in S. When we evaluate x B y, where x and y are in S, we have k + 1 (where $k \geq 2$) exclusive results of $x R_0 y$, $x R_1 y$, ..., $x R_k y$, where $x R_0 y$ denotes $x = y$. It is shown that for any sequence W consisting of $r_1 r_2 \ldots r_k + 1$ distinct elements in S, where $r_1, r_2, \ldots, r_k$ are positive integers, there is an integer s with $1 \leq s \leq k$ such that W has a subsequence of $w_1, w_2, \ldots, w_{r\_s + 1}$ satisfying $w_1 R_s w_2$, $w_2 R_s w_3$, ..., $w_{r\_s} R_s w_{r\_s + 1}$. The above result generalizes the well-known Erdős-Szekeres theorem in [1] (A combinatorial problem in geometry, Compositio Mathematica 2 (1935) 463-470.)

Keywords: Erdős-Szekeres Theorem, Generalization

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1. Introduction

In 1935, Erdős and Szekeres [1] proved the following theorem.

Theorem 1.1. Every sequence of $r_1 r_2 + 1$ distinct real numbers contains a monotonically increasing subsequence of length $r_1 + 1$ or a monotonically decreasing subsequence of length $r_2 + 1$.

In 1959, Seidenberg [2] gave a simple proof of the Erdős-Szekeres theorem. Motivated by Seidenberg's proofs in [2], we attempt to generalize the Erdős-Szekeres theorem in this note. To state our result, we need the following definitions or notations. Let S be an infinite set. Suppose B is a binary operator which can be applied to any two elements in S. The set S is called a k-result set based on B if the evaluations of x B y, where x and y are any two elements of S, produce k + 1 (where $k \geq 2$) exclusive results of $x R_0 y$, $x R_1 y$, ..., $x R_k y$, where $x R_0 y$ denotes $x = y$. For instance, the set of real numbers equipped with the binary comparison operator between any two real numbers is a 2-result set. Two elements x and y in a k-result set based on a binary operator are called distinct if xR_0y does not hold. A sequence $W = w_1 w_2 \ldots w_p$ is an R_q sequence of a k-result set S if $p = 1$ or $p \geq 2$ and $w_1 R_q w_2$, $w_2 R_q w_3$, ..., $w_{p-1} R_q w_p$, where q is an integer with $1 \leq q \leq k$. The main result of this note is as follows.

Theorem 1.2. Let S be a k-result ($k \geq 2$) set based a binary operator B. Then for any sequence W consisting of $r_1 r_2 \ldots r_k + 1$ distinct elements in S, where $r_1, r_2, \ldots, r_k$ are positive integers, there is an integer s with $1 \leq s \leq k$ such that W has a subsequence of $x_1, x_2, \ldots, x_{r\_s + 1}$ satisfying $x_1 \, Rs \, x_2, x_2 \, Rs \, x_3, \ldots, x_{r\_s} \, Rs \, x_{r\_s + 1}$.

Remark. Let S and B be, respectively, the set of real numbers and the binary comparison operator between two real numbers. Then S is a 2-result set with R_0, R_1, and R_2, respectively, the equality relation, denoted " = ", between two real numbers, the less than relation, denoted " < ", between two real numbers, and the greater than relation, denoted " > "', between two real numbers. Then we can use Theorem 1.2 to derive Theorem 1.1. Thus, Theorem 1.2 is a generalization of Theorem 1.1.

2. Proofs

Proof of Theorem 1.2. Let $W := w_1 w_2 \ldots w_h$, where $h := r_1 r_2 \ldots r_k + 1$, be any sequence consisting of h distinct elements in S. For each w_i, where $1 \leq i \leq h$, we define a k-tuple of $(z(w_i, R_1), z(w_i, R_2), \ldots, z(w_i, R_k))$, where $z(w_i, R_j)$ is the length of the longest R_j subsequence of W starting at w_i, where j is any integer with $1 \leq j \leq k$. Suppose Theorem 1.2 is not true. Then $z(w_i, R_j) \leq r_j$, for each i with $1 \leq i \leq h$ and each j with $1 \leq j \leq k$. Thus, the total number of k-tuples of $((z(w_i, R_1), z(w_i, R_2), \ldots, z(w_i, R_k))$ is less than or equal to $r_1 r_2 \ldots r_k$. Notice that there are $h := r_1 r_2 \ldots r_k + 1$ distinct elements in W, and for each of them we define a k-tuple. By the Pigeonhole principle, we have that there exist two elements w_s and w_t in W such that $(z(w_s, R_1), z(w_s, R_2), \ldots, z(w_s, R_k)) = (z(w_t, R_1), z(w_t, R_2), \ldots, z(w_t, R_k))$ which implies that $z(w_s, R_1) = z(w_t, R_1)$, $z(w_s, R_2) = z(w_t, R_2)$, ..., $z(w_s, R_k) = z(w_t, R_k)$. If w_s appears before w_t in W, since S is a k-result set based on a binary operator B and the elements in W are distinct, there is an integer u such that $w_s \, R_u \, w_t$, where $1 \leq u \leq k$. Prefixing w_s to the longest R_u subsequence of W starting at w_t, we obtain an R_u subsequence of W starting at w_s, denoted $W_1(w_s)$. Thus, we have that $z(w_s, R_u) \geq$ the number of elements in $W_1(w_s) = z(w_t, R_u) + 1$, a contradiction. If w_t appears before w_s in W, since S is a k-result set based on a binary operator B and the elements in W are distinct, there is an integer v such that $w_t \, R_v \, w_s$, where $1 \leq v \leq k$. Prefixing w_t to the longest R_v subsequence of W starting at w_s, we obtain an R_v subsequence of W starting at w_t, denoted $W_2(w_t)$. Thus, we have that $z(w_t, R_v) \geq$ the number of elements in $W_2(w_t) = z(w_s, R_v) + 1$, a contradiction again. This completes the proof of Theorem 1.2.

3. Conclusions

In this note, we present a generalization of the well-known Erdős-Szekeres theorem. We will look for applications of our result in the future.

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Author's Contribution: The author is solely responsible for the conceptualization, methodology, analysis, and preparation of the manuscript.

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