Annals of Pure and Applied Mathematics
Vol. 33, No. 1, 2026, 21-31
ISSN: 2279-087X (P), 2279-0888(online)
Published on 20 March 2026
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v33n1a04935

Annals of
Pure and Applied
Mathematics

Weighted Residual Method for the Numerical Solution of Some Class of Boundary Value Problems Using Euler Wavelets

L. M. Angadi

Department of Mathematics
Shri Siddeshwar Government First Grade College & P. G. Studies Centre,
Nargund – 582207, India
Email: angadi.lm@gmail.com

Received 2 February 2026; accepted 18 March 2026

Abstract. Boundary value problems (BVPs) occur frequently in the fields of engineering and science, such as gas dynamics, nuclear physics, atomic structures and chemical reactions. In most cases, these problems do not always find the exact solutions via analytical methods. In this paper, a weighted residual method for the numerical solution of BVPs using Euler wavelets (WRMEW) is proposed. Here, Euler wavelets are used as weight functions that assume basis elements which allow us to obtain the numerical solution of the BVPs. Obtained numerical solutions using this method are compared with existing methods and exact solutions. Some BVPs are taken to demonstrate the validity and applicability of the proposed method.

Keywords: Weighted residual method; Euler wavelets; Function approximation; Boundary value problems.

AMS Mathematics Subject Classification (2010): 65T60, 97N40, 30E25

1. Introduction

It is vital to note that most of the differential equations arising from the modelling of physical phenomena do not always have known analytical solutions. Thus, the need for the development of numerical approaches to find approximate solutions becomes essential [1].

Recently, some of the numerical methods have been used for the numerical solutions of differential equations. For example, Hermite wavelet-based Galerkin method [2], Laguerre wavelet-Galerkin Method [3], the Galerkin method using Gegenbauer wavelets [4] etc.

The subject of wavelets has received much attention because of the comprehensive mathematical power and the good application potential of wavelets in many interesting physical problems. Wavelet functions have generated significant interest from both theoretical and applied research over the last few years. The name wavelet comes from the requirement that they should integrate to zero, waving above and below the *x*-axis. However, wavelet analysis is a numerical concept which allows one to represent a function in terms of a set of base functions, called wavelets, which are localized both in location and scale [5].

As we have noted earlier, spectral bases are infinitely differentiable but have global support. On the other hand, base functions used in finite-element methods have small compact support but poor continuity properties. Already we know that spectral methods have good spectral localization but poor spatial localization, while finite element methods have good spatial localization, but poor spectral localization. Wavelet bases execute to combine the advantages of both spectral and finite element bases. We can look forward to numerical methods based on wavelet bases capable of attaining good spatial and spectral resolutions. The representation of a smooth function in terms of a series expansion using orthogonal polynomials is a fundamental idea in approximation theory and forms the basis of spectral methods of solution of differential equations with functional arguments. An approach to studying differential equations is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element type methods. The weighted residual methods represent a broader set of approaches that encompasses Galerkin's technique. The Galerkin method is considered the most widely used in applied mathematics because of its implementation and simplicity [6 - 7].

In this paper, I developed a weighted residual method using Euler wavelets for the numerical solution for BVPs. This method is based on expanding the solution by Euler wavelets with unknown coefficients. The properties of Euler wavelets, together with the weighted residual method i.e. the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution of the BVPs is obtained.

The organization of the paper is as follows. Euler wavelets and function approximation is given in section 2. Section 3 deals with a weighted residual method using Euler wavelets (WRMEW) for the solution of BVPs. Numerical Experiment is given in section 4. Finally, the conclusions of the proposed work are discussed in section 5.

2. Euler wavelets and Function approximation

Euler wavelets are defined [8 – 9] as,

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} E_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}} \\ 0 \qquad , & otherwise \end{cases} \tag{2.1}$$

With

$$E_m(x) = \begin{cases} 1 \qquad , & m = 0 \\ \dfrac{1}{\sqrt{\dfrac{2(-1)^{m-1}(m!)^2}{(2m)!} E_{2m+1}(0)}} E_m(x) \, , & m > 0 \end{cases} \tag{2.2}$$

Here, $n = 1,2,......2^{k-1}$, $m = 0, 1, 2, 3, \ldots.., M-1$ & $k > 0$

For normality the coefficient is $\dfrac{1}{\sqrt{\dfrac{2(-1)^{m-1}(m!)^2}{(2m)!}E_{2m+1}(0)}}$,

$a = 2^{-(k-1)}$ is the dilation parameter and $b = (n-1)2^{-(k-1)}$ is the translation parameter.

The following generating functions can be used to define the well-known Euler polynomials $E_m(x)$ of order m

$$\frac{2e^{xs}}{e^s + 1} = \sum_{m=0}^{\infty} E_m(x)\frac{s^m}{m!} \quad (|s| < \pi) \tag{2.3}$$

In perticular, the rational numbers $E_m = 2^m E_m\left(\frac{1}{2}\right)$ are the familiar Euler numbers. Additionally, the following relation can be used to create the first kind Euler polynomials for $m = 0, 1, 2, 3, \ldots.., N$

$$\sum_{m=0}^{N}\binom{m}{k}E_k(x) + E_m(x) = 2t^m \ \& \ \binom{m}{k} \text{ is a binomial coefficient.}$$

The first few fundamental polynomials are represented explicitly by

$$E_0(x) = 1,\ E_1(x) = x - \frac{1}{2},\ E_2(x) = x^2 - x,\ E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4},\ldots\ldots..$$

The following formula satisfies these polynomials.

$$\int_0^1 E_m(x)E_n(x)dx = (-1)^{n-1}\frac{m!\,(n+1)!}{(m+n+1)!}E_{m+n+1}(0), \quad m, n \geq 1 \tag{2.4}$$

Over the interval $[0, 1)$, Euler polynomials form a full basis.

If $x = 0$, the Euler polynomials are $E_0(0) = 1,\ E_1(0) = -\frac{1}{2},\ E_3(0) = \frac{1}{4}, \ldots\ldots.$

For $k = 1$ & $M = 3$ in (2.1) and (2.2), then the Euler wavelets are given by

$\psi_{1,0}(x) = 1,\quad \psi_{1,1}(x) = 4x - 1,\quad \psi_{1,2}(x) = \sqrt{6}\left(4x^2 - 2x\right),$

$\psi_{1,3}(x) = \dfrac{4\sqrt{5\sqrt{17}}}{17}\left(8x^3 - 6x^2 + \dfrac{1}{4}\right)$ and so on.

Function approximation:

Suppose $y(x) \in L^2[0, 1)$ is expressions of Euler wavelets as:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x) \tag{2.5}$$

Truncating the above infinite series, we get

$$y(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) \tag{2.6}$$

Convergence of Euler wavelets

Theorem 1.1. If a continuous function $y(x) \in L^2(R)$ defined on $[0, 1)$ be bounded, i.e. $y(x) \le K$, then the Euler wavelets expansion of $y(x)$ converges uniformly to it [8].

3. Method of solution

Consider the boundary value of the problem is of the form,

$$y'' + \alpha y' + \beta y = f(x) \tag{3.1}$$

With boundary conditions $y(0) = a, \quad y(1) = b$ (3.2)

where α, β are constants and $f(x)$ be a continuous function.

Write the Eq. (3.1) as

$$R(x) = y'' + \alpha y' + \beta y - f(x) \tag{3.3}$$

where $R(x)$ is the residual of the Eq. (3.1). When $R(x) = 0$ for the exact solution and $y(x)$ will satisfy the boundary conditions.

Consider the trail series solution of the Eq. (3.1), $y(x)$ defined over [0, 1) can be expanded as a modified Euler wavelet satisfying the given boundary conditions which involve unknown parameters as follows:

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \tag{3.4}$$

where $c_{n,m}$'s are unknown coefficients to be determined.

Accuracy in the solution is increased by choosing higher degree Euler wavelet polynomials.

Differentiating Eq. (3.4) twice with respect to x and substitute the values of in y , y' , y'' Eq. (4.3).

To find $c_{n,m}$'s we choose weight functions as assumed bases elements and integrate on boundary values together with the residual to zero [10].

i.e. $$\int_0^1 \psi_{1,m}(x)\, R(x)\, dx = 0, \quad m = 0, 1, 2, \ldots\ldots$$

then obtained a system of linear algebraic equations and on solving this system, get the unknown parameters. Then substitute these unknowns in the trail solution i.e. Eq. (3.4), we obtained numerical solution of Eq. (3.1).

In order to know the accuracy of WRMEW for the test problems, we use the maximum absolute error as an error metric. Here are the formulas for calculating the (i) maximum absolute error, (ii) L_2 - Norm, (iii) L_∞ - Norm.

(i) Maximum absolute error $= E_{\max} = \max \left| y(x)_e - y(x)_n \right|$,

where $y(x)_e$ and $y(x)_n$ are exact and numerical solution

(ii) L_2 - Norm $= \left\| \left(\sum_{m=1}^{n} E_m^2 \right)^{1/2} \right\|$

(iii) L_∞ - Norm $= \left\| Max(E_m) \right\|, \quad m = 1, 2, \ldots\ldots, 9$

4. Numerical Implementation

Problem 4.1. First, consider the boundary value problem [11],

$$y'' - y = x - 1, \quad 0 \le x \le 1 \tag{4.1}$$

With boundary conditions: $y(0) = 0, \quad y(1) = 0$ (4.2)

The implementation of the Eq. (4.1) as per the method explained in section 3 is as follows:

The residual of Eq. (4.1) can be written as

$$R(x) = y'' - y - (x - 1) \tag{4.3}$$

Now, choosing the weight function $w(x) = x(1 - x)$ for Euler wavelet bases to satisfy the given boundary condition (Eq. (4.2)), i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1 - x) = x(1 - x)$$

$$\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1 - x) = (4x - 1)\, x(1 - x)$$

$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1 - x) = \sqrt{6}\left(4x^2 - 2x\right) x(1 - x)$$

Assuming the trail solution of Eq. (4.1) for $k = 1$ and $M = 3$ is given by

$$y(x) = c_{1,0}\, \psi_{1,0}(x) + c_{1,1}\, \psi_{1,1}(x) + c_{1,2}\, \psi_{1,2}(x) \tag{4.4}$$

Then the Eq. (4.4) becomes

$$y(x) = c_{1,0}\{x(1 - x)\} + c_{1,1}\{(4x - 1)\, x(1 - x)\} + c_{1,2}\left\{\sqrt{6}\left(4x^2 - 2x\right) x(1 - x)\right\}$$

$$\Rightarrow y(x) = c_{1,0}\left(x - x^2\right) + c_{1,1}\left(5x^2 - 4x^3 - x\right) + c_{1,2}\sqrt{6}\left(6x^3 - 4x^4 - 2x^2\right) \tag{4.5}$$

Differentiating Eq. (4.5) twice w.r.t. x i.e.

$$y'(x) = c_{1,0}(1 - 2x) + c_{1,1}\left(10x - 12x^2 - 1\right) + c_{1,2}\sqrt{6}\left(18x^2 - 16x^3 - 4x\right)$$

$$\& \ y''(x) = c_{1,0}(-2) + c_{1,1}(10 - 24x) + c_{1,2}\sqrt{6}\left(36x - 48x^2 - 4\right)$$

Substitute the values of y , y'' in Eq. (4.3), to get the residual of Eq. (4.1). The "weight functions" are the same as the basis functions.

Then by the weighted Galerkin method, consider the following:

$$\int_0^1 \psi_{1,j}(x)\,R(x)\,dx = 0, \quad j = 0, 1, 2 \tag{4.6}$$

For $j = 0, 1, 2$ in Eq. (4.6),

i.e.
$$\left.\begin{aligned}\int_0^1 \psi_{1,0}(x)\,R(x)\,dx &= 0\\ \int_0^1 \psi_{1,1}(x)\,R(x)\,dx &= 0\\ \int_0^1 \psi_{1,2}(x)\,R(x)\,dx &= 0\end{aligned}\right\} \tag{4.7}$$

From Eq. (4.7), we have system of algebraic equations with unknown coefficients i.e. $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. Solving this by Gauss elimination method, we obtain the values for $c_{1,0} = 0.2692$, $c_{1,1} = -0.0429$ and $c_{1,2} = 0.0019$. Substituting these values in Eq. (4.5), we get the numerical solution. The comparison of numerical solution and the absolute errors are presented in table 1 and numerical solution with exact solution of Eq. (4.1) is $y(x) = -\left(\frac{1}{1-e^2}\right)e^x + \left(\frac{e^2}{1-e^2}\right)e^{-x} - x + 1$ in figure 1.

Table 1: Comparison of numerical solution and absolute error with exact solution of the problem 4.1.

x	Numerical solution			Exact solution	Absolute error		
	Ref[11]	**Ref[12]**	**WRMEW**		**Ref[11]**	**Ref[12]**	**WRMEW**
0.1	0.0276352	0.0264712	0.0264776	0.0265183	1.12e-03	4.70e-05	4.10e-05
0.2	0.0453501	0.0443444	0.0442661	0.0442945	1.06e-03	5.00e-05	2.80e-05
0.3	0.0545619	0.0546184	0.0544956	0.0545074	5.50e-05	1.10e-04	1.20e-05
0.4	0.0566876	0.0583436	0.0582517	0.0582599	1.57e-03	8.40e-05	8.20e-06
0.5	0.0531447	0.0565875	0.0565928	0.0565906	3.45e-03	3.10e-06	2.20e-06
0.6	0.0453501	0.0504023	0.0504617	0.0504834	5.13e-03	8.10e-05	2.20e-05
0.7	0.0347212	0.0407912	0.0408631	0.0408782	6.16e-03	8.70e-05	1.50e-05
0.8	0.0226751	0.0286751	0.0286861	0.0286795	6.00e-03	4.40e-06	6.60e-06
0.9	0.0106289	0.0148592	0.0147926	0.0147663	4.14e-03	9.30e-05	2.60e-05

Problem 4.2. Next, consider the boundary value problem [12],

$$y'' - y' = -1, \quad 0 \le x \le 1 \tag{4.8}$$

With boundary conditions: $y(0) = 0, \quad y(1) = 0$ (4.9)

As explained in section 3 and in the previous problem, we obtain the values of $c_{1,0} = 0.4544$, $c_{1,1} = 0.0356$ and $c_{1,2} = 0.0041$. Substituting these values in Eq. (4.5), we get the numerical solution. The comparison of numerical solution and the absolute errors are presented in table 2 and numerical solution with exact solution of Eq. (4.8) is $y(x) = x - \left(\frac{e^x - 1}{e - 1}\right)$ in figure 2.

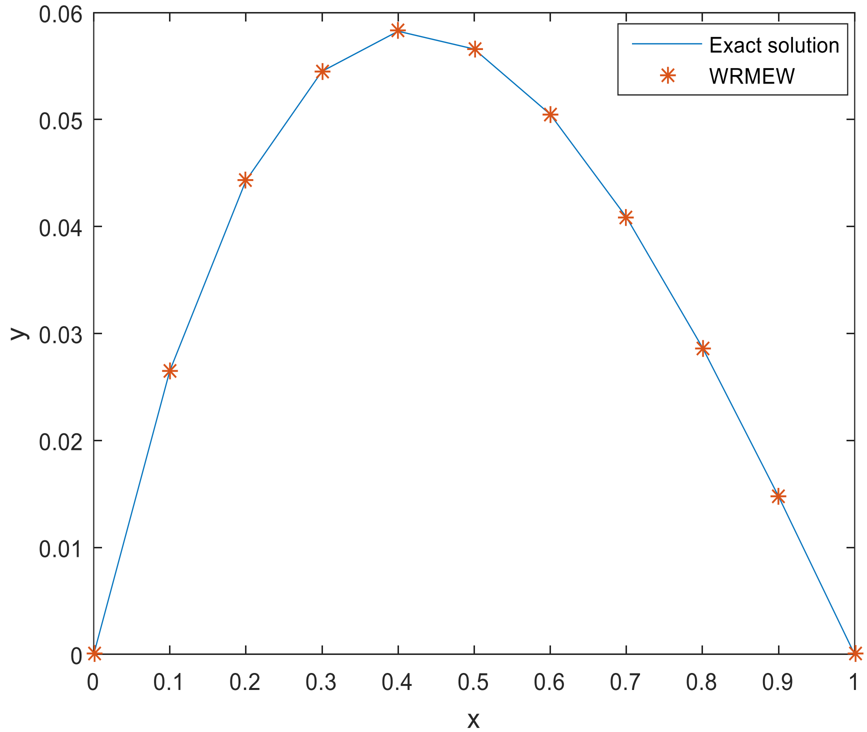

Fig. 1. Comparison of numerical solution with exact solution of the problem 4.1.

Table 2: Comparison of numerical solution and absolute error with exact solution of the problem 4.2.

x	Numerical solution			Exact solution	Absolute error		
	FDM	Ref[12]	WRMEW		FDM	Ref[12]	WRMEW
0.1	0.310289	0.038684	0.038829	0.038793	1.27e-03	1.09e-04	3.60e-05
0.2	0.590204	0.071009	0.071179	0.071149	1.43e-03	5.00e-05	3.00e-05
0.3	0.812347	0.096232	0.096413	0.096390	3.33e-03	1.58e-04	2.30e-05
0.4	0.954971	0.113656	0.113797	0.113769	3.92e-03	1.13e-04	2.80e-05
0.5	1.004126	0.122420	0.122490	0.122459	4.13e-03	3.90e-05	3.10e-05
0.6	0.954971	0.121367	0.121596	0.121546	3.92e-03	1.79e-04	5.00e-05
0.7	0.812347	0.109825	0.110062	0.110020	3.33e-03	1.95e-04	4.20e-05
0.8	0.590204	0.086853	0.086778	0.086764	2.42e-03	8.90e-05	1.40e-05
0.9	0.310289	0.050414	0.050528	0.050545	1.27e-03	1.31e-04	1.70e-05

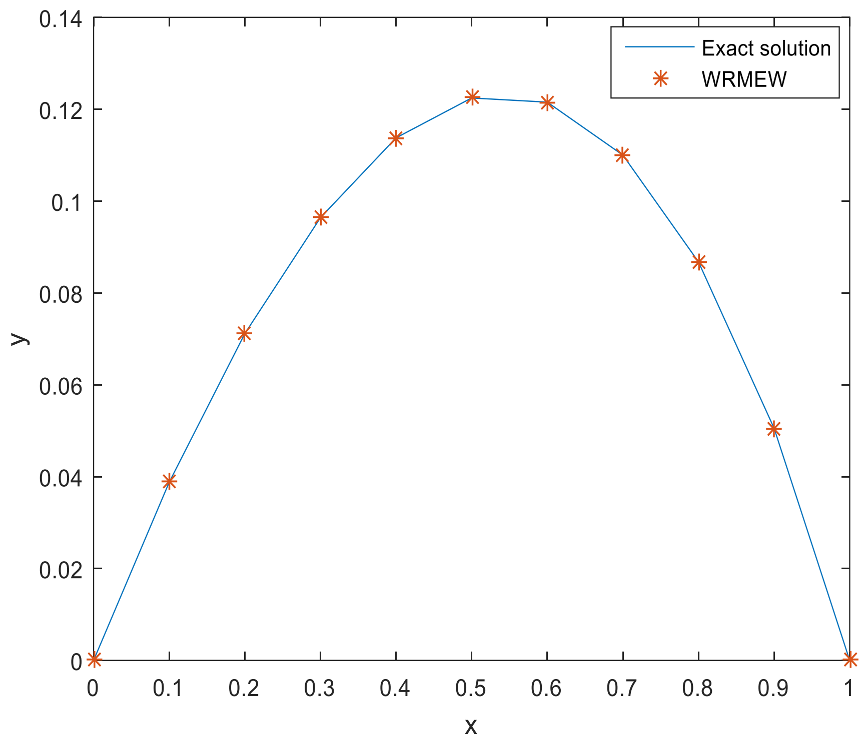

Fig. 2. Comparison of numerical solution with exact solution of the problem 4.2.

Problem 4.3. Next, consider another SBVP of the form [13]

$$\left(1-\frac{x}{2}\right)\frac{d^2y}{dx^2}+\frac{3}{2}\left(\frac{1}{x}-1\right)\frac{dy}{dx}+\left(\frac{x}{2}-1\right)y= 5-\frac{29}{2}x+\frac{13}{2}x^2+\frac{3}{2}x^3-\frac{1}{2}x^4,\quad 0\le x\le 1 \tag{4.10}$$

With boundary conditions: $y(0)=0,\ y(1)=0$ (4.11)

In section 3 and in the preceding problem, the values of $c_{1,0}=0.2506$, $c_{1,1}=0.2497$ and $c_{1,2}=0.0001$ are derived. By substituting these values into Eq. (4.5), we arrive at the numerical solution. The comparison between the numerical solution and the absolute errors is outlined in table 3, figure 3 shows the contrast between numerical solution with the exact solution of Eq. (4.10) is $y(x)=x^2-x^3$.

Table 3: Comparison of numerical solution and absolute error in relation to the exact solution for problem 4.3.

x	Numerical solution WRMEW	Exact solution	Absolute error
0.1	0.0090667	0.009000	6.70e-05
0.2	0.0320962	0.032000	9.60e-05

0.3	0.0631011	0.063000	1.00e-04
0.4	0.0960914	0.096000	9.10e-05
0.5	0.1250750	0.125000	7.50e-05
0.6	0.1440573	0.144000	5.70e-05
0.7	0.1470414	0.147000	4.10e-05
0.8	0.128028	0.128000	2.80e-05
0.9	0.0810155	0.081000	1.60e-05

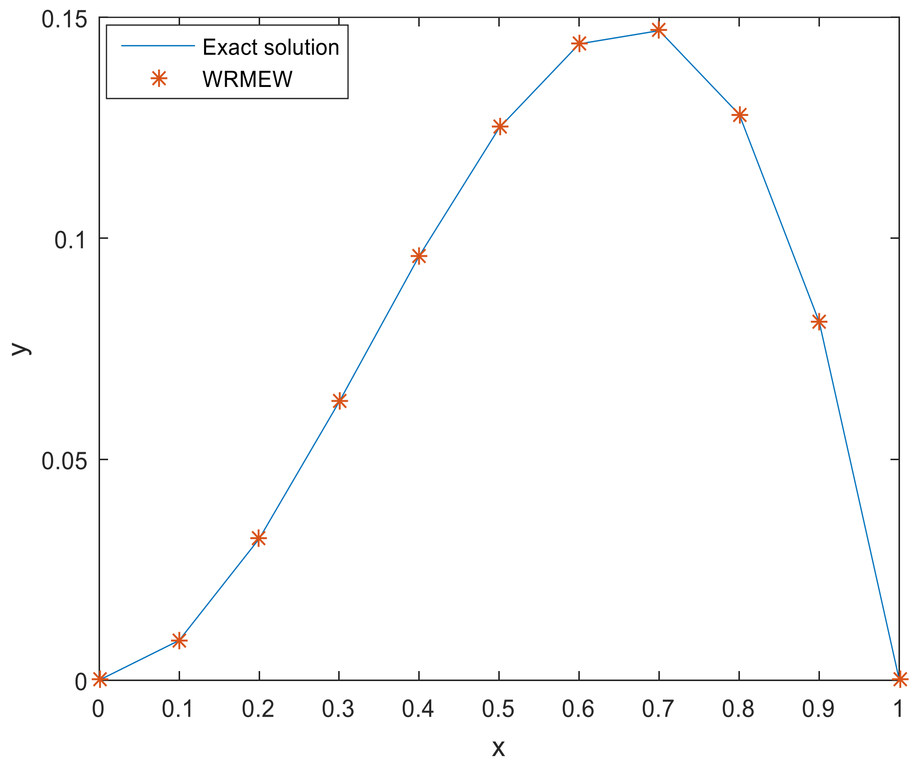

Figure 3: Comparison of numerical solution with exact solution of the problem 4.3.

Table 4: Comparison for error norms L_2, L_∞ to compare with exact solutions for problem 4.1 and 4.2

Method	L_2 **norm**	L_∞ **norm**
	Problem 4.1	
Ref [11]	1.16e-02	6.20e-03
Ref [12]	2.16e-04	1.10e-04
WRMEW	6.41e-05	4.10e-05
	Problem 4.2	
FDM	9.00e-03	4.10e-03
Ref [12]	3.86e-04	1.95e-04
WRMEW	9.60e-05	5. 00e-05

5. Conclusion

In this paper, proposed the WRMEW for the numerical solution of boundary value problems. From the above tables and figures, it is observed that:

- The numerical solutions obtained by the proposed method are better than the existing methods (FDM, Ref [11] & Ref [12]) and nearer to the exact solution.
- The absolute errors, L_2 and L_∞ norms of this method are very less as compared with the existing methods (FDM, Ref [11] & Ref [12]).

Hence, the WRMEW is very effective for solving boundary value problems.

Acknowledgement. The authors express their gratitude to the anonymous reviewers for their insightful suggestions and comments, which significantly enhanced the clarity and presentation of the work.

Conflicts of Interest. The author declares that there are no conflicts of interest related to the publication of this work.

Author's Contribution. The author is solely responsible for the conceptualization, methodology, analysis, and preparation of the manuscript.

REFERENCES

1. Md. Kamruzzaman and Md. Mehedi Hasan, A new numerical approach for solving initial value problems of ordinary differential equations, *Annals of Pure and Applied Mathematics*, 17(2) (2018) 157–162.
2. L.M. Angadi, Numerical solution of singular boundary value problems by Hermite wavelet based Galerkin method, *Annals of Pure and Applied Mathematics*, 23(2) (2021) 101–110.
3. S.C. Shiralashetti, L.M. Angadi and S. Kumbinarasaiah, Laguerre wavelet-Galerkin method for the numerical solution of one dimensional partial differential equations, *International Journal of Mathematics and its Applications*, 6(1-E) (2018) 939–949.
4. L.M. Angadi, Galerkin method for the numerical solution of one-dimensional differential equations using Gegenbauer wavelets, *Journal of Physical Sciences*, 29 (2024) 13–22.
5. L.M. Angadi, Numerical solution of singular boundary value problems by Galerkin's method utilizing Taylor wavelets, *Annals of Pure and Applied Mathematics*, 31(2) (2025) 147–155.
6. K. Amaratunga and J.R. Williams, Wavelet-Galerkin solutions for one dimensional partial differential equations, *International Journal of Numerical Methods in Engineering*, 37 (1994) 2703–2716.
7. J.W. Mosevic, Identifying differential equations by Galerkin's method, *Mathematics of Computation*, 31 (1977) 139–147.
8. Y. Wang and L. Zhu, Solving nonlinear Volterra integrodifferential equations of fractional order by using Euler wavelet method, *Advances in Difference Equations*, 27 (2017) 1–16.
9. S.C. Shiralashetti and S.I. Hanaji, Euler wavelet based numerical scheme for the solutions of parabolic partial differential equations, *Malaya Journal of Matematik*, 1 (2020) 173–176.

10. J.E. Cicelia, Solution of weighted residual problems by using Galerkin's method, *Indian Journal of Science and Technology*, 7(3) (2014) 52–54.
11. S. Kostadinova, J.V. Buralieva and K.H. Saneva, Wavelet-Galerkin solution of some ordinary differential equations, *Proceedings of the XI International Conference ETAI 2013*, Ohrid, Republic of Macedonia, 26–28 September 2013.
12. S.C. Shiralashetti, L.M. Angadi, S. Kumbinarasaiah and B.S. Hoogar, Laguerre wavelet based Galerkin method for the numerical solution of elliptic problems, *International Journal of Engineering, Science and Mathematics*, 6(8) (2017) 81–87.
13. S.B.A. Sami, A.K. Alomari and I. Hashim, Approximate solutions of singular two-point BVPs using Legendre operational matrix of differentiation, *Journal of Applied Mathematics*, (2013) Article ID 547502, 1–6.