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Short Communication

On the Diophantine Equation $p^x + n^{3y} = z^3$

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Abstract. In this paper, we show that the Diophantine equation $p^x + n^{3y} = z^3$, where n is a positive integer and p is a prime number with $p \neq 3$, $p \neq 7$ and $n \equiv 1 \pmod{p}$, has no solutions in non-negative integers x, y and z.

Keywords: Diophantine equation; integer solution; congruence

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In 2017, Burshtein [1] showed that the Diophantine equation $p^3 + q^2 = z^3$, where p is a prime number and $q > 1$, has exactly four positive integer solutions in all of which $p = 7$. The solutions (p,q,z) are $(7,13,8), (7,7^2,14), (7,3\cdot7^2,28)$ and $(7,3\cdot7^2\cdot13,154)$. After that, Burshtein [2] proved that the positive integer solutions of the Diophantine equation $p^3 + q = z^3$, where p and q are prime numbers, are $(p,q,z) = (p, 3p^2+3p+1, p+1)$. In 2020, Burshtein [3] studied the Diophantine equation $p^x + q^y = z^3$, where p and q are prime numbers and $1 \le x, y \le 2$ are integers. In 2021, Burshtein [4] showed that the Diophantine equation $p^3 + q^y = z^3$, where p, q are distinct odd prime numbers and $y > 3$, has no solutions in positive integers.

In 2022, Mina and Bacani [5] investigated the positive integer solutions of the Diophantine equation $p^x + (p+4)^y = z^3$, where p and $p+14$ are prime numbers with $p \equiv 1 \pmod 3$. In 2023, Tadee [6] presented all non-negative integer solutions of the Diophantine equation $8^x + p^y = z^3$, where p is a prime number. In 2024, Tadee [7] found some conditions for the non-existence of non-negative integer solutions (x,y,z) of the Diophantine equation $3^x + n^y = z^3$, where n is a positive integer. In the same year, Khanom, Wananiyakul and Tongsomporn [8] investigated the non-negative integer solutions of the Diophantine equation $3^x + a^y = z^3$, where a is a positive integer, using elementary techniques. In 2025, Tadee [9] showed that if a is a positive integer with $a \equiv 3,9,10 \pmod{13}$, then the Diophantine equation $13^x + a^y = z^3$ has no non-negative integer solution. In 2026, Yiamras and Tadee [10] presented all non-negative integer solutions (x, y, z) of the Diophantine equation $7^x + n^y = z^3$, where n is a positive integer with $n \equiv 2,3,4,5 \pmod{7}$.

In this paper, we will consider the Diophantine equation $p^x + n^{3y} = z^3$, where n is a positive integer and p is a prime number with $p \neq 3$, $p \neq 7$ and $n \equiv 1 \pmod{p}$.

2. Main results

In this section, we present our research results.

Theorem 2.1. Let n be a positive integer and p be a prime number with $p \neq 3$, $p \neq 7$ and $n \equiv 1 \pmod{p}$. Then the Diophantine equation

$$p^x + n^{3y} = z^3 \qquad (1)$$

has no non-negative integer solution.

Proof: Assume that there exists a non-negative integer solution (x, y, z) of Equation (1).

Case 1. $x = 0$. From Equation (1), we have $1 + n^{3y} = z^3$ or $z^3 - n^{3y} = 1$. This implies that $(z - n^y)(z^2 + zn^y + n^{2y}) = 1$. It implies that

$$z - n^y = 1 \qquad (2)$$

and

$$z^2 + zn^y + n^{2y} = 1. \qquad (3)$$

Since z is a non-negative integer, we can consider these in two cases as follows:

Subcase 1.1. $z = 0$. From Equation (2), we have $n^y = -1$. This is impossible.

Subcase 1.2. $z \geq 1$. Then $z^2 + zn^y + n^{2y} > 1$. This contradicts with Equation (3).

Case 2. $x \geq 1$. From Equation (1), we get $z^3 - n^{3y} = p^x$ or $(z - n^y)(z^2 + zn^y + n^{2y}) = p^x$. Since p is a prime number, there exists a non-negative integer u such that

$$z - n^y = p^u \qquad (4)$$

and

$$z^2 + zn^y + n^{2y} = p^{x-u}. \qquad (5)$$

From Equation (4) and (5), we have

and

$$p^{2u} + 3p^u n^y + 3n^{2y} = p^{x-u}. \qquad (6)$$

Since u is a non-negative integer, we can consider these in two cases as follows:

Subcase 2.1. $u = 0$. From Equation (6), we have

$$1 + 3n^y + 3n^{2y} = p^x. \tag{7}$$

Since $x \geq 1$ and $n \equiv 1 \pmod{p}$, it follows that $p^x \equiv 0 \pmod{p}$ and $1 + 3n^y + 3n^{2y} \equiv 7 \pmod{p}$, respectively. From Equation (7), we get $7 \equiv 0 \pmod{p}$. Therefore $p = 7$. This is impossible since $p \neq 7$.

Subcase 2.2. $u \geq 1$. From Equation (6), we get $x - u > 0$. Then $p^{x-u} \equiv 0 \pmod{p}$. Since $n \equiv 1 \pmod{p}$, it follows that $p^{2u} + 3p^u n^y + 3n^{2y} \equiv 3 \pmod{p}$. From Equation (6), we have $3 \equiv 0 \pmod{p}$. Thus $p = 3$. This is impossible since $p \neq 3$.

From the two cases mentioned above, we can conclude that the Equation (1) has no non-negative integer solution.

Example 2.1. The Diophantine equation $2^x + 27^y = z^3$ has no solutions in non-negative integers x, y and z.

Example 2.2. The Diophantine equation $5^x + 216^y = z^3$ has no solutions in non-negative integers x, y and z.

Corollary 2.2. Let m, n be positive integers and p be a prime number with $p \neq 3$, $p \neq 7$ and $n \equiv 1 \pmod{p}$. Then the Diophantine equation $p^x + n^{3y} = z^{3m}$ has no non-negative integer solution.

Proof: Assume that there exists a non-negative integer solution (x, y, z) of the equation $p^x + n^{3y} = z^{3m}$ or $p^x + n^{3y} = (z^m)^3$. Then (x, y, z^m) is a non-negative integer solution of the Equation (1). This contradicts with Theorem 2.1. Hence, the Diophantine equation $p^x + n^{3y} = z^{3m}$ has no non-negative integer solution.

3. Conclusion

By using elementary methods, we prove that the Diophantine equation $p^x + n^{3y} = z^3$, when n is a positive integer and p is a prime number with $p \neq 3$, $p \neq 7$ and $n \equiv 1 \pmod{p}$, has no non-negative integer solution.

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