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Some Notes on Injective S-acts

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Abstract. Let S be a monoid. This article consists of two parts. The first part mainly investigates that when $S = G \dot{\cup} I$, where G is a group and I a right ideal of S, the n-principally weakly injectivity of S-acts can be determined by the n-principally weakly injectivity of I^1-acts, which generalizes the existing results. In the second part, the existing issues in the study of injectivity discussed in reference [1] are primarily addressed, and correct results are provided.

Keywords: S-act; n-principally weakly injective

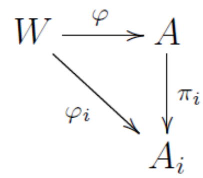
AMS Mathematics Subject Classification (2020): 13C11

1. Introduction

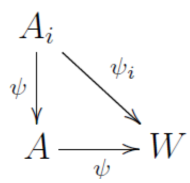
This article introduces the definition of n-principally weakly injective, that is, Definition 2.1, then decomposes S into a disjoint union of the group G and the ideal I of S, and some properties are obtained, namely Proposition 2.3 and Proposition 2.5. Later, the definition of the kernel ideal is restated, leading to Theorem 3.3. As for the remaining results, we are concerned with the corrections in reference [4] and the correct proof process provided.

Let S be a monoid with identity 1. Recall that a right S-act [2] A is a set together with a map $f: A \times S \longrightarrow A$, $(a, s) \longmapsto as$, called its action, such that, denoting $\lambda(a, s)$ by as, We have $a1 = a$ and $a(st) = (as)t$ for all $s, t \in S$ and $a \in A$. Let A and B are right S-acts. A function $g: A \rightarrow B$ is called a right S-homomorphism, if for any $a \in A$ and $s \in S$, we have $g(as) = g(a)s$.

Let $\mathcal{C}$ be a category. Recalled $A = \prod \{A_i, i \in I\}$ is a product, $\{A_i, i \in I\}$ is a family of S-acts. If there is a homomorphism $\pi_i: A \rightarrow A_i$, for all $i \in I$, and for all $W \in \mathcal{C}$, if there exists a S-homomorphism $\varphi_i: W \rightarrow A_i$, then exists an unique S-homomorphism φ, we have



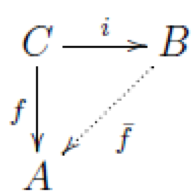
is followed. Similarly. $A = \coprod \{A_i, i \in I\}$ is called a coproduct, $\{A_i, i \in I\}$ is a family of S-acts, if there is a homomorphism $\varepsilon_i: A_i \to A$, for all $i \in I$, and $W \in \mathcal{C}$, if there exists a S-homomorphism $\psi_i: A_i \to W$, $i \in I$, then there exists a unique S-homomorphism $\psi: A \to W$,



is followed.

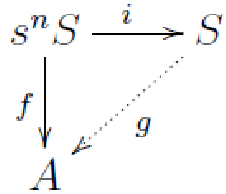
Let A be a right S-act. An element θ of an S-act is called a zero or a fixed element if $\theta s = \theta$ for all $s \in S$. A S-subact A of a S-act B is called large in B if any S-homomorphism $f: B \to C$ whose restriction $f|_A$ to A is a monomorphism, is itself a monomorphism. A S-act B_S is a retract subact of S-act A_S if and only if there exists a subact W of A_S and S-epimorphism $f: A_S \to W$ such that $B_S \cong W$ and $f(W) = w$ for every $w \in W$. A set $\{(a, a') \in A \times A | f(a) = f(a')$ is called a kernel of f with $f: A \to B$ is a S-homomorphism, denoting by $kerf$. Obviously, $kerf$ is called an identically congruent if it satisfies $kerf = 1_A$. Let A be a right S-act, B be a non-empty set of A. Recalled B is a subact of A, if it satisfies $bs \in B$ for all $b \in B$ and $s \in S$, denoting by $B \leq A$. Let A be a right S-act. An element $a \in A$ is called divisible by $s \in S$ if there exists $b \in A$ such that $bs = a$.

A right S-act A_S is called an injective right S-act, if for any right S-act B, any right subact C of B and any homomorphism $f \in Hom(C, A)$, there exists a homomrphism $\bar{f} \in Hom(B, A)$ which extends f, that is, $\bar{f}|_C = f$



2. n-principally weakly injective

Definition 2.1. [3] Let S be a monoid, n be a positive integer. A is called n-principally weakly injective, if for any S-homomorphism $f: s^n S \to A$, there exists S-homomorphism $g: S \to A$ which extends f, that is, $\bar{f}|_C = f$.



Lemma 2.2. [3] The following statements are equivalent for any S-act A over a monoid S.

(1) The S-act A is n-principally weakly injective;

(2) For every $s \in S$ and every S-homomorphism $f: s^nS \to A$, there exists $z \in A$ such that $f(x) = zx$ for every $\forall x \in s^nS$;

(3) Foe every $s \in S$, $a \in A$ with $ker\lambda_{s^n} \le ker\lambda_a$, then $a \in A$ is divisible by s^n, that is, $a = zs^n$ for some $z \in A$.

In the following sections, we show that $S = G \dot{\cup} I$ is a monoid, where G is a group and I is an ideal of S, and $I^1 = I \dot{\cup} 1$, since I is a subsemigroup of S, every S-act can be considered as an I^1-act.

Proposition 2.3. Let $S = G \dot{\cup} I$ be a monoid and A be an S-act. A is n-principally weakly injective as an S-act whenever it is n-principally weakly injective as an I^1-act.

Proof: Since A is n-principally weakly injective as an I^1-act, by Lemma 2.4, there exists $z \in A$, such that $a = zi^n$ for $i^n \in I^1$, $a \in A$, and we have $ker\lambda_{i^n} \le ker\lambda_a$. If $S = G$ is a group, then A is n-principally weakly injective as a G-act. Hence, for every $g^n \in G$, we have $ker\lambda_{g^n} \le ker\lambda_a$, and there exists $b \in A$ such that $a = bg^n$. So, for each $s \in S$, $a \in A$ with $ker\lambda_{s^n} \le ker\lambda_a$, there exists $z \in A$ such that $a = zs^n$. That is, A is n-principally weakly injective.

Corollary 2.4.[1] Let $S = G \dot{\cup} I$ be a monoid and A be an S-act. If A is n-principally weakly injective as an $I1$-act, then A is n-principally weakly injective as an S-act.

Proposition 2.5. Let A be a right S-act. Any retract of an n-principally weakly injective act is also n-principally weakly injective.

Proof: Let A be an n-principally weakly injective act and B be a retract of A, f:$snS \to B$ be an S-homomorphism for all $s \in S$ S. Since A is an n-principally weakly injective act, there exists an S-homomorphism $g: S \to A$ and $g \circ i = i_B \circ f$, where $i: s^nS \to S$ is the inclusion map, and $i_B: B \to A$ is also the inclusion map. Let $h = \pi_A \circ g$ and π_A is the map from A to B, then we have $h \circ i = \pi_A \circ g \circ i = \pi_A \circ i_B \circ f = I_A \circ f = f$. Hence B is n-principally weakly injective.

3. Correction of problems in injective research

In [1], we research some injective properties about S-act, but some conclusions need to be corrected. In the [1], there are some problems from Definition 2.1 to Theorem 2.3 and for the purpose of narration, the original content is listed below as it is.

A monoid S is called weakly left zero if for every $s \in S$, there exists $t \in S$ such that $st = s$. (Definition 2.1 from [1]).

A monoid S is called a kernel monoid if S is weakly left zero and for every $s \in S$, there exists $t \in S$ such that $ker\lambda_s \leq ker\lambda_t$. (Definition 2.1 from [1]).

Let $S = G \dot{\cup} I$ be a monoid. Then every S-act is principally weakly injective whenever I^1 is a kernel monoid and principally weakly self-injective. (Theorem 2.3 from [1])

Proof: Since I^1 is a kernel monoid, then for every $i \in I^1$, there exists $j \in I^1$ such that $Ker\lambda_i \leq Ker\lambda_j$ and $ij = i$, and I^1 is principally weakly self-injective. Hence, by Theorem 1.2, j is divisible by i. So there exists $x \in I^1$ such that $j = xi$. Now, we have $i = ixi$, that is, i is a regular element. Hence, I^1 is a regular monoid, then S is regular, by Theorem 4.1.6 of [4], every S-act is principally weakly injective.

About Definition 2.1 of [1]. Since S is a monoid, $st = s$ is not quite appropriate in this definition. If we let $t = 1$, then we have $s = s$, that is, it is trivial.

About Definition 2.2 of [1]. Noting about the definition of a kernel monoid, both of these conditions are followed, but for every s, it is not necessary to find the same element t such that $s = st$ and $ker\lambda_s \leq ker\lambda_t$.

About Theorem 2.3 of [1]. The definition of a kernel monoid was wrongly used in the proof. The author holds in the proof that for every $i \in I^1$, there exists $j \in I^1$ such that $ker\lambda_i \leq ker\lambda_j$ and $i = ij$.

Based on the above reasons and in combination with the main results of the paper, we will correct Definitions 2.1 and 2.2 and Theorem 2.3 of [1].

Definition 3.1. [5] Let S be a monoid and I be a proper right ideal of S, if for every $i \in I$, there exists $j \in I$ such that $i = ij$, then I is called left-stable.

Definition 3.2. Let I be a proper ideal of a monoid S, I is called a kernel ideal which is for every $i \in I$, there exists $j \in I$ such that $ker\lambda_i \leq ker\lambda_j, i = ij$.

Theorem 3.3. Let $S = G \dot{\cup} I$ be a monoid, if I is a kernel ideal and $I1$ is principally weakly self-injective, then every S-act is principally weakly injective.

Proof: Since I is a kernel ideal, then for every $i \in I$, there exists $j \in I$ such that $ker\lambda_i \leq ker\lambda_j$ and $ij = i$, and I^1 is principally weakly self-injective. Hence, by Theorem 1.2, j is divisible by i, there exists $x \in I$ such that $j = xi$. Now, we have $i = ixi$, that is, i is a regular element. So I^1 is a regular monoid. Hence, S is regular, then by Theorem 4.1.6 of [4]. Every S-act is principally weakly injective.

The second step is the correction of Theorem 3.4 of [1]. Its main contents are as follows:

Let $S = G \dot{\cup} I$ be a monoid with zero element θ, and $GI = \{0\}$. Then, each S-act $A \in Act_0$-S is an injective S-act whenever it is injective as an I^1-act.

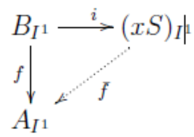
In this theorem, if $GI = \{0\}$, we have $I = \{0\}$. Therefore, this conclusion is actually to suppose $S = G^0$, but for every S-act is injective on the 0-group. Hence, this conclusion is trivial. So, it is more reasonable for us to replace $GI \subset I$ with $GI = \{0\}$ in Theorem 3.4 of [1], and its proof needs to be revised. Thus, we have the following Theorem 3.4.

Theorem 3.4. Let $S = G \dot{\cup} I$ be a monoid with zero element θ, and $GI \subset I$. Then, each

S-act $A \in Act_0$-S is an injective S-act whenever it is injective as an I^1-act.

Proof: Suppose A is an injective I^1-act. Since A contains a zero θ, by Theorem 1.8 of [5], so, we prove that every S-homomorphism $f: B \to A$ is extended to $\bar{f}: xS \to A$. So, the following possible cases can occur.

Case 1. If $xI \cap xG \neq \emptyset$, suppose $m \in xI \cap xG$, then $m = xg$, $g \in G$, $m \in xI$, since xI is a subact of xS and $(xg)g^{-1} = x(gg^{-1}) = x \in xI$. Hence, we have $xS \subseteq xI$, and $xI \subseteq xS$, then $xI = xS$. Considering xS and A as I^1-act, i, f as I^1-homomorphism, the following commutative diagram is followed.

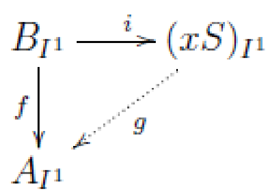


Now, we will prove $\bar{f}: xS \to A$ is an S-homomorphism:

For all $s \in I$, then $\vec{f}((xt)s) = \bar{f}(xt)s$;

For all $s \in G$, then $\bar{f}((xt)s) = \bar{f}(x(ts)) = \bar{f}(x)ts = \bar{f}(xt)s$.

Case 2. If $xI \cap xG = \emptyset$, then



Now, we define the map $\bar{f}: xS \to A$ to be:

If $xs \in xI$, then $\bar{f}(xs) = g(xs)$;

If $xs \in xG$, then $\bar{f}(xs) = \theta$.

Obviously, $\bar{f}$ is well-defined. Thus, to prove $\bar{f}$ is an S-homomorphism:

For every $t \in I$, $xs \in xI$, then $\bar{f}((xs)t) = \bar{f}(x(st)) = g(x(st)) = g((xs)t) = g(xs)t = \bar{f}(xs)t$;

For every $t \in I$, $xs \in xG$, then $\bar{f}((xs)t) = \bar{f}(x(st)) = g(x(st)) = g((xs)t) = g(xs)t = \bar{f}(xs)t$;

For every $t \in G$, $xs \in xG$, then $\bar{f}((xs)t) = \bar{f}(x(st)) = \theta_{A_s} = \theta_{A_s}t = \bar{f}(xs)t$;

For every $t \in G$, $xs \in xI$, then $\bar{f}((xs)t) = \bar{f}(x(st)) = g(x(st)) = g(x)(st) = g(xs)t = \bar{f}(xs)t$.

So $A \in Act_0$-S is an injective S-act.

Next, we discuss the errors in Theorem 3.5 of [1]. Its content description is as follows:

Theorem 3.5. [1] Let $S = G \dot{\cup} I$ be a monoid whose idempotents are central. Then, every S-act with a unique zero is injective whenever every I^1-act is so.

The content and proof method of this theorem are equivalent to those of Theorem 3.6.3 of [2]. The content of Theorem 3.6.3 is as follows: The following statements are equivalent.

(1) S is a completely injective monoid;

(2) S contains a zero element; any of its left and right ideals can be generated by idempotent elements.

The condition (2) in Theorem 3.6.3 is actually equivalent to the idempotent elements being central elements in Theorem 3.5 by Proposition 4.4. of [9]. A semigroup S, every S-act is weakly injective. If every right ideal of S has an idempotent generator. Hence, the proof of Theorem 3.5 is actually (2)⇒(1) of Theorem 3.6.3, but this proof has already been provided from [5]. So, this theorem is trivial.

Finally, there is the correction of the content of Theorem 3.6 from [1]. The content description of Theorem 3.6.

Theorem 3.6. [1] Let $S = G \dot{\cup} I$ and $h: S \to I^1$ be a nontrivial semigroup homomorphism with $h(1) = 1$. Then, A is an injective I^1-act if and only if it is an injective S-act.

There are some problems in the proof provided in this literature, so the correct proof process is given.

Proof: Suppose A is not an injective I^1-act, since an injective S-act A is injective if and only if A has not essential extension, then A has a proper essential extension I^1-act, that is, A is a large I^1-subact of B, for every S-homomorphism $f: B \to C$, considering a monomorphism from $f|_A$ to A as I^1-homomorphism, that A is a large S-subact of B. Hence, B is a proper essential extension of A; this contradicts the injectivity of A as an S-act. So A is an injective S-act.

Conversely, we first consider A, B and C as an I^1-act, f as an I^1-homomorphism, the existence of an I^1-homomorphism $\bar{f}: C \to A$ which completes the diagram follows from the hypothesis.

$$\begin{array}{ccc} B_{I^1} & \xrightarrow{i} & C_{I^1} \\ f \downarrow & \swarrow \bar{f} & \\ A_{I^1} & & \end{array}$$

Now, for every $a \in A$ and $s \in S$, we have $a \cdot s = ah(s)$, by Proposition 2.1 of [4]. Suppose T and S are semigroups and $f: T \to S$, thus we define $f^*: Act_S \to Act_T$ by $a \otimes t \to af(t)$, where $f(t) \in S$, then the action of $f(t)$ on a is transformed into a right S-act. Hence the semigroup homomorphism is converted into a S-homomorphism, so for every $c \in C$, $s \in S$, we have $\bar{f}(c \cdot s) = \bar{f}(c \cdot h(s)) = \bar{f}(c)h(s) = \bar{f}(c)s$, that $\bar{f}$ is a S-homomorphism. So A is an injective S-act.

Remark 3.7. The proof of Theorem 3.6 from [1] should clearly specify how a semigroup homomorphism is transformed into an S-homomorphism. Otherwise, $\bar{f}(c \cdot h(s)) = \bar{f}(c)h(s)$ can not hold; it is impossible to prove f is an S-homomorphism.

Remark 3.8. Theorem 2 of [3] proves the following results.

Let S be a monoid and n be a positive integer, then the following statements are equivalent£º

(1) All right S-acts are n-principally weakly injective;
(2) All right ideals of S are n-principally weakly injective;

(3) All finitely generated right ideals of S are n-principally weakly injective;
(4) All principal right ideals of S are n-principally weakly injective;
(5) For all $s \in S$, there exists $x \in S$ such that $s^n = s^n x s^n$.

Obviously, a group satisfies condition (5) in this theorem. Hence, Lemma 2.1 of [1] proves that. Let S be a group; then every S-act is principally weakly injective. In fact, Theorem 2 of [3] obtains a more general conclusion; it is only necessary to take $n = 1$.

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