Annals of Pure and Applied Mathematics
Vol. 32, No. 2, 2025, 49-55
ISSN: 2279-087X (P), 2279-0888(online)
Published on 31 October 2025
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v32n2a01975

Annals of Pure and Applied Mathematics

Hesitant Intuitionistic Fuzzy Prime Submodule of R-Module M

Rajaa Raihan Rasool[1]*, Mohammed Jassim Mohammed[2] and Ahmed H.Alwan[3]

[1,3]Department of Mathematics, College of Education for Pure Sciences,
Thi-Qar University, Thi-Qar, Iraq.
[2]Department of Mathematics, College of Education,
Al Ayen University, Thi-Qar, Iraq.
*Corresponding author: ahmedalnasry983@gmail.com

Received; accepted

Abstract. In this paper, the concepts of hesitant intuitionistic fuzzy submodule of R-module M and hesitant intuitionistic fuzzy prime submodule of R-module M are introduced. And it was concluded that the intersection and the union of any two hesitant intuitionistic fuzzy prime submodules is itself a hesitant intuitionistic fuzzy prime submodule, and present some related results.

Keywords: hesitant fuzzy set, hesitant intuitionistic fuzzy set, hesitant fuzzy submodule of R-module M, hesitant intuitionistic fuzzy submodule of R-module M, hesitant intuitionistic fuzzy prime submodule of R-module M.

1. Introduction

Zadeh [12] in (1965) introduced the concept of fuzzy set (FS) in a set X. The intuitionistic fuzzy set (in short, IFS) on X was introduced by Atanassov [1] in1986, and prove some properties. Torra [11] in (2010) introduced the concept of hesitant fuzzy set (HFS) and he defined the complement, union and intersection of HFSs. Isaac and John in 2011 introduced the concept on Intuitionistic fuzzy submodules of a module [5]. Poonam and Gagandeep in 2018 studied some of the properties of intuitionistic fuzzy prime submodules with the help of residual quotient in [10]. Azizi in (2006) introduced the concept of weakly prime submodule and prime submodule [2]. Khashan, in (2012) introduced the concept of almost prime submodule [6]. Fadhil, Mohammed and Hadi [3] in (2021) introduced on hesitant fuzzy prime modules.

In this work we write the definition of a hesitant intuitionistic fuzzy prime submodule of R-module M and prove some results about them.

2. Preliminaries

In this section, we list some definitions, and results needed in the later sections.

Definition 2.1. [13] Let X be a non – empty set and let I be the closed interval of real numbers $I = [0,1]$. A fuzzy set μ in X is a function $\mu : X \to I$. $\mu(x)$ is the degree of membership of the element x to the set X. The family of all fuzzy sets in X is denoted by I^x.

Definition 2.2. [1] Let X be a non-empty set an intuitionistic fuzzy set (in short, IFS) A is an object have the form :
A = { < a, $\mu_A(a)$,$v_A(a)$ > : $a \in$ X } where the functions μ_A: X → [0,1]iand v_A: X → [0,1] denoted the degree of membership and non-membership for each element a ∈ X to the set A (respectively) and $0 \le \mu_A(a) + v_A(a) \le 1$ for all $a \in$ X . The family of all intuitionistic fuzzy sets denoted by IF(X). Furthermore, we call:
$\pi_A(a) = 1 - \mu_A(a) - v_A(a)$, $\forall a \in$ X the intuitionistic index or hesitancy degree of A . It is obvious that, $0 \le \pi_A \le 1$, for all $a \in$ X.

Definition 2.3. [12] Let X be a reference set. A hesitant fuzzy set on X is defined in term of a function h that when applied to X return a subset of [0,1], that is, h: X → P([0,1]), where A= $\{< x, h_A(x) >: x \in X\}$ and $h_A(x)$ is a set of some values in [0,1] , denoting the possible membership degree of the element $x \in$X to the set A.

Definition 2.4. [9] Let X be a non-empty set. Suppose μ and v are functions from X to the collection of subsets of [0, 1]. A **hesitant intuitionistic fuzzy set** (HIFS) on X is a set A= { < x, $\grave{\mu}_{h_A}(x)$, $\grave{v}_{h_A}(x)$ > / $x \in X$ }, where values $\grave{\mu}_{h_A}(x)$ and $\grave{v}_{h_A}(x)$ denote the possible membership and non-membership values of the element $x \in X$ to the set A respectively, which satisfy $\grave{\mu}_{h_A}{}^{-}(x) + \grave{v}_{h_A}{}^{+}(x) \le 1$ and $\grave{\mu}_{h_A}{}^{+}(x) + \grave{v}_{h_A}{}^{-}(x) \le 1$.

Definition 2.5. [7] Let A and B are two be HIFSs of the forms
A= $\{< x, \grave{\mu}_{h_A}(x), \grave{v}_{h_A}(x) >/x \in X\}$, $B = \{< x, \grave{\mu}_{h_B}(x), \grave{v}_{h_B}(x) >/ x \in X\}$ on X, then:
1) $A \subset B$ if and only if $\grave{\mu}_{h_A}(x) \subseteq \grave{\mu}_{h_B}(x)$ *and* $\grave{v}_{h_A}(x) \supseteq \grave{v}_{h_B}(x)$ *for all* $x \in X$.
2) $A = B$ *if and only if* $A \subseteq B$ *and* $B \supseteq A$.
3) $A^c = \{< x, \grave{v}_{h_A}(x), \grave{\mu}_{h_A}(x) >: x \in X\}$
4) $A \cup B = \{< x, \grave{\mu}_{h_A}(x) \cup \grave{\mu}_{h_B}(x), \grave{v}_{h_A}(x) \cap \grave{v}_{h_B}(x) >: x \in X\}$
where
$= \cup_{y_1 \in \grave{\mu}_{h_A}(x), y_2 \in \grave{\mu}_{h_B}(x)} max\{y_1, y_2\}$ and $(\grave{\mu}_{h_A} \cup \grave{\mu}_{h_B})(x)$
$= \cup_{y_1 \in \grave{v}_{h_A}(x), y_2 \in \grave{v}_{h_B}(x)} min\{y_1, y_2\} (\grave{v}_{h_A} \cap v_{h_B})(x)$
5) $A \cap B = \{< x, \grave{\mu}_{h_A}(x) \cap \grave{\mu}_{h_B}(x), \grave{v}_{h_A}(x) \cup \grave{v}_{h_B}(x) >: x \in X\}$
where
$= \cup_{y_1 \in \grave{\mu}_{h_A}(x), y_2 \in \grave{\mu}_{h_B}(x)} min\{y_1, y_2\}$ and $(\grave{\mu}_{h_A} \cap \grave{\mu}_{h_B})(x)$
$(\grave{v}_{h_A} \cup \grave{v}_{h_B})(x) = \cup_{y_1 \in \grave{v}_{h_A}(x), y_2 \in \grave{v}_{h_B}(x)} max\{y_1, y_2\}$

Definition 2.6. [8] Let $\alpha, \beta \subseteq [0,1]$. Then a hesitant intuitionistic fuzzy point (HIFP , in short) $x_{(\alpha,\beta)}$ of X is a HIFs of X defined by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & : \quad y = x \\ (\emptyset, [0,1]) & : \quad y \ne x \end{cases}$$

In this case x is said to be the support of $x_{(\alpha,\beta)}$and α, β are called a value and a non value of $x_{(\alpha,\beta)}$ respectively. A HIFP $x_{(\alpha,\beta\,)}$ is called belong to a HIFs A of X denoted by $x_{(\alpha,\beta)} \in$ A, $\alpha \subseteq \grave{\mu}_{h_A}(x)$ and $\beta \supseteq \grave{v}_{h_A}(x)$

Definition 2.7. [8] Let X anon-empty set and A be a hesitant intuitionistic fuzzy set in X, then the (α, β)-level set defined by :

$A_{H_{(\alpha,\beta)}} = \{a \in X : \grave{\mu}_{h_A}(a) \supseteq \alpha\, , \grave{v}_{h_A}(a) \subseteq \beta\}$ where $\alpha, \beta \subseteq [0,1]$.

Definition 2.8. [3] Let h be an hesitant fuzzy set over R-module M then h is said to be hesitant fuzzy module (in short, HFM) over R-module M if for all $x, y \in M, r \in R$.

(i) $h(x - y) \supseteq h(x) \cap h(y)$ (ii) $h(rx) \supseteq h(x)$

Definition 2.9. [5] Let M be a module over a ring R. An IFS A = (μ_A, v_A) in M is called an intuitionistic fuzzy submodule (IFSM) of M if

1. $\mu_A(0) = 1$, $v_A(0) = 0$
2. $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$, $v_A(x + y) \leq v_A(x) \vee v_A(y)$, ∀ x, y ∈ M
3. $\mu_A(rx) \geq \mu_A(x)$, $v_A(rx) \leq v_A(x)$,∀ $x \in$ M, $r \in$ R

Definition 2.10. Let M be a modules over a ring R. A hesitant intuitionistic fuzzy set A = $(\grave{\mu}_{h_A}, \grave{v}_{h_A})$ of M is called hesitant intuitionistic fuzzy submodule (HIFSM) if

1) $\grave{\mu}_{h_A}(\theta) = [0,1]$, $\grave{v}_{h_A}(\theta) = \emptyset$, where θ is a zero element of M;

2) $\grave{\mu}_{h_A}(x + y) \supseteq \grave{\mu}_{h_A}(x) \cap \grave{\mu}_{h_A}(y)$ and $\grave{v}_{h_A}(x + y) \subseteq \grave{v}_{h_A}(x) \cup \grave{v}_{h_A}(y)$, ∀ $x, y \in$ M.

3) $\grave{\mu}_{h_A}(rx) \supseteq \grave{\mu}_{h_A}(x)$ and $\grave{v}_{h_A}(rx) \subseteq \grave{v}_{h_A}(x)$, ∀ $x \in$ M, $r \in$ R.

*The set of all hesitant intuitionistic fuzzy submodules of M is denoted by HIFSM(M).

Remark 2.11. By saying that A = $(\grave{\mu}_{h_A}, \grave{v}_{h_A})$ is a hesitant intuitionistic fuzzy module (HIFM) we mean that A = $(\grave{\mu}_{h_A}, \grave{v}_{h_A})$ is a hesitant intuitionistic fuzzy submodule of some R module M.

Definition 2.13. Let X be a set and A subset of X. The characteristic hesitant intuitionistic fuzzy set A is defined to be the structure $\chi_{h_A}(a) = < \mu_{h\chi_A}(a), v_{h\chi_A}(a) : a \in X >$ where

$$\mu_{h\chi_A}(a) = \begin{cases} [0,1] & \text{if } a \in A \\ \emptyset & \text{if } a \notin A \end{cases} \qquad v_{h\chi_A}(a) = \begin{cases} \emptyset & \text{if } a \in A \\ [0,1] & \text{if } a \notin A \end{cases}$$

3. New results on hesitant intuitionistic fuzzy prime submodule of R-module M

In the section, we introduced definition of hesitant intuitionistic fuzzy prime submodule M and prove some result about them.

Definition 3.1. Let A, B∈ HIFSM(M) and C ∈ HIFSM(R), define the residual quotient (A : B) and (A : C) as follows:

(A : B) = $\cup\{D | D \subseteq HIFSM(R)$ such that $D \cdot B \subseteq A\}$

$= \cup\{\, r_{(\alpha,\beta)} : r \in R \text{ and } \alpha, \beta \subseteq [0, 1]$, such that $r_{(\alpha,\beta)}\, B \subseteq A\}$

And

(A: C) = $\bigcup\{E \mid E \subseteq HIFSM(M)$ such that $C \cdot E \subseteq A\}$

$= \bigcup\{x_{(\alpha,\beta)} : x \in M \text{ and } \alpha, \beta \subseteq [0,1]$, such that $C . x_{(\alpha,\beta)} \subseteq A\}$

Definition 3.2. Let A $= \left(\mu_{\grave{h}_A}, v_{\grave{h}_A}\right)$ be a hesitant intuitionistic fuzzy submodules of R-module M . Then we define rA as follows:
rA = $\{(\mu_{\grave{h}_{rA}}(a), v_{\grave{h}_{rA}}(a)) : a \in \mathrm{M}\}$ where for each $a \in$ M,
$\mu_{\grave{h}_{rA}}(a) = \cup\{\mu_{\grave{h}_A}(b) : b \in M,\ a = rb\}$, and
$v_{\grave{h}_{rA}}(a) = \cap\{v_{\grave{h}_A}(b) : b \in M,\ a = rb\}$

Definition 3.3. Let A $= \left(\mu_{\grave{h}_A}, v_{\grave{h}_A}\right)$ and B $= \left(\mu_{\grave{h}_B}, v_{\grave{h}_B}\right)$ are two hesitant intuitionistic fuzzy submodules of R-module M, and $r_{(\alpha,\beta)} \in$ HIFP(R). Then, for all m ∈ M:

$$\mu_{h'_{r_{(\alpha,\beta)}\cdot B}}(y) = \begin{cases} \cup\{\alpha \cap \mu_{\grave{h}_B}(m) & if \quad y = rm, \quad r \in R, m \in M\} \\ \quad \emptyset & if \quad other\ wise \end{cases}$$

$$v_{h'_{r_{(\alpha,\beta)}\cdot B}}(y) = \begin{cases} \cap\{\beta \cup v_{\grave{h}_B}(m) & if \quad y = rm,\ r \in R, m \in M\} \\ \quad \emptyset & if \quad other\ wise \end{cases}$$

Definition 3.4. Let M be R-module . If $r_{(\delta,\gamma)}$ be hesitant intuitionistic fuzzy point of R and $y_{(\sigma,\tau)}$, $x_{(\alpha,\beta)}$ be hesitant intuitionistic fuzzy points of M, then:
1) $r_{(\delta,\gamma)}x_{(\alpha,\beta)} = (rx)_{(\delta\cap\alpha,\gamma\cup\beta)}$
2)$x_{(\alpha,\beta)} + y_{(\sigma,\tau)} = (x+y)_{(\alpha\cap\sigma,\beta\cup\tau)}$

Definition 3.5. [4] A proper submodule N of an R-module M, N is called a prime submodule if whenever $a \in R$, $x \in$ M, with $ax \in N$ implies that $x \in N$ or $a \in (N : \mathrm{M})$.
Where $(\boldsymbol{N}:\boldsymbol{M}) = \{\boldsymbol{a} : \boldsymbol{aM} \subseteq \boldsymbol{N}\}$

Definition 3.6. Let A be hesitant intuitionistic fuzzy submodule of R-module M, then A is called an hesitant intuitionistic fuzzy prime submodule (P-HIFSM(M), in short) of M, if for each $r_{(\delta,\gamma)} \in$ HIFP(R), $x_{(\alpha,\beta)} \in$ HIFP(M) $\{r \in R, x \in M . \alpha, \beta, \delta, \gamma \subseteq [0,1]\}$
Such that $r_{(\delta,\gamma)}x_{(\alpha,\beta)} \subseteq A$, then either $x_{(\alpha,\beta)} \subseteq A$ or $r_{(\delta,\gamma)}\ \chi_{h_M} \subseteq A$.

Theorem 3.7. Let A be a P-HIFSM of B. If $\mathrm{A}_{H_{(\alpha,\beta)}} \neq \mathrm{B}_{H_{(\alpha,\beta)}}$, if A be a P-HIFSM of B. $\alpha, \beta \subseteq [0,1]$, then $\mathrm{A}_{H_{(\alpha,\beta)}}$ is prime submodule of $\mathrm{B}_{H_{(\alpha,\beta)}}$

Proof: Let $\mathrm{A}_{H_{(\alpha,\beta)}} \neq \mathrm{B}_{H_{(\alpha,\beta)}}$ and $rm \in \mathrm{A}_{H_{(\alpha,\beta)}}$, for some r $\in R$ and $m \in M$
since $rm \in \mathrm{A}_{H_{(\alpha,\beta)}}$ imples $\mu_{\grave{h}_A}(rm) \supseteq \alpha$ and $v_{\grave{h}_A}(rm) \subseteq \beta$ [By definition 2.7]
Then $(rm)_{(\alpha,\beta)} = r_{(\alpha,\beta)}m_{(\alpha,\beta)} \subseteq$ A [By definition 3.4]
Since A be a P-HIFSM of B, then
Either $m_{(\alpha,\beta)} \subseteq$ A or $r_{(\alpha,\beta)}B \subseteq$ A.
Case (i): If $m_{(\alpha,\beta)} \subseteq$ A then, $\mu_{\grave{h}_A}(m) \supseteq \alpha$ and $v_{\grave{h}_A}(m) \subseteq \beta$ [By definition 2.6]
So, $m \in \mathrm{A}_{H_{(\alpha,\beta)}}$
Case (ii) : If $r_{(\alpha,\beta)}B \subseteq$ A, then for any $y \in r\mathrm{B}_{H_{(\alpha,\beta)}}$, $y = rx$, for some $x \in \mathrm{B}_{H_{(\alpha,\beta)}}$
So, $\mu_{\grave{h}_B}(x) \supseteq \alpha$ and $v_{\grave{h}_B}(x) \subseteq \beta$ [By definition 2.7]

Now, $\alpha = \alpha \cap \grave{\mu}_{h_B}(x) \subseteq \cup \{\alpha \cap \grave{\mu}_{h_B}(x) : y = rx\} = \mu_{h'_{r_{(\alpha,\beta)}.B}}(y) \subseteq \grave{\mu}_{h_A}(y)$ [By definition 3.3]

Similarly,
we have $\beta = \beta \cup \grave{v}_{h_B}(x) \supseteq \cap \{\beta \cup \grave{v}_{h_B}(x) : y = rx\} = v_{h'_{r_{(\alpha,\beta)}.B}}(y) \supseteq \grave{v}_{h_A}(y)$ [By definition 3.3]. Then, $y \in A_{H_{(\alpha,\beta)}}$
Thus, $rB_{H_{(\alpha,\beta)}} \subseteq A_{H_{(\alpha,\beta)}}$
Hence $A_{H_{(\alpha,\beta)}}$ is prime submodule of $B_{H_{(\alpha,\beta)}}$

Theorem 3.8. Let M be a modules, and A, B are be P-HIFSM of M, where A = $(\grave{\mu}_{h_A}, \grave{v}_{h_A})$ and B = $(\grave{\mu}_{h_B}, \grave{v}_{h_B})$. Then A ∩ B ∈ P-HIFSM(M).
Proof: Suppose that A , B ∈ P-HIFSM(M), and for each $r_{(\delta,\gamma)} \in$ HIFP(R), $x_{(\alpha,\beta)} \in$ HIFP(M) $\{r \in R, x \in M . \alpha, \beta, \delta, \gamma \subseteq [0,1]\}$ such that $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq A \cap B$,
Then $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq A \ \wedge \ r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq B$
Since A, B ∈ P-HIFSM(M)
Thus either $x_{(\alpha,\beta)} \subseteq A$ or $r_{(\delta,\gamma)}\, \chi_{h_M} \subseteq A$ and either $x_{(\alpha,\beta)} \subseteq B$ or $r_{(\delta,\gamma)}\, \chi_{h_M} \subseteq B$
So, either $x_{(\alpha,\beta)} \subseteq A \cap B$ or $r_{(\delta,\gamma)}\, \chi_{h_M} \subseteq A \cap B$
Implies A ∩ B ∈ P-HIFSM(M).

Corollary 3.9. Let $\{A_{h_i}, i \in I\}$ be a family of a P-HIFSM of M, then $(\bigcap_{i \in I} A_{h_i})$ is a P-HIFSM of M.

Theorem 3.10. Let M be a modules over a ring R and A, B are be P-HIFSM of M, where A= $(\grave{\mu}_{h_A}, \grave{v}_{h_A})$ and B = $(\grave{\mu}_{h_B}, \grave{v}_{h_B})$. Then A ∪ B ∈ P-HIFSM(M).
Proof: Suppose that A , B ∈ P-HIFSM(M), and for each $r_{(\delta,\gamma)} \in$ HIFP(R), $x_{(\alpha,\beta)} \in$ HIFP(M) $\{[r \in R, x \in M . \alpha, \beta, \delta, \gamma \subseteq [0,1]\}$ such that $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq A \cup B$,
Then $r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq A \ \vee \ r_{(\delta,\gamma)} x_{(\alpha,\beta)} \subseteq B$
Since A, B ∈ P-HIFSM(M)
Thus either $x_{(\alpha,\beta)} \subseteq A$ or $r_{(\delta,\gamma)}\, \chi_{h_M} \subseteq A$ or either $x_{(\alpha,\beta)} \subseteq B$ or $r_{(\delta,\gamma)}\, \chi_{h_M} \subseteq B$
So, either $x_{(\alpha,\beta)} \subseteq A \cup B$ or $r_{(\delta,\gamma)}\, \chi_{h_M} \subseteq A \cup B$
Implies A ∪ B ∈ P-HIFSM(M).

Corollary 3.11. Let $\{A_{h_i}, i \in I\}$ be a family of a P-HIFSM of M, then $(\bigcup_{i \in I} A_{h_i})$ is a P-HIFSM of M.

Theorem 3.12. Let A = $(\grave{\mu}_{h_A}, \grave{v}_{h_A})$ be P-HIFSM of M, then $A_{H_{(\alpha,\beta)}}$ is prime submodule of M.
Proof: Let $r \in$ R, $x \in M$ such that $rx \in A_{H_{(\alpha,\beta)}}$
Since $rx \in A_{H_{(\alpha,\beta)}}$ implies $\grave{\mu}_{h_A}(rx) \supseteq \alpha$ and $\grave{v}_{h_A}(rx) \subseteq \beta$ [By definition 2.7]
So, $(rx)_{(\alpha,\beta)} \subseteq A$ [By definition 2.6]. Implies $r_{(\alpha,\beta)} x_{(\alpha,\beta)} \subseteq A$ [by definition 3.3]
Since A be is a P-HIFSM of M. Thus, either $x_{(\alpha,\beta)} \subseteq A$ or $r_{(\alpha,\beta)} . \chi_{h_M} \subseteq A$

If $x_{(\alpha,\beta)} \subseteq A$ implies $\mu\grave{}_{h_A}(x) \supseteq \alpha$ and $v\grave{}_{h_A}(x) \subseteq \beta$ [By definition 2.6]
Then $x \in A_{H_{(\alpha,\beta)}}$
If $r_{(\alpha,\beta)}.\chi_{h_M} \subseteq A$. So, $\mu_{h_{r'_{(\alpha,\beta)}.\chi_{h_M}}}(rm) \subseteq \mu\grave{}_{h_A}(rm)$, then [By definition 3.3]we get
$\mu_{h_{r'_{(\alpha,\beta)}.\chi_{h_M}}}(y) = \cup\left\{\alpha \cap \mu_{h\chi'_{h_M}}(m): y = rm, r \in R, m \in M\right\} = \cup\{\alpha \cap [0,1]\} = \alpha$
Implies $\mu\grave{}_{h_A}(rm) \supseteq \alpha$
Similarly,
$v_{h_{r'_{(\alpha,\beta)}.\chi_{h_M}}}(y) = \cap\left\{\beta \cup v_{h\chi'_{h_M}}(m): y = rm, r \in R, m \in M\right\} = \cap\{\beta \cup [0,1]\} = \beta.$
Implies $v\grave{}_{h_A}(rm) \subseteq \beta$. Thus $rm \in A_{H_{(\alpha,\beta)}}$ implies $rM \subseteq A_{H_{(\alpha,\beta)}}$.
Then $A_{H_{(\alpha,\beta)}}$ is prime submodule of R-module M .

Proposition 3.13. Let A be HIFSM(M), and $y_{(\sigma,\tau)}, x_{(\alpha,\beta)} \subseteq$ A, then:
1) $x_{(\alpha,\beta)} + y_{(\sigma,\tau)} \subseteq A$ 2) $r_{(\delta,\gamma)}x_{(\alpha,\beta)} \subseteq A$, for $r_{(\delta,\gamma)} \in$ HIFP(R)
Proof:
1) Since $x_{(\alpha,\beta)} \subseteq A$, then $\mu\grave{}_{h_A}(x) \supseteq \alpha$, $v\grave{}_{h_A}(x) \subseteq \beta$ and $y_{(\sigma,\tau)} \subseteq A$, then $\mu\grave{}_{h_A}(y) \supseteq \sigma$, $v\grave{}_{h_A}(y) \subseteq \tau$ [By definition 2.6]
Since $x_{(\alpha,\beta)} + y_{(\sigma,\tau)} = (x+y)_{(\alpha\cap\sigma,\beta\cup\tau)}$[By definition 3.4] and A be a HIFSM(M), then:
$\mu\grave{}_{h_A}(x + y) \supseteq \mu\grave{}_{h_A}(x) \cap \mu\grave{}_{h_A}(y) \supseteq \alpha \cap \sigma$ and $v\grave{}_{h_A}(x + y) \subseteq v\grave{}_{h_A}(x) \cup v\grave{}_{h_A}(y) \subseteq \beta \cup \tau$ $\forall\ x, y \in$ M.
Thus $\mu\grave{}_{h_A}(x + y) \supseteq \alpha \cap \sigma$ and $v\grave{}_{h_A}(x + y) \subseteq \beta \cup \tau$
Implies $(x+y)_{(\alpha\cap\sigma,\beta\cup\tau)} \subseteq A$ [By definition 2.6].
So, $x_{(\alpha,\beta)} + y_{(\sigma,\tau)} \subseteq A$ [By definition 3.4]
2) Since $r_{(\delta,\gamma)}x_{(\alpha,\beta)} = (rx)_{(\delta\cap\alpha,\gamma\cup\beta)}$[By definition 3.4], and A be a HIFSM(M), then
$\mu\grave{}_{h_A}(rx) \supseteq \mu\grave{}_{h_A}(x) \supseteq \alpha$ and $v\grave{}_{h_A}(rx) \subseteq v\grave{}_{h_A}(x) \subseteq \beta$, $\forall\ x \in$ M, $r \in$ R.
Implies $\mu\grave{}_{h_A}(rx) \supseteq \alpha$ and $v\grave{}_{h_A}(rx) \subseteq \beta$
Since $\mu\grave{}_{h_A}(rx) \supseteq \alpha \supseteq \alpha \cap \sigma$ and $v\grave{}_{h_A}(rx) \subseteq \beta \subseteq \beta \cup \tau$
Hence $\mu\grave{}_{h_A}(rx) \supseteq \alpha \cap \sigma$ and $v\grave{}_{h_A}(rx) \subseteq \beta \cup \tau$
Therefore $(rx)_{(\delta\cap\alpha,\gamma\cup\beta)} \subseteq A$[By definition 3.4]
Implies $r_{(\delta,\gamma)}x_{(\alpha,\beta)} \subseteq A$.

Definition 3.14. Let M be a module, and A be hesitant intuitionistic fuzzy submodule of M, and $r_{(\delta,\gamma)}$ be a hesitant intuitionistic fuzzy point of R and $y_{(\alpha,\beta)}$ be a hesitant intuitionistic fuzzy point of M. Define the residual quotient (A : $y_{(\alpha,\beta)}$) and (A : $r_{(\delta,\gamma)}$) as follows:
(A: $y_{(\alpha,\beta)}$) = $\cup\left\{x_{(\varepsilon,\tau)} \middle| x_{(\varepsilon,\tau)} \subseteq HIFP(R)\right.$ such that $x_{(\varepsilon,\tau)}\ y_{(\alpha,\beta)} \subseteq A\}$, and
(A: $r_{(\delta,\gamma)}$) =$\cup\left\{a_{(\varepsilon,\tau)} \middle| a_{(\varepsilon,\tau)} \subseteq HIFP(M)\right.$such that $r_{(\delta,\gamma)}a_{(\varepsilon,\tau)} \subseteq A\}$

Proposition 3.15. Let A be a P-HIFSM(M), and $x_{(\varepsilon,\tau)} \in$ HIFP(R), then (A : $x_{(\varepsilon,\tau)}$) be a P-HIFSM(M).
Proof: Suppose that A ∈ P-HIFSM(M), and for each $r_{(\delta,\gamma)} \in$ HIFP(R), $y_{(\alpha,\beta)} \in$ HIFP(M) $\{r \in R, y \in M\ .\alpha, \beta, \delta, \gamma \subseteq [0,1]\}$ such that $r_{(\delta,\gamma)}y_{(\alpha,\beta)} \subseteq \left(A : x_{(\varepsilon,\tau)}\right)$

So, $\left(x_{(\varepsilon,\tau)}\, r_{(\delta,\gamma)}\right) y_{(\alpha,\beta)} \subseteq A$
Since A ∈ P-HIFSM(M), then $y_{(\alpha,\beta)} \subseteq A$ or $x_{(\varepsilon,\tau)} r_{(\delta,\gamma)} \chi_{h_M} \subseteq A$
If $x_{(\varepsilon,\tau)} r_{(\delta,\gamma)} \chi_{h_M} \subseteq A$ then $r_{(\delta,\gamma)} \chi_{h_M} \subseteq \left(A : x_{(\varepsilon,\tau)}\right)$
$y_{(\alpha,\beta)} \subseteq A$ and $x_{(\varepsilon,\tau)} \in$ HIFP(R). Implies $x_{(\varepsilon,\tau)} y_{(\alpha,\beta)} \subseteq A$ [By proposition 3.13]
Then $y_{(\alpha,\beta)} \subseteq (A : x_{(\varepsilon,\tau)})$
So, $y_{(\alpha,\beta)} \subseteq (A : x_{(\varepsilon,\tau)})$ or $r_{(\delta,\gamma)} \chi_{h_M} \subseteq \left(A : x_{(\varepsilon,\tau)}\right)$
Then $(A : x_{(\varepsilon,\tau)})$ be a P-HIFSM(M).

Acknowledgements: The author expresses sincere gratitude to the anonymous referee for their valuable comments, constructive feedback, and insightful suggestions, which have significantly contributed to improving the clarity, quality, and overall presentation of this paper.

Author's Contributions: All authors have contributed equally to the research and preparation of this work.

Conflicts of interest: The authors declare no conflicts of interest.

REFERENCES

1. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems,* 20 (1986) 86-96.
2. A. Azizi, Weakly prime submodules and prime submodules, *Glasgow Mathematical Journal*, 48(2) (2006) 343–346.
3. A. Fadhil and J. Mohammad, On hesitant fuzzy modules, M.Sc. Thesis, University of Thi-Qar, (2021).
4. E. A. thab, Prime and semiprime submodules, M.Sc. Thesis, University of Baghdad, (1996).
5. Pisac and P.John, On intuitionistic fuzzy submodules of a module, *International Journal of Mathematical Sciences and Applications,* 1(3) (2011) 1447–1454.
6. H. Khashan, On almost prime submodules, *Acta Mathematica Scientia*, 32(2) (2012) 645–651.
7. A. Nazra, R. Lestari and G. Wicaksono, Hesitant intuitionistic fuzzy soft sets, *Journal of Physics: Conference Series*, 890(1) (2017) 012118 .
8. R. Raihan and J. Mohammad, New Results Related with Hesitant Intuitionistic Fuzzy Ideals of a Ring, M.Sc. Thesis, University of Thi-Qar, (2022).
9. T. Rashid and I. Beg, Hesitant intuitionistic fuzzy set, *International Journal of Fuzzy Logic and Intelligent Systems*, 14(3) (2014) 181-187.
10. P. Sharma and G. Kaur, On intuitionistic fuzzy prime submodules, *Notes on Intuitionistic Fuzzy Sets*, 24(4) (2018) 97–112.
11. V. Torra, Hesitant fussy set, *International Journal of Intelligent Systems,* 25(6) (2010) 529-539.
12. L.Z adeh, Fuzzy sets, *Information and Control*, 8 (1965) 338-353.