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F-Bridge Domination in Fuzzy Graphs using Strong Arcs

O.T.Manjusha

PG and Research Department of Mathematics
Govt Arts and Science College, Kondotty
Malappuram-673641, India
E-mail: manjushaot@gmail.com

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Abstract. A vital part of communication networks is reliable node connectivity. The network's stability could be in danger if there is a lack of solid connectivity between nodes. Fuzzy graphs are designed to ensure network stability by identifying different dominating sets with strong arcs. This study introduces the notion of F-bridge dominating set and F-bridge domination number in fuzzy graphs and its uses in complex networks. A few notable characteristics of F-bridge domination numbers are obtained, and pertinent instances are used to study them. F-bridge domination numbers of, complete fuzzy graphs, bipartite fuzzy graphs, fuzzy cycles, and fuzzy trees are identified. In the event of a node failure, the application of F-bridge domination in a partial mesh topology to ensure network continuation is demonstrated.

Keywords: Fuzzy graph, strong arcs, strong domination, fuzzy bridges, weight of arcs.

AMS Mathematics Subject Classification (2010): 05C69

1. Introduction

Most real-world issues on marketing, economics, technology, diagnosis of illnesses, the environment, etc., can be modeled as networks. As networks get larger, tracking all their nodes and connections becomes more difficult. Determining which nodes control the network as a whole is therefore essential. The concept of domination is now one of the latest developments in graph theory to tackle these issues. Numerous dominant uses can be found in parallel computing, neural networks, communication networks, and picture compression. This paper investigates several features of the fuzzy bridge domination number and presents the concept of the fuzzy bridge-dominating set of a fuzzy graph with strong edges. Furthermore, the fuzzy bridge dominating set and the corresponding fuzzy bridge domination number are found for complete fuzzy graphs, complete bipartite fuzzy graphs, and fuzzy trees. It demonstrates how to use fuzzy bridge-dominating sets in partial-mesh topology to assess a network's strong connectedness. In networking, the idea of fuzzy bridge domination is proposed to enable data retrieval from systems or servers even in the event of a breakdown. In subsequent work, we can use an algorithmic approach to determine the fuzzy bridge domination number in a fuzzy graph.

2. Preliminaries

Graphs are simply models of relations, as is widely known. An easy approach to display information about the relationships between objects is with a graph. Vertices represent the objects, while edges describe the relations. Creating a "fuzzy graph model" is a necessary step when there is vagueness in the object's description, its relationships, or both. A concise synopsis of some fundamental definitions of fuzzy graphs can be found in [9, 10, 12-13, 15-18, 21, 26-29].

A fuzzy graph is denoted by $G:(V,\sigma,\mu)$ where V is a node-set, σ and μ are mappings defined as $\sigma: V \to [0,1]$ and $\mu: V\times V \to [0,1]$, where σ and μ represent the membership values of a vertex and an edge respectively. For any fuzzy graph $\mu(x,y) \leq \min\{\sigma(x),\sigma(y)\}$. We consider fuzzy graph G with no loops and assume that V is finite and non-empty, μ is reflexive (i.e., $\mu(x,x)=\sigma(x)$ for all x) and symmetric (i.e., $\mu(x,y)=\mu(y,x)$ for all (x,y)). In all the examples, σ is chosen suitably. Also, we denote the underlying crisp graph by $G^*:(\sigma^*,\mu^*)$ where $\sigma^*=\{u\in V|\sigma(u)>0\}$ and $\mu^*=\{(u,v)\in V\times V|\mu(u,v)>0\}$. Throughout we assume that $\sigma^*=V$. The fuzzy graph $H:(\tau,\nu)$ is said to be a partial fuzzy subgraph of $G:(\sigma,\mu)$ if $\nu\subseteq\mu$ and $\tau\subseteq\sigma$. In particular, we call $H:(\tau,\nu)$ a fuzzy subgraph of $G:(\sigma,\mu)$ if $\tau(u)=\sigma(u)$ for all $u\in\tau^*$ and $\nu(u,v)=\mu(u,v)$ for all $(u,v)\in\nu^*$. A fuzzy graph $G:(V,\sigma,\mu)$ is called trivial if $|\sigma^*|=1$. Two nodes u and v in a fuzzy graph G are said to be adjacent if $\mu(u,v)>0$.

A path P of length n is a sequence of distinct nodes $u_0,u_1,\cdots,u_n$ such that $\mu(u_{i-1},u_i)>0, i=1,2,\cdots,n$ and the degree of membership of the weakest arc is defined as its strength. If $u_0=u_n$ and $n\geq 3$, then P is called a cycle, and P is called a fuzzy cycle if it contains more than one weakest arc. The strength of a cycle is the strength of the weakest arc in it. The strength of connectedness between two nodes x and y is defined as the maximum of the strength of all paths between x and y and is denoted by $CONN_G(x,y)$.

A fuzzy graph $G:(V,\sigma,\mu)$ is connected if for every x,y in $\sigma^*, CONN_G(x,y)>0$.

An arc (u,v) of a fuzzy graph is called an effective arc (M-strong arc) if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called the effective neighborhood of u and is denoted by $EN(u)$.

A fuzzy graph G is said to be complete if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ for all $u,v\in\sigma^*$.

The order p and size q of a fuzzy graph $G:(\sigma,\mu)$ are defined to be $p=\sum_{x\in V}\sigma(x)$ and $q=\sum_{(x,y)\in V\times V}\mu(x,y)$.

Let $G:(V,\sigma,\mu)$ be a fuzzy graph and $S\subseteq V$. Then the scalar cardinality of S is defined to be $\sum_{v\in S}\sigma(v)$ and it is denoted by $|S|_s$. Let p denote the scalar cardinality of V, also called the order of G.

The complement of a fuzzy graph G, denoted by $\overline{G}$ is defined to be $\overline{G}=(V,\sigma,\overline{\mu})$ where $\overline{\mu}(x,y)=\sigma(x)\wedge\sigma(y)-\mu(x,y)$ for all $x,y\in V$ [30].

An arc of a fuzzy graph is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted. Depending on

$CONN_G(x,y)$ of an arc (x,y) in a fuzzy graph G, Mathew and Sunitha [28] defined three different types of arcs. Note that $CONN_{G-(x,y)}(x,y)$ is the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x,y). An arc (x,y) in G is $\alpha-$strong if $\mu(x,y) > CONN_{G-(x,y)}(x,y)$. An arc (x,y) in G is $\beta-$strong if $\mu(x,y) = CONN_{G-(x,y)}(x,y)$. An arc (x,y) in G is $\delta-$arc if $\mu(x,y) < CONN_{G-(x,y)}(x,y)$.

Thus an arc (x,y) is a strong arc if it is either $\alpha-$strong or $\beta-$strong. A path P is called a strong path if P contains only strong arcs. If $\mu(u,v) > 0$, then u and v are called neighbors. The set of all neighbors of u is denoted by $N(u)$. Also v is called strong neighbor of u if arc (u,v) is strong. The set of all strong neighbors of u is called the strong neighborhood of u and is denoted by $N_s(u)$. The closed strong neighborhood $N_s[u]$ is defined as $N_s[u] = N_s(u) \cup \{u\}$.

An arc is called a fuzzy bridge of a fuzzy graph $G: (V,\sigma,\mu)$ [17]if its removal reduces the strength of connectedness between some pair of nodes in G.

The strong degree of a node $v \in V$ is defined as the sum of membership values of all strong arcs incident at v. It is denoted by $d_s(v)$. That is $d_s(v) = \sum_{u \in N_s(v)} \mu(u,v)$.

The minimum strong degree of G is $\delta_s(G) = \wedge \{d_s(v): v \in V\}$ and the maximum strong degree of G is $\Delta_s(G) = \vee \{d_s(v): v \in V\}$.

The strong neighborhood degree of a node v is defined as $d_{SN}(v) = \sum_{u \in N_s(v)} \sigma(u)$. The minimum strong neighborhood degree of G is $\delta_{SN}(G) = \wedge \{d_{SN}(v): v \in V\}$ and the maximum strong neighborhood degree of G is $\Delta_{SN}(G) = \vee \{d_{SN}(v): v \in V\}$.

A fuzzy graph G is said to be bipartite [27] if the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1,v_2) = 0$ if $v_1,v_2 \in V_1$ or $v_1,v_2 \in V_2$. Further if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$, then G is called a complete bipartite graph and is denoted by K_{σ_1,σ_2}, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2.

A node u is said to be isolated if $\mu(u,v) = 0$ for all $v \neq u$.

3. F-Bridge domination in fuzzy graphs

Nagoorgani and Chandrasekaran [14] introduced the concept of domination using strong arcs in fuzzy graphs. According to Nagoorgani a node v in a fuzzy graph G is said to strongly dominate itself and each of its strong neighbors, i.e., v strongly dominates the nodes in $N_s[v]$. A set D of nodes of G is a strong dominating set of G if every node of $V(G) - D$ is a strong neighbor of some node in D. They defined a minimum strong dominating set in a fuzzy graph G as a strong dominating set with a minimum number of nodes [14]. These concepts motivated researchers to reformulate some of the concepts in domination more effectively.

Also in [15] Nagoorgani defined a minimum strong dominating set as a strong dominating set of minimum scalar cardinality. The scalar cardinality of a minimum strong dominating set is called the strong domination number of G.

Manjusha and Sunitha [9] defined strong domination numbers using membership values (weights) of arcs in fuzzy graphs as follows.

Definition 3.1. [9] *The weight of a strong dominating set D is defined as $W(D) = \sum_{u \in D} \mu(u,v)$, where $\mu(u,v)$ is the minimum of the membership values* (weights) of strong arcs incident on u. The strong domination number of a fuzzy graph G is defined as the minimum weight of strong dominating sets of G and it is denoted by $\gamma_s(G)$ or simply γ_s. A minimum strong dominating set in a fuzzy graph G is a strong dominating set of minimum weight.

Let $\gamma_s(\overline{G})$ or $\overline{\gamma_s}$ denote the strong domination number of the complement of a fuzzy graph G.

Now we define F- Bridge domination in fuzzy graphs using strong arcs as follows.

Definition 3.2. *A strong dominating set D of a fuzzy graph $G:(V,\sigma,\mu)$ is an F-bridge dominating set of G if the induced fuzzy subgraph $< D >$ is connected and each arc in $< D >$ is a fuzzy bridge.*

Remark 3.3. *Note that a fuzzy graph $G:(V,\sigma,\mu)$ contains an F-bridge dominating set if and only if G is connected.*

Definition 3.4. *The weight of an F-bridge dominating set D is defined as $W(D) = \sum_{u\in D} \mu(u,v)$, where $\mu(u,v)$ is the minimum of the membership values(weights) of strong arcs incident on u. The F-bridge domination number of a fuzzy graph G is defined as the minimum weight of all F-bridge dominating sets of G and it is denoted by $\gamma_{Fb}(G)$ or simply γ_{Fb}. A minimum F-bridge dominating set in a fuzzy graph G is an F-bridge dominating set of minimum weight.*

Let $\gamma_{Fb}(\overline{G})$ or $\overline{\gamma_{Fb}}$ denote the F-Bridge domination number of the complement of a fuzzy graph G.

Example 3.5. *Consider the fuzzy graph in Figure 1. In this fuzzy graph (a,c) and (e,d) are $\delta-$ arcs, and all others are strong arcs. Hence $D = \{b,c,d,f\}$ is a minimum F-bridge dominating set, and F-bridge domination number is $\gamma_{Fb}(G) = 0.2 + 0.2 + 0.2 + 0.5 = 1.1$.*

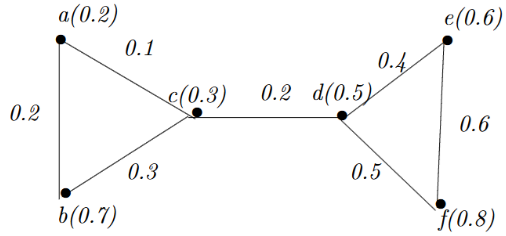

Figure 1: Illustration of F-bridge domination

Proposition 3.6. *Any F-bridge dominating set of a fuzzy graph $G:(V,\sigma,\mu)$ is a strong dominating set of G.*

Remark 3.7. *The converse of Proposition 3.6 need not be true as seen in the following example.*

Example 3.8. *Consider the fuzzy graph in Figure 2. In this fuzzy graph,* $D = \{u, x\}$ *is a strong dominating set, but not an F-bridge dominating set, since the induced fuzzy subgraph* $< D >$ *is not connected and so no fuzzy bridge in* $< D >$.

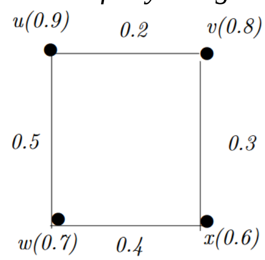

Figure 2: Example of a strong dominating set but not an F-bridge dominating set

Since an F-bridge dominating set is necessarily a strong dominating set, the following result is obvious.

Proposition 3.9. *For any connected fuzzy graph* $G: (V, \sigma, \mu)$ $\gamma_s(G) \leq \gamma_{Fb}(G)$.

4. F-bridge domination number for classes of fuzzy graphs

In this section, we have determined the F-bridge domination number of complete fuzzy graphs, complete bipartite fuzzy graphs, fuzzy cycles, and join of a fuzzy graph with a complete fuzzy graph.

Proposition 4.1. *If* $G: (V, \sigma, \mu)$ *is a complete fuzzy graph, then*

$$\gamma_{Fb}(G) = \wedge \{2\mu(u, v): u, v \in \sigma^*\}.$$

Proof: Since G is a complete fuzzy graph, all arcs are strong [25] and each node is adjacent to all other nodes. Hence $D = \{u, v\}$ is an F-bridge dominating set for any $u, v \in \sigma^*$. Hence the result follows.

Proposition 4.2. *For any complete bipartite fuzzy graph* K_{σ_1,σ_2}),

$$\gamma_{Fb}(K_{\sigma_1,\sigma_2}) = \{2\mu(u, v): u \in V_1, v \in V_2\}.$$

where $\mu(u, v)$ is the weight of a weakest arc in K_{σ_1,σ_2}.

Proof: In K_{σ_1,σ_2}, all arcs are strong. Also, each node in V_1 is adjacent to all nodes in V_2. Hence in K_{σ_1,σ_2}, the F-bridge dominating sets are any sets containing at least 2 nodes, one in V_1 and the other in V_2. Hence the set $\{u, v\}$ of nodes of any weakest arc (u, v) in K_{σ_1,σ_2} forms an F-bridge dominating set.

Hence $\gamma_{Fb}(K_{\sigma_1,\sigma_2}) = \mu(u, v) + \mu(u, v) = 2\mu(u, v)$. Hence the result.

Theorem 4.3. *Let* $G: (V, \sigma, \mu)$ *be a fuzzy cycle where* G^* *is a cycle. Then,* $\gamma_{Fb}(G) = \wedge$ $\{W(D): D$ *is a strong connected dominating set in* G *with* $|D| \geq (n-2)\}$, *where* n *is the number of nodes in* G.

Proof: In a fuzzy cycle, every arc is strong. Also, the number of nodes in an F-bridge dominating set of G and G^* are the same because each arc in both graphs is strong. In graph G^*, the F- Bridge domination number of G^* is obtained as $(n-2)$ since every strong connected dominating set contains $(n-2)$ nodes [19]. Hence the minimum number of nodes in an F- Bridge dominating set of G is $(n-2)$. Hence the result follows.

Definition 4.4. ([11,12]) *Union of two fuzzy graphs: Let* $G_1:(\sigma_1,\mu_1)$ *and* $G_2:(\sigma_2,\mu_2)$ *be two fuzzy graphs with* $G_1^*:(V_1,E_1)$ *and* $G_2^*:(V_2,E_2)$ *with* $V_1 \cap V_2 = \phi$ and let $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ be the union of G_1^* and G_2^*.

Then the union of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G:(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & if \ u \in V_1 \setminus V_2 \\ \sigma_2(u) & if \ u \in V_2 \setminus V_1 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(u,v) = \begin{cases} \mu_1(u,v) & if \ (u,v) \in E_1 \setminus E_2 \\ \mu_2(u,v) & if \ (u,v) \in E_2 \setminus E_1 \end{cases}$$

Definition 4.5. ([11,12]) *Join of two fuzzy graphs: Consider the join* $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ *of graphs where* E' *is the set of all arcs joining* the nodes of V_1 and V_2 where we assume that $V_1 \cap V_2 = \phi$. Then the join of two fuzzy graphs G_1 and G_2 is a fuzzy graph

$G = G_1 + G_2 : (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ defined by

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u), \ u \in V_1 \cup V_2$$

and

$$(\mu_1 + \mu_2)(u,v) = \begin{cases} (\mu_1 \cup \mu_2)(u,v) & if \ (u,v) \in E_1 \cup E_2 \ and \\ \sigma_1(u) \wedge \sigma_2(v) & if \ (u,v) \in E' \end{cases}$$

Theorem 4.6. *For any fuzzy graph* $G:(V,\sigma,\mu)$, $\gamma_{Fb}(K_\sigma + G) = 2\mu(u,v)$ *where* $\mu(u,v)$ *is the weight of the weakest arc incident on* u *for any node* $u \in K_\sigma$.

Proof: For any fuzzy graph G, any node in K_σ is adjacent to all other nodes in K_σ and G and note that all such arcs are strong arcs. Hence any set $D = \{u,v\}$ for each node u in K_σ and any node v in G, is an F- bridge dominating set of $K_\sigma + G$. Hence $\gamma_{Fb}(K_\sigma + G) = \mu(u,v) + \mu(u,v)$ where $\mu(u,v)$ is the weight of the weakest arc incident on u for any node $u \in K_\sigma$.

5. Minimal F-bridge domination in fuzzy graphs

In this section, we have defined minimal F-bridge dominating sets and discussed some properties.

Definition 5.1. *An F-bridge dominating set* D *of a connected fuzzy graph* $G:(V,\sigma,\mu)$ *is called a minimal F-bridge dominating set if no proper subset of* D *is an F-bridge dominating set of* G.

Remark 5.2. *Every minimum F-bridge dominating set is minimal but not conversely.*

Example 5.3. *Consider the fuzzy graph in Figure 3.*

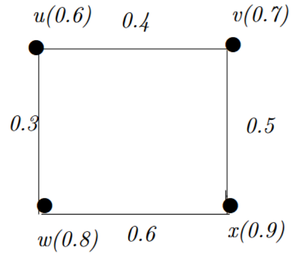

Figure 3: Illustration of minimal F-bridge domination

In the fuzzy graph of Figure 3, $(u,v),(v,x),(x,w)$ are strong arcs and (u,w) is a $\delta-$ arc. $D=\{w,x\}$ is a minimal F-bridge dominating set but not a minimum F-bridge dominating set since the set $\{u,v\}$ forms a minimum F-bridge dominating set with $\gamma_{Fb}(G)=0.8$, but $W(D)=1.1$.

Note that in a complete fuzzy graph, the minimum and minimal F-bridge dominated sets are the same and any set containing two nodes is the minimum F-bridge dominating set. Hence the following theorems are obvious.

Theorem 5.4. *Every non-trivial complete fuzzy graph G has an F-bridge dominating set D whose complement $V\backslash D$ also contains an F-bridge dominating set.*

Theorem 5.5. *Let G be a complete fuzzy graph. If D is a minimal F-bridge dominating set then $V\backslash D$ also contains an F-bridge dominating set.*

Note that in a complete bipartite fuzzy graph, the end nodes of any weakest arc form a minimal F-bridge dominating set. Hence the following theorems are obvious.

Theorem 5.6. *Every non-trivial complete bipartite fuzzy graph G has an F-bridge dominating set D of two elements whose complement $V\backslash D$ also contains an F-bridge dominating set.*

Theorem 5.7. *Let G be a complete bipartite fuzzy graph. If D is a minimal F-bridge dominating set of two elements then $V\backslash D$ also contains an F-bridge dominating set of at least two elements.*

Remark 5.8. *Theorems 5.4 to 5.7 are not true in general connected fuzzy graphs as seen in the following example.*

Example 5.9. *Consider the fuzzy graph given in Figure 4.*

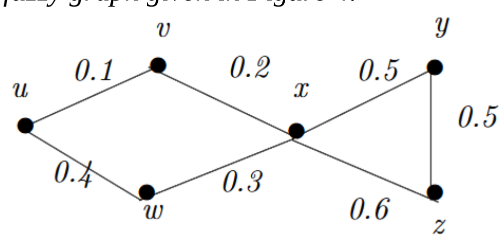

Figure 4: Example of an F-bridge dominating set D such that $V\backslash D$ does not contain an F-bridge dominating set

In this fuzzy graph, all node weights are taken as 1. $D = \{x, w\}$ is a minimal F-bridge dominating set. But $V \backslash D = \{u, v, y, z\}$ does not contain an F-bridge dominating set.

6. F-bridge domination in fuzzy trees

Note that in the definition of a fuzzy tree, F is the unique maximum spanning tree (MST) of G [27].

Definition 6.1. ([17]) *A fuzzy cut node w is a node in G whose removal from G reduces the strength of connectedness between some pair of nodes other than w.*

Definition 6.2. ([3]) *A node z is called a fuzzy end node if it has exactly one strong neighbor in G. A non-trivial fuzzy tree G contains at least two fuzzy end nodes and every node in G is either a fuzzy cut node or a fuzzy end node.*

Definition 6.3. ([2,27]) *In a fuzzy tree G an arc is strong if and only if it is an arc of F where F is the associated unique maximum spanning tree of G.*

Note that these strong arcs are α-strong and there are no β-strong arcs in a fuzzy tree [24]. Also note that in a fuzzy tree G an arc (x, y) is α-strong if and only if (x, y) is a fuzzy bridge of G [24].

Theorem 6.4. *In a non-trivial fuzzy tree $G: (V, \sigma, \mu)$, each node of an F-bridge dominating set is incident on an $\alpha -$strong arc (fuzzy bridge) of G.*

Proof: Let D be an F-bridge dominating set of G. Let $u \in D$. Since D is a strong dominating set, there exists $v \in V \backslash D$ such that (u, v) is a strong arc. Then (u, v) is an arc of the unique MST F of G [2, 27]. Hence (u, v) is an $\alpha -$strong arc or a fuzzy bridge of G [17]. Since u is arbitrary, this is true for every node of the F-bridge dominating set of G. This completes the proof.

Proposition 6.5. *In a non-trivial fuzzy tree $G: (V, \sigma, \mu)$, no node of an F-bridge dominating set is an end node of a $\beta -$strong arc.*

Proof: Note that a fuzzy graph is a fuzzy tree if and only if it has no $\beta -$strong arcs [24]. Hence the proposition.

Theorem 6.6. *In a non-trivial fuzzy tree $G: (V, \sigma, \mu)$, except K_2, the set of all fuzzy cut nodes is an F-bridge dominating set.*

Proof: Let D be the set of all fuzzy cut nodes of a non-trivial fuzzy tree $G: (V, \sigma, \mu)$. Then D is a strong connected dominating set in G [10]. Note that the internal nodes of F are the fuzzy cut nodes of G [24]. Also, note that all the strong arcs in G are fuzzy bridges. Hence D is an F-bridge dominating set of G.

Remark 6.7. *The set of all fuzzy end nodes need not be an F-bridge dominating set in a non-trivial fuzzy tree $G: (V, \sigma, \mu)$ except K_2.*

Theorem 6.8. *In a fuzzy tree $G: (V, \sigma, \mu)$, each node of every F-bridge dominating set is contained in the unique maximum spanning tree of G.*

Proof: Since G is a fuzzy tree, G has a unique maximum spanning tree F which contains all the nodes of G[2, 27]. In particular, F contains all nodes of every F-bridge dominating set of G. This completes the proof.

Theorem 6.9. *In a non-trivial fuzzy tree $G:(V,\sigma,\mu)$ except K_2, $\gamma_{Fb}(G)=W(S)$ where S is the set of all fuzzy cut nodes of G.*

Proof: Note that the set S of all fuzzy cut nodes of G is an F-bridge dominating set of G (Theorem 6.6). Here we have to prove that S is a minimum F-bridge dominating set. Suppose if possible S is not a minimum F-bridge dominating set. Then there exists an F-bridge dominating set S' such that $W(S')<W(S)$. Then S' has 4 choices.

a) S' contains all fuzzy cut nodes and at least one fuzzy end node.
b) At least one fuzzy cut node say w is not contained in S' and S' contains no fuzzy end node.
c) S' is a combination of fuzzy cut nodes and fuzzy end nodes.
d) S' contains only fuzzy end nodes.

In case 1 it is obvious that $W(S')>W(S)$.

In case 2 $<S'>$ (the fuzzy subgraph induced by S') is not connected if w is an internal node of $<S>$ (the fuzzy subgraph induced by S) or S' is not a strong dominating set if w is an end node of the fuzzy subgraph $<S>$ for,

A fuzzy tree contains at least 2 fuzzy end nodes. If w is an end node of $<S>$ then one neighboring node of w is a fuzzy end node say u in G and w is the only strong neighbor of u in G. Therefore, if w is not contained in $<S'>$ then u is not strongly dominated by any node in G. Hence S' is not a strong dominating set of G.

Case 3 has 3 possibilities.

a) G has a unique maximum weighted arc adjacent to any fuzzy end node, then $W(S')>W(S)$ since the weight of maximum arc is contributed to $W(S')$ but not to $W(S)$
b) The unique maximum weighted arc is adjacent to any fuzzy cut node then $W(S')\geq W(S)$
c) G has more than one maximum weighted arc and one of these is adjacent to a fuzzy cut node and the other is adjacent to a fuzzy end node then $W(S')>W(S)$.

In case 4 we can consider cases a, b, and c as in case 3, we get similar results.

Therefore in all the cases, we get a contradiction. Hence the minimum F-bridge dominating set of G is the set of all fuzzy cut nodes of G.

Hence, $\gamma_{Fb}(G)=W(S)$.

7. F-bridge domination in complement of fuzzy graphs

Sunitha and Vijayakumar [26] have defined the present notion of the complement of a fuzzy graph. Sandeep and Sunitha have studied the connectivity concepts in a fuzzy graph and its complement [21].

Definition 7.1. ([26]) *The complement of a fuzzy graph G, denoted by* $\overline{G}$ *or* G^c *is defined to be* $\overline{G} = (V, \sigma, \overline{\mu})$ *where* $\overline{\mu}(x,y) = \sigma(x) \wedge \sigma(y) - \mu(x,y)$ *for all* $x, y \in V$.

Bhutani has defined the isomorphism between fuzzy graphs [1].

Definition 7.2. ([1]) *Consider the fuzzy graphs* $G_1:(V_1,\sigma_1,\mu_1)$ *and* $G_2:(V_2,\sigma_2,\mu_2)$ *with* $\sigma_1^* = V_1$ *and* $\sigma_2^* = V_2$. *An isomorphism between two fuzzy graphs* G_1 *and* G_2 *is a bijective map* $h: V_1 \to V_2$ *that satisfies*

$$\sigma_1(u) = \sigma_2(h(u)) \text{ for all } u \in V_1.$$

$\mu_1(u,v) = \mu_2(h(u),h(v))$ for all $u, v \in V_1$ and we write $G_1 \approx G_2$.

Definition 7.3. ([26]) *A fuzzy graph* G *is self-complementary if* $G \approx \overline{G}$.

Theorem 7.4. *If* G *is a connected fuzzy graph with no* M *−strong arcs then* G *and* G^c *contain at least one F-bridge dominating set.*
Proof: If G is a connected fuzzy graph with no $M-$strong arcs then G^c is also connected [21]. Hence both G and G^c contain at least one F-bridge dominating set.

Remark 7.5. *There are fuzzy graphs which contain* M *−strong arcs such that* G *and* G^c *contain F-bridge dominating set [Example 7.6].*

Example 7.6. *Consider the fuzzy graph in Figure 5. Here* (v,w) *is the only* $M - strong$ *arc in* G *and* G *and* G^c *are connected. In* G, $D = \{v,w\}$ *is an F-bridge dominating set and in* G^c, $D = \{u,w\}$ *is an F-bridge dominating set.*

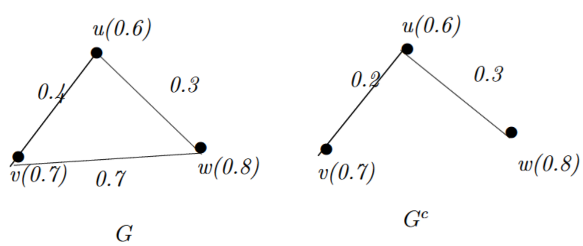

Figure 5: Illustration of an F-bridge dominating set in a fuzzy graph and its complement

Theorem 7.7. *Let* $G:(V,\sigma,\mu)$ *be a fuzzy graph. Then each of* G *and* G^c *contain at least one F-bridge dominating set if and only if* G *contains at least one connected spanning fuzzy subgraph with no* M *−strong arcs.*
Proof: Note that for a fuzzy graph $G:(V,\sigma,\mu)$, G and G^c are connected if and only if G contains at least one connected spanning fuzzy subgraph with no M −strong arcs [21]. Thus, it follows that for a fuzzy graph $G:(V,\sigma,\mu)$, G and G^c contains at least one F-bridge dominating set if and only if G contains at least one connected spanning fuzzy

subgraph with no $M-$strong arcs.

Theorem 7.8. *Let $G:(V,\sigma,\mu)$ be a fuzzy graph. Then each of G and G^c contain at least one F-bridge dominating set if and only if G contains at least one fuzzy spanning tree having no $M-$strong arcs.*

Proof: Note that for a fuzzy graph $G:(V,\sigma,\mu)$, G and G^c are connected if and only if G contains at least one fuzzy spanning tree having no $M-$strong arcs [21]. Thus it follows that for a fuzzy graph $G:(V,\sigma,\mu)$, each of G and G^c contain at least one F-bridge dominating set if and only if G contains at least one fuzzy spanning tree having no $M-$strong arcs.

Theorem 7.9. *If $G:(V,\sigma,\mu)$ is a connected self-complementary fuzzy graph, then each of G and G^c contain at least one F-bridge dominating set.*

Proof: Since G is self complementary G is isomorphic to G^c. Also, G^c is connected since G is connected. Hence the result.

Corollary 7.10. *If G is a connected fuzzy graph such that $\mu(u,v)=\frac{1}{2}(\sigma(u)\wedge\sigma(v))$ for all $u,v\in\sigma^*$ then each of G and G^c contain at least one F-bridge dominating set.*

Proof: Since $\mu(u,v)=\frac{1}{2}(\sigma(u)\wedge\sigma(v))$ for all $u,v\in\sigma^*$, G is self complementary [26]. Hence the result follows.

8. Practical application

The mesh topology is a fundamental network configuration for establishing connections between network devices or computers to facilitate data sharing among all devices. It aims to prevent the entire network from failing if a device or network experiences issues. The network components are interconnected to a certain extent, and no central control exists. Most devices are connected to two or three network devices or computers, which play a dominant role. Essentially, every device is connected to at least one of the dominant networks, minimizing the risk of data loss due to this network topology.

Consider the network presented in Table 1 for illustrating the use of fuzzy bridge domination in a partial-mesh topology. Nodes A, B, C, D, and E can be defined as follows:

Table 1. Nodes comprising the network.

Node	System
A	Data warehouses
B	Information management systems
C	Transaction processing systems
D	Executive information systems
E	Decision support systems

Assume that each system is a node and that the relationships between the systems are represented by the arcs. Prioritizing the amount of data collected from the systems is achieved by assigning weights to arcs.

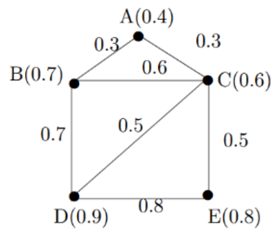

Figure 6: Partial mesh topology representation of computer networks

From the Fig. 6, $\{A,B,C,D,E\}$, $\{A,B,D,C\}$, $\{A,B,D\}$, $\{B,D,E\}$, $\{B,D\}$ are some strong dominating sets in G. Further, $(B,C),(B,D)$ and (D,E) are fuzzy bridges of G.

In this application, the F-bridge dominating set with induced fuzzy subgraph is $\{B,D\}$.

If suppose one of the devices $A, C,$ or E breaks down, then the rest of the networks remain intact due to their connection with the strongly connected dominating elements B and D through the fuzzy bridge. The same procedure applies to complex network connections.

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