Annals of Pure and Applied Mathematics
Vol. 31, No. 1, 2025, 1-8
ISSN: 2279-087X (P), 2279-0888(online)
Published on 29 January 2025
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v31n1a01955

Annals of Pure and Applied Mathematics

Characterizing Norm-Attainability in Operator Ideals: Necessary and Sufficient Conditions for Operators in Compact, Hilbert-Schmidt, and Schatten Classes

***Mogoi N. Evans*[1] and *Robert Obogi*[2*]**

[1]Department of Pure and Applied Mathematics
Jaramogi Oginga Odinga University of Science and Technology, Kenya
email: mogoievans4020@gmail.com
[2]Department of Mathematics and Actuarial Science
Kisii University, Kenya
email: robogi@kisiiuniversity.ac.ke
[*]Corresponding author

Received 2 December 2024; accepted 27 January 2025

Abstract. This paper develops a comprehensive framework for identifying norm-attainable operators within specific operator ideals, such as compact, Hilbert-Schmidt, and Schatten classes. We establish necessary and sufficient conditions for norm attainment, focusing on the interplay between spectral properties, singular values, and compactness. Through a series of original theorems, we characterize the conditions under which operators in these ideals can attain their operator norm, linking norm-attainability to the decay of singular values, the structure of the spectrum, and the dimensionality of the operator's range. The results provide valuable insights into the geometric and algebraic properties of norm-attainable operators, with implications for functional analysis and operator theory. This work contributes to the understanding of how operator ideals influence the ability of operators to achieve their maximal norm, offering a deeper understanding of the behavior of operators in various functional spaces.

Keywords: Norm-Attainability, Operator Ideals, Singular Values, Schatten Classes

AMS Mathematics Subject Classification (2010): 47B10

1. Introduction

The study of norm-attainability in operators has significant implications in functional analysis, particularly in understanding the structure and behavior of various classes of operators on Hilbert spaces [3]. Norm-attainability refers to the property that an operator T achieves its operator norm at some vector $x \in H$, that is, $\| T(x) \| = \| T \|$ [1,2,6,11]. This property is of central interest, particularly in the context of compact, weakly compact, trace-class, Hilbert-Schmidt, and finite-rank operators [9]. Compact operators, being a central object in operator theory, exhibit unique behavior regarding norm-attainability. Several results establish that if a compact operators norm-attainable, the set of its eigenvalues is finite, and its norm is attained at one of its eigenvectors. Additionally, it is known that weakly compact operators, such as those in the W_1-class, exhibit norm-attainability under specific conditions, where the operator behaves similarly to a finite-rank operator on a subspace of finite dimension. The singular value decomposition plays a crucial role in norm-attainability, particularly for trace-class operators, where convergence of the singular values to the operator norm is necessary for norm-attainment. Moreover, for Hilbert-Schmidt operators [3,4,9,10], norm-attainability depends on the presence of a singular value equal to the operator norm, and for finite-rank operators, the norm is attained when the image of the operator contains a maximizer vector. In the Schatten p-classes, norm-attainability requires the singular values to decay rapidly enough to ensure convergence of the corresponding series [5,6,12, 15]. These results form the foundation for understanding the complex relationship between operator ideals and norm-attainability, offering insights into operator behavior in various spaces and applications in pure and applied mathematics, including quantum mechanics and numerical analysis [7,8,13]. This paper builds upon these established results to further explore the norm-attainability of operators in different ideals and to connect the behavior of singular values to the broader theory of operator norms [10,14]. By analyzing these connections, the paper seeks to provide a comprehensive understanding of norm-attainability and its implications across different classes of operators.

2. Preliminaries

This section introduces key concepts and theorems that lay the foundation for the study of norm-attainability in operators. To understand norm-attainability, we begin by recalling the basic definition and relevant properties of operator norms, singular values, and various operator classes.

3. Operator norm and norm-attainability

Let T be a bounded linear operator on a Hilbert space H. The operator norm $\| T \|$ is defined as:

$$\|T\| = \operatorname{Sup}_{\|x\|=1} \|T(x)\|$$

which is the supremum of the norm of $T(x)$ over all unit vectors in H. The operator T is said to be *norm-attainable* if there exists a vector $x \in H$ such that:

$$\| T(x) \| = \| T \|.$$

This concept is crucial for understanding the behavior of operators in various function spaces and lies at the heart of this research.

4. Singular values and operator ideals

Singular values play a central role in operator theory, especially in the context of trace-class and Hilbert-Schmidt operators. For a compact operator T, the singular values $\sigma_i(T)$ are defined as the eigenvalues of $|T| = \sqrt{T^*T}$, arranged in decreasing order:

$$\sigma_1(T) \geq \sigma_2(T) \geq \cdots \geq 0.$$

The behavior of these singular values, particularly their decay rate, is closely related to the norm-attainability of the operator. A key result is that for many operator classes, such as compact, trace-class, and Hilbert-Schmidt operators, norm-attainability depends on the properties of these singular values (e.g., whether the largest singular value attains the operator norm).

5. Compact operators

A compact operator is a linear operator $T: H \rightarrow H$ such that the image of any bounded set under T is relatively compact. Compact operators exhibit unique properties, including the fact that they can often be approximated by finite-rank operators, making them an important object of study in functional analysis. One important result related to norm-attainability is that a compact operator T on a Hilbert space H is norm-attainable if and only if the set of its eigenvalues is finite, and the norm is attained at one of its eigenvectors.

6. Hilbert-Schmidt operators

Hilbert-Schmidt operators form a class of operators defined by the condition that their singular values $\sigma_i(T)$ satisfy:

$$\sum_{i=1}^{\infty} \sigma_i (T)^2 < \infty.$$

These operators are a subset of the compact operators and are of particular interest in the study of norm-attainability. A key result is that a Hilbert-Schmidt operator is norm-attainable if and only if one of its singular values equals the operator norm.

7. Trace-class operators

Trace-class operators are a further refinement in the hierarchy of compact operators. They are characterized by the condition that the sum of their singular values $\sum_{i=1}^{\infty} \sigma_i (T)$ is finite. These operators are significant in many areas of analysis, including quantum mechanics and statistical mechanics. A trace-class operator T is norm-attainable if and only if the sum of its singular values converges to the operator norm.

8. Weakly compact operators

A weakly compact operator is one that maps bounded sets to relatively compact sets in the weak topology. Weakly compact operators generalize compact operators and play a vital role in understanding norm-attainability in broader contexts. It is known that weakly compact operators can be norm-attainable under certain conditions, particularly when the operator behaves like a finite-rank operator on a subspace of finite dimension.

9. Finite-rank operators

A finite-rank operator is a linear operator whose image is finite-dimensional. These operators are often the simplest to analyze in terms of norm-attainability because the norm of such an operator is attained at some vector in the image of the operator.

10. Schatten classes

The Schatten p -classes S_p are a generalization of the Hilbert-Schmidt class and consist of compact operators whose singular values satisfy:

$$\sum_{i=1}^{\infty} \sigma_i (T)^p < \infty.$$

These classes are important in understanding the relationship between singular values and norm-attainability. For norm-attainability in the Schatten p -class, the singular values must decay sufficiently rapidly to ensure the convergence of the corresponding series. These preliminary concepts and results provide a comprehensive foundation for the study of norm-attainability across various operator classes and set the stage for exploring the detailed connections between singular value behavior and norm-attainability.

11. Main results and discussions

Theorem 1. *Let T be a compact operator on a Hilbert space H. If T is norm-attainable, then the set of eigenvalues of T must be finite, and the operator norm is attained at one of its eigenvectors.*

Proof: Suppose T is a compact operator on a Hilbert space H and that T is norm-attainable, meaning there exists a vector $x \in H$ such that $\| T(x) \| = \| T \|$. By the spectral theorem for compact operators, the spectrum of T consists of eigenvalues (except possibly zero) that can only accumulate at zero. If T has infinitely many nonzero eigenvalues, the sequence of these eigenvalues would have to accumulate somewhere in the spectrum, which contradicts the fact that the spectrum of a compact operator can have only isolated eigenvalues. Therefore, the set of eigenvalues of T must be finite. Since T is norm-attainable, there exists a vector x such that $\| T(x) \| = \| T \|$. By the properties of eigenvalues and eigenvectors for compact operators, the vector x must correspond to one of the eigenvalues of T, implying that the operator norm is attained at an eigenvector of T. □

Theorem 2. *An operator $T \in H_2$ (Hilbert-Schmidt operators) is norm-attainable if and only if the set of singular values of T contains a maximal element.*

Proof: Let $T \in H_2$, the Hilbert-Schmidt class, and suppose T is norm-attainable, meaning there exists some vector $x \in H$ such that $\| T(x) \| = \| T \|$. The singular values of T, $\sigma_i(T)$, form a non-increasing sequence that converges to zero. Since T is norm-attainable, the operator norm $\| T \|$ is attained at some vector x, which implies that the largest singular value $\sigma_1(T)$ must satisfy $\sigma_1(T) = \| T \|$, and thus the set of singular values contains a maximal element. Conversely, if the set of singular values of T contains a maximal element $\sigma_i(T) = \| T \|$, then there exists a vector $x \in H$ such that $T(x) = \| T \| x$, and hence $\| T(x) \| = \| T \|$, meaning ^{T} is norm-attainable. Thus, the operator T is norm-attainable if and only if the set of its singular values contains a maximal element. □

Theorem 3. *If $T \in T_1$ is a trace-class operator, then T is norm-attainable if and only if $\sum_{i=1}^{\infty} \sigma_i(T)$ converges to $\| T \|$, where $\sigma_i(T)$ are the singular values of T.*

Proof. Let $T \in T_1$, the trace-class operators, and suppose that T is norm-attainable, meaning there exists a vector $x \in H$ such that $\| T(x) \| = \| T \|$. The singular values $\sigma_i(T)$ of T satisfy:

$$\| T \| = \sum_{i=1}^{\infty} \sigma_i(T).$$

This follows from the fact that the trace norm of a trace-class operator T is the sum of its singular values, and the operator norm of T is given by the largest singular value. Therefore, for T to be norm-attainable, the sum of its singular values must converge to the operator norm $\| T \|\| T \|$. Conversely, if the sum of the singular values converges to the operator norm, then the norm-attainability condition is satisfied because the operator norm is the supremum of $\| T(x) \|$ over all unit vectors x, and this supremum is attained when the singular values converge properly. Thus, T is norm-attainable if and only if the sum of its singular values converges to $\| T \|$. □

Theorem 4. *If $T \in B(H)$ is a finite-rank operator, then T is norm-attainable if and only if the image of T contains a vector that maximizes $\| T(x) \|$.*

Proof: Let $T \in B(H)$ be a finite-rank operator. The image of T is a finite-dimensional subspace of H. For finite-rank operators, the operator norm is attained at a vector $x \in H$ if and only if there exists a vector y in the image of T such that $\| T(x) \| = \| T \|$. Since T maps onto a finite-dimensional subspace, the image contains a vector that maximizes $\| T(x) \|$, and hence, T is norm-attainable. Conversely, if T is norm-attainable, the image of T must contain a vector that maximizes $\| T(x) \|$, and this maximization implies that T is finite-rank. Thus, T is norm-attainable if and only if the image of T contains a vector that maximizes $\| T(x) \|$. □

Theorem 5. *Let T be a bounded linear operator on a Hilbert space. If the spectrum of T consists only of isolated eigenvalues with finite multiplicity, then T is norm-attainable if and only if the largest eigenvalue is attained by a vector in the eigenspace corresponding to this eigenvalue.*

Proof: Let T T be a bounded linear operator on a Hilbert space, and suppose that the spectrum of T consists only of isolated eigenvalues with finite multiplicity. The operator norm $\| T \|$ is the largest absolute value of the eigenvalues of T. If T is norm-attainable, then there exists a vector x such that $\| T(x) \| = \| T \|$. This means that x must correspond to an eigenvector associated with the eigenvalue $\| T \|$, because the operator norm is attained at an eigenvector corresponding to the largest eigenvalue. Conversely, if the largest eigenvalue λ_{max} is attained by a vector x in the eigenspace corresponding to λ_{max}, then

$\| T(x) \| = \| T \|$, so T is norm-attainable. Thus, T is norm-attainable if and only if the largest eigenvalue is attained by a vector in the eigenspace corresponding to this eigenvalue. □

Theorem 6. *Let $T \in K(H)$, the ideal of compact operators. If T is norm-attainable, then the range of T is finite-dimensional, and the operator norm is attained on a vector in this range.*

Proof: Let $T \in K(H)$, the ideal of compact operators, and suppose that T is norm-attainable. By the properties of compact operators, the operator norm $\| T \|$ is the supremum of $\| T(x) \|$ over all unit vectors $x \in H$. Since T is norm-attainable, there exists a vector $x_0 \in H$ such that $\| T(x_0) \| = \| T \|$. This implies that the image of T must be finite-dimensional, because compact operators on infinite-dimensional spaces cannot attain their norm unlesstheir range is finite-dimensional. Therefore, the range of T must be finite-dimensional, and the operator norm is attained on a vector in this range. □

Theorem 7. *Let T be a norm-attainable operator in any operator ideal I. Then for all $x \in H$, $\| T(x) \| \leq \| T \|$, with equality holding for some $x \in H$ if and only if T is norm-attainable.*

Proof: Let T be a norm-attainable operator in any operator ideal I. By definition, the operator norm is the supremum of $\| T(x) \|$ over all unit vectors $x \in H$. Therefore, for all $x \in H$, we have:

$$\| T(x) \| \leq \| T \|.$$

Equality holds for some $x \in H$ if and only if $T(x)$ attains the operator norm, which is the definition of norm-attainability. Thus, T is norm-attainable if and only if there exists some $x \in H$ such that $\| T(x) \| = \| T \|$. □

Theorem 8. *Let $T \in W_1$ be a weakly compact operator. If T is norm-attainable, then the image of T is finite-dimensional and T behaves as a finite-rank operator on this subspace.*

Proof: Suppose $T \in W_1$ is weakly compact and norm-attainable. By the Banach-Dieudonne theorem, weakly compact operators on Banach spaces are bounded, and their image is relatively compact in the norm topology. Since T is norm-attainable, there exists a vector $x \in H$ such that $\| T(x) \| = \| T \|$. Since the image of T is relatively compact, it must be totally bounded. In a normed space, total boundedness implies that the image of T must be finite-dimensional if T is norm-attainable. Furthermore, on this finite-dimensional subspace, T behaves as a finite-rank operator. Thus, the image of T is finite-dimensional, and T behaves as a finite-rank operator on this subspace. □

Theorem 9. *If T is a Hilbert-Schmidt operator with singular values $\sigma_i(T)$, then T is norm-attainable if and only if there exists an index i such that $\sigma_i(T) = \| T \|$.*

Proof: Let $T \in H_2$, the Hilbert-Schmidt class, and suppose that T is norm-attainable. This means that there exists a vector $x \in H$ such that $\| T(x) \| = \| T \|$. The singular values of T form a non-increasing sequence $\{\sigma_i(T)\}$ that converges to zero. For T to be norm-attainable, there must exist an index i such that the largest singular value satisfies $\sigma_i(T) = \| T \|$, as the operator norm is equal to the largest singular value of a Hilbert-Schmidt operator. Conversely, if there exists an index i such that $\sigma_i(T) = \| T \|$, then there exists a vector $x \in H$ such that $T(x) = \| T \| x$, and thus $\| T(x) \| = \| T \|$, implying that T is norm-attainable. Thus, T is norm-attainable if and only if there exists an index i such that $\sigma_i(T) = \| T \|$. □

Theorem 10. *Let $T \in S_p$, the Schatten p -class of operators. If T is norm-attainable, then the singular values of T must decay rapidly enough such that the sum $\sum_{i=1}^{\infty} \sigma_i (T)^p$ converges.*

Proof: Let $T \in S_p$, the Schatten p -class, and suppose that T is norm-attainable, meaning there exists a vector $x \in H$ such that $\| T(x) \| = \| T \|$. The singular values $\sigma_i(T)$ of T form a sequence that decays to zero. For T to be in the Schatten p-class, the series $\sum_{i=1}^{\infty} \sigma_i (T)^p$ must converge. Since T is norm-attainable, the operator norm $\| T \|$ is equal to the largest singular value. For the singular values of T to decay rapidly enough for T to be in S_p, it is necessary that the sum $\sum_{i=1}^{\infty} \sigma_i (T)^p$ converges. This convergence ensures that the singular values decay sufficiently fast for the operator to remain in the Schatten p -class. Thus, if T is norm-attainable, the singular values of T must decay rapidly enough such that the sum $\sum_{i=1}^{\infty} \sigma_i (T)^p$ converges. □

12. Conclusion

This research explores the norm-attainability of operators in various operator ideals such as compact, weakly compact, trace-class, and Hilbert-Schmidt operators. It establishes key connections between singular value decay and norm-attainment, offering conditions under which norm-attainment occurs or fails. The findings indicate that compact operators with rapidly decaying singular values may not be norm-attainable, while weakly compact and Hilbert-Schmidt operators have more specific conditions. The research lays a foundation for future studies in broader contexts, such as non-Hilbert spaces and unbounded operators, with implications for fields like quantum mechanics, functional analysis, and numerical methods.

Acknowledgements. The author expresses sincere gratitude to the anonymous referee for their valuable comments, constructive feedback, and insightful suggestions, which have significantly contributed to improving the clarity, quality, and overall presentation of this paper.

Author's Contributions: All authors have contributed equally to the research and preparation of this work.

Conflicts of interest. The authors declare no conflicts of interest.

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