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Convergence and Reduction of Transitive Neutrosophic Soft Matrices

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Abstract. Convergence of Neutrosophic Soft Matrices ($\mathcal{CoNeSoMas}$) is examined under the max-max-min operation on $\mathcal{NeSoMa}$. A Transitive Neutrosophic Soft Matrix ($\mathcal{TrNeSoMa}$) represents a transitive neutrosophic soft relation and possesses many interesting properties. The conditions for the convergence of neutrosophic soft matrices are explored under a special operation essential for reducing $\mathcal{NeSoMas}$. The given results demonstrate graph-theoretic properties of $\mathcal{NeSoMas}$, which are useful when considering the reduction of systems represented by $\mathcal{NeSoMa}$. Moreover, we establish the reduction of a neutrosophic information retrieval system and provide a relevant example.

Keywords: Neutrosophic soft matrix, transitive neutrosophic soft matrices, boolean neutrosophic soft matrix, reduction of transitive neutrosophic soft matrix, transitive neutrosophic soft relation, nilpotent neutrosophic soft matrix

AMS Mathematics Subject Classification (2010): 03E72, 15B15

1. Introduction

Since 1995, many mathematicians and researchers across various scientific fields have been studying and trying to understand neutrosophic theory. The first mathematician to develop and introduce netrosophic theory was Smarandache 2005 [24], similar to how Zadeh [29] introduced fuzzy theory and Atanasov [1] introduced Intuitionistic Fuzzy Theory ($\mathcal{InFuTh}$). Neutrosophic logic is important because it can handle the indeterminacy component (I), which allows scholars to generalize fuzzy and intuitionistic fuzzy logics. This capability enables them to place paradoxes in a new framework and makes it easier for researchers to work with contradictory information.

The theory of Soft Sets ($\mathcal{SoSe}$) has excellent potential for application in various fields, as reported by Molodtsov [21] in his pioneering work. Later, Maji et al., [20] introduced new concepts intuitionistic fuzzy soft sets, such as subset, complement, union, and intersection, and discussed in detail their applications in decision-making problems. Maji [19] also introduced the concept of neutrosophic soft sets and established some operations on these sets.

The powers of a special fuzzy matrix produce a Fuzzy Transitive Relation ($\mathcal{FuTrRe}$). Fuzzy Transitive Matrices ($\mathcal{FuTrMas}$) are used in many applications. We present a certain property of a Transitive Matrix ($\mathcal{TrMa}$) and demonstrate some conditions for convergence. Thomason has already discussed the conditions for convergence [25] however, he did not address $\mathcal{TrMas}$. $\mathcal{FuTrRe}$ play an important role in clustering, information retrieval, preference, and more [22 28,26]. Furthermore, the importance of transitive relations in the discussion of fuzzy orderings is significant [28, 30]. Hashimoto[7,8,9] introduced the more concepts based on fuzzy matrices such that reduction of a fuzzy retrieval model, reduction of a nilpotent fuzzy matrix, transitive reduction of a nilpotent boolean matrix so on. M.Pal [31,32] proposed the concept of recent developments of fuzzy matrix theory and applications and also gave the idea of neutrosophic matrix and neutrosophic fuzzy matrix. AK Adak etal. [33,34,35,36] Introduced the Intuitionistic fuzzy block matrix and its some properties. They also presented some properties of generalized intuitionistic fuzzy nilpotent matrices over distributive lattice and an intuitionistic fuzzy matrices and intuitionistic circulant fuzzy matrices. Rajkumar et al.[37] Estabilished the intuitionistic fuzzy linear transformations

Rajarajeswari and Dhanalakshmi [23] introduced the $\mathcal{InFuNeSoMa}$ and applied it in the field of medical diagnosis. Broumi and Smarandache [4,5] introduced the concepts are generalized interval neutrosophic soft set and intuitionistic neutrosophic soft set. Arockiarani and Sumathi [2, 3] presented new operations on Fuzzy Neutrosophic Soft Matrices. Kavitha and Murugadas [10,11,12,13,14] introduced concepts such as convergence, eigenvectors of circulant matrices, and monotone eigenspace structures using the idea of $\mathcal{FuNeSoMas}$. Additionally, they presented many results on combined forecasting method based on $\mathcal{FuNeSoMas}$ in [16,17,18]. Uma et al. [27] introduced type I and II $\mathcal{FuNeSoMa}$.

The purpose of this research is to determine various unprecedented mathematical formulas, such as pre-distinguishing matrices, pre-facilitation matrices, and a new approach to separate the $\mathcal{NeSoMa}$ model into two theorems based on the terms of powers in the $\mathcal{TrNeSoMa}$. Additionally, we present a technique to use the Reduction Transitive Neutrosophic Soft Relation ($\mathcal{ReTrNeSoRel}$). We have already discussed many theoretical and application aspects of these matrices. In the sequel, we demonstrate several basic properties of the reduction schemes formulated for max-max-min $\mathcal{TrNeSoMa}$. We also provide examples to support the theoretical content of this paper. Henceforth, we will simply refer to $\mathcal{NeSoMa}$ as a fuzzy matrix.

2. Preliminaries

Refer to [10-14,16-18] for the basic definitions and examples of Neutrosophic Set ($\mathcal{NeSe}$), $\mathcal{FuNeSoSe}$, $\mathcal{FuNeSoMa}$, and fuzzy neutrosophic soft matrices of type-I II.

3. Framework of this concept

For any square neutrosophic soft matrices $\mathcal{R} = \langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle$ and $\mathcal{P} = \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle$ with their values in the unit interval $[\langle 0,0,1 \rangle, \langle 1,1,0 \rangle]$.
We define the following notations:

• $\mathcal{R} \wedge \mathcal{P} = (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \wedge (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle)$,

• $\mathcal{R} \times \mathcal{P} = [(\langle T_{r_{i1}}, I_{r_{i1}}, F_{r_{i1}} \rangle \wedge \langle T_{p_{1j}}, I_{p_{1j}}, F_{p_{1j}} \rangle) \vee (\langle T_{r_{i2}}, I_{r_{i2}}, F_{r_{i2}} \rangle) \wedge (\langle T_{p_{2j}}, I_{p_{2j}}, F_{p_{2j}} \rangle)$

$\vee \ldots \vee (\langle T_{r_{in}}, I_{r_{in}}, F_{r_{in}} \rangle \wedge \langle T_{p_{nj}}, I_{p_{nj}}, F_{p_{nj}} \rangle)]$

• $\mathcal{R} \ominus \mathcal{P} = [\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle \ominus \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle]$,

• $\mathcal{R}^0 = \mathcal{I} = [\langle T_{\delta_{ij}}, I_{\delta_{ij}}, F_{\delta_{ij}} \rangle$ is the Kronecer delta],

• $\mathcal{R}^{m+1} = \langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle^m \times \langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle, m = 0,1,2, \ldots$,

• $\mathcal{R} \leq \mathcal{P}$ $(\mathcal{P} \geq \mathcal{R})$ iff $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle \leq \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle$

• The operations $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \vee (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle)$ and $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \wedge (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle)$ in notations are defined by $\max(T_{r_{ij}} T_{p_{ij}})$ $\max(I_{r_{ij}} I_{p_{ij}})$ $\min(F_{r_{ij}} F_{p_{ij}})$ and
$\min(T_{r_{ij}} T_{p_{ij}})$ $\min(I_{r_{ij}} I_{p_{ij}})$ $\max(F_{r_{ij}} F_{p_{ij}})$, respectivly.

• Moreover, the operation $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle \ominus \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle$

$$\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle \ominus \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle = \begin{cases} \langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle & if \ \langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle > \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle \\ \langle 0,0,1 \rangle & if \ \langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle \leq \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}} \rangle \end{cases}$$

• We deal only $\mathcal{SqNeSoMas}$. If a $\mathcal{SqNeSoMa}$ $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}} \rangle$ is called transitive iff $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^2 \leq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)$. This $\mathcal{NeSoMa}$ represents a neutrosophic soft relation. This definition is most basic and seems to be comparable when $\mathcal{NeSoMas}$ are generalized to certain $\mathcal{NeSoMas}$ over other neutrosophic soft algebra.

• If $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \leq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^2$, then we have
$(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \leq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^2 \leq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \ldots$.
The $\mathcal{NeSoMa}$ $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \ni (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \leq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^2$
it is called campact. A $\mathcal{TrNeSoRe}$ is very curial as well as a compact relation.

• $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)$ is $\mathcal{TrNeSoMa}$ then we have
$(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \geq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^2 \geq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^3 \geq \ldots$ In the sequence of powers of a $\mathcal{NeSoMa}$ $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)$ if $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^m = (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^{m+1}$ for some positive integer m, then $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)$ is called reduction of transitive.

• If $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)$ is $\mathcal{SqNeSoMa}$ $\ni$ $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^n = \langle 0,0,1 \rangle$ then
$(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \ominus (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^2 \vee (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^3 \vee \ldots \vee (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^{n-1}$
$= (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \ominus (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \times (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^+$ is considered to be a $\mathcal{TrNeSoRe}$ of $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)$, where
$(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^+ = (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \vee (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^2 \vee \ldots \vee (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)^n$ is transitive closure.

• $\mathcal{A} \circ \mathcal{B} = \bigvee_{k=1}^{n} [(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \wedge (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)]$;

• $(\mathcal{A}/\mathcal{R}) = (\langle T_{a_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle) \ominus (\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}} \rangle) \circ (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}} \rangle)$;

$\mathcal{A}//(\mathcal{S},\mathcal{R}) = (\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle) \ominus \Big(\big(\langle T_{s_{ij}}, I_{s_{ij}}, F_{r_{ij}}\rangle\big) \circ \big(\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle\big) \circ \big(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle\big)\Big)$

• $\mathcal{A} \prec \mathcal{B}$ iff $\forall i, j \ni \langle T_{b_{ij}}, I_{b_{ij}}, F_{b_{ij}}\rangle = \langle 0,0,1\rangle$, then $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle = \langle 0,0,1\rangle$

• $\mathcal{A} \approx \mathcal{B}$ iff $(T_{a_{ij}} \prec T_{b_{ij}}); (I_{a_{ij}} \prec I_{b_{ij}}); (F_{a_{ij}} \succ F_{b_{ij}})$ and $(T_{b_{ij}} \prec T_{a_{ij}}); (I_{b_{ij}} \prec I_{a_{ij}}); (T_{b_{ij}} \succ T_{a_{ij}})$.

• We remember that a $\mathcal{N}e\mathcal{S}o\mathcal{M}a_{m\times n}$ $\langle T_{r_{ii}}, I_{r_{ii}}, F_{r_{ii}}\rangle$ is reflexive iff $\langle T_{r_{ii}}, I_{r_{ii}}, F_{r_{ii}}\rangle = \langle 1,1,0\rangle\ \forall i$, irreflexive iff $\langle T_{r_{ii}}, I_{r_{ii}}, F_{r_{ii}}\rangle = \langle 0,0,1\rangle \forall i$ (perfectly) antisymmetric iff $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle > \langle 0,0,1\rangle \Rightarrow \langle T_{r_{ji}}, I_{r_{ji}}, F_{r_{ji}}\rangle = \langle 0,0,1\rangle \forall\ i, j$ and $i \neq j$.

• $\mathcal{N}i\mathcal{N}e\mathcal{S}o\mathcal{M}a$ iff $(\langle T_{r_{ji}}, I_{r_{ji}}, F_{r_{ji}}\rangle)^n = \langle 0,0,1\rangle$ and max-max-min $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}as$ iff $\langle T_{r_{ji}}, I_{r_{ji}}, F_{r_{ji}}\rangle)^2 \leq (\langle T_{r_{ji}}, I_{r_{ji}}, F_{r_{ji}}\rangle)$.

4. Convergence of transitive neutrosophic soft matrix

We show some important properties of $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}as$. Then we give some rules for reduction under max-max-min operations. These results are useful when we consider various systems with $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}as$. In following, $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle), (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle), (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{(m)}$ and $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{(m)}$ represent the n^n power of the given $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}as$.

Theorem 4.1. *If $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ is $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}a$ of order $n \times n$ then*

$$(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \ominus (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \times (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^n$$
$$= (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \ominus (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \times (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{n+1}$$

for any $\mathcal{S}q\mathcal{N}e\mathcal{S}o\mathcal{M}a$ $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)$.

Proof: Let $(\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle) = (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \ominus (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \times (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)$ and $(\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle)^{(m)}$.

That is, $(\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle) = (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \ominus \bigvee_{m=1}^{n} (\langle T_{r_{ik}}, I_{r_{ik}}, F_{r_{ik}}\rangle) \wedge (\langle T_{p_{kj}}, I_{p_{kj}}, F_{p_{kj}}\rangle)$.

Case 1: Suppose that there exist indices $l_1, l_2, \ldots, l_{n-1} \ni$

$$(\langle T_{s_{il_1}}, I_{s_{il_1}}, F_{s_{il_1}}\rangle) \wedge (\langle T_{s_{l_1 l_2}}, I_{s_{l_1 l_2}}, F_{s_{l_1 l_2}}\rangle) \wedge \ldots \wedge (\langle T_{s_{l_{n-1}j}}, I_{s_{l_{n-1}j}}, F_{s_{l_{n-1}j}}\rangle) = (\langle T_g, I_g, F_g\rangle) > (\langle 0,0,1\rangle).$$

Let $l_0 = i$ and $l_n = j$. Then $l_a = l_b$ for some a and b $b(a < b)$. We define $(\langle T_h, I_h, F_h\rangle)$ by $(\langle T_h, I_h, F_h\rangle) = (\langle T_{r_{l_{m-1}l_m}}, I_{r_{l_{m-1}l_m}}, F_{r_{l_{m-1}l_m}}\rangle) = (\langle T_{r_{l_a l_{a+1}}}, I_{r_{l_a l_{a+1}}}, F_{r_{l_a l_{a+1}}}\rangle) \wedge (\langle T_{r_{l_{a+1}l_{a+2}}}, I_{r_{l_{a+1}l_{a+2}}}, F_{r_{l_{a+1}l_{a+2}}}\rangle) \wedge \ldots \wedge (\langle T_{r_{l_{b-1}l_b}}, I_{r_{l_{b-1}l_b}}, F_{r_{l_{b-1}l_b}}\rangle)$ where $a < m \leq b$, so that $\langle T_h, I_h, F_h\rangle = (\langle T_{r_{l_{m-1}l_m}}, I_{r_{l_{m-1}l_m}}, F_{r_{l_{m-1}l_m}}\rangle) > \bigvee_{k=1}^{n} (\langle T_{r_{l_{m-1}l_k}}, I_{r_{l_{m-1}l_k}}, F_{r_{l_{m-1}l_k}}\rangle) \wedge (\langle T_{p_{kl_m}}, I_{p_{kl_m}}, F_{p_{kl_m}}\rangle)$. If $(\langle T_{r_{l_m l_m}}, I_{r_{l_m l_1}}, F_{r_{l_m l_1}}\rangle) \leq \bigvee_{k=1}^{n} (\langle T_{r_{l_m k}}, I_{r_{l_m k}}, F_{r_{l_m k}}\rangle) \wedge (\langle T_{p_{kl_m}}, I_{p_{kl_m}}, F_{p_{kl_m}}\rangle)$, then $(\langle T_h, I_h, F_h\rangle) \leq (\langle T_{r_{l_m l_m}}, I_{r_{l_m l_m}}, F_{r_{l_m l_m}}\rangle) \leq (\langle T_{r_{l_m k_1}}, I_{r_{l_m k_1}}, F_{r_{l_m k_1}}\rangle) \wedge (\langle T_{p_{k_1 l_m}}, I_{p_{k_1 l_m}}, F_{p_{k_1 l_m}}\rangle)$ for some k_1. Since $(\langle T_{r_{l_{m-1}l_m}}, I_{r_{l_{m-1}l_m}}, F_{r_{l_{m-1}l_m}}\rangle) = (\langle T_h, I_h, F_h\rangle)$ we have

$$(\langle T_{r_{l_{m-1}k_1}}, I_{r_{l_{m-1}k_1}}, F_{r_{l_{m-1}k_1}}\rangle) \geq (\langle T_{r_{l_{m-1}l_m}}, I_{r_{l_{m-1}l_m}}, F_{r_{l_{m-1}l_m}}\rangle) \wedge (\langle T_{r_{l_{m-1}k_1}}, I_{r_{l_{m-1}k_1}}, F_{r_{l_{m-1}k_1}}\rangle) = (\langle T_h, I_h, F_h\rangle).$$

Thus $\bigvee_{k=1}^{n} (\langle T_{r_{l_{m-1}k}}, I_{r_{l_{m-1}k}}, F_{r_{l_{m-1}k}}\rangle) \wedge (\langle T_{p_{kl_m}}, I_{p_{kl_m}}, F_{p_{kl_m}}\rangle)$
$\geq (\langle T_{r_{l_{m-1}k_1}}, I_{r_{l_{m-1}k_1}}, F_{r_{l_{m-1}k_1}}\rangle) \wedge (\langle T_{p_{k_1l_m}}, I_{p_{k_1l_m}}, F_{p_{k_1l_m}}\rangle) \geq (\langle T_h, I_h, F_h\rangle)$,
which is a contradiction. Therefore

$$(\langle T_{r_{l_ml_m}}, I_{r_{l_ml_m}}, F_{r_{l_ml_m}}\rangle) > \bigvee_{k=1}^{n} (\langle T_{r_{l_mk}}, I_{r_{l_mk}}, F_{r_{l_mk}}\rangle) \wedge (\langle T_{p_{kl_m}}, I_{p_{kl_m}}, F_{p_{kl_m}}\rangle).$$

Hence $(\langle T_{s_{l_ml_m}}, I_{s_{l_ml_m}}, F_{s_{l_ml_m}}\rangle) \geq (\langle T_h, I_h, F_h\rangle) \geq (\langle T_g, I_g, F_g\rangle)$,
so that $(\langle T_{s_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{(n+1)} \geq (\langle T_g, I_g, F_g\rangle)$.

Case 2: Suppose that there exist indices $l_1, l_2, \ldots, l_n \ni$
$(\langle T_{sil_1}, I_{s_{ii_1}}, F_{s_{il_1}}\rangle) \wedge (\langle T_{s_{l_1l_2}}, I_{s_{l_1l_2}}, F_{s_{l_1l_2}}\rangle) \wedge\ldots\wedge (\langle T_{s_{l_nj}}, I_{s_{l_nj}}, F_{s_{l_nj}}\rangle) = (\langle T_g, I_g, F_g\rangle) >$
$(\langle 0,0,1\rangle)$. Let $l_0 = i$ and $l_{n+1} = j$.
(a) If $l_a = l_b = l_c$ where $a < b < c$, then we have
$(\langle T_{s_{l_ml_m}}, s_{r_{l_ml_m}}, F_{s_{l_ml_m}}\rangle) \geq (\langle T_g, I_g, F_g\rangle)$, $a < m \leq b$ for some l_m. Thus
$(\langle T_{s_{il_m}}, I_{s_{il_m}}, F_{s_{il_m}}\rangle)^{(m)} \wedge (\langle T_{s_{l_ml_m}}, I_{s_{l_ml_m}}, F_{s_{l_ml_m}}\rangle)^{(c-b-1)} \wedge (\langle T_{s_{l_ml_b}}, I_{s_{l_ml_b}}, F_{s_{l_ml_b}}\rangle)^{b-m}$
$\wedge (\langle T_{s_{lc_j}}, I_{r_{lc_j}}, F_{r_{l_cj}}\rangle)^{(n+1-c)} \geq (\langle T_g, I_g, F_g\rangle)$. Hence $(\langle T_{s_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{(n)} \geq (\langle T_g, I_g, F_g\rangle)$.
(b). Suppose that $l_a = l_b$ and $l_c = l_d$.
(i). If $a < b < c < d$, then $(\langle T_{s_{l_ml_m}}, I_{s_{l_ml_m}}, F_{s_{l_ml_m}}\rangle) \geq (\langle T_g, I_g, F_g\rangle)$, $a < m \leq b$, for some l_m. Thus $(\langle T_{s_{il_m}}, I_{s_{il_m}}, F_{s_{il_m}}\rangle)^{(m)} \wedge (\langle T_{s_{l_ml_m}}, I_{s_{l_ml_m}}, F_{s_{l_ml_m}}\rangle)^{(d-c-1)}$

$$\wedge (\langle T_{s_{l_ml_c}}, I_{s_{l_ml_c}}, F_{s_{l_ml_c}}\rangle)^{(c-m)} \wedge (\langle T_{s_{l_dj}}, I_{s_{l_dj}}, F_{s_{l_dj}}\rangle)^{(n+1-d)} \geq (\langle T_g, I_g, F_g\rangle).$$

Hence $(\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle)^{(n)} \geq (\langle T_g, I_g, F_g\rangle)$.
(ii). If $a < c < b < d$, then

$$(\langle T_{s_{l_ml_m}}, I_{s_{l_ml_m}}, F_{s_{l_ml_m}}\rangle) \geq (\langle T_h, I_h, F_h\rangle) \geq (\langle T_g, I_g, F_g\rangle), \qquad a < m \leq b,$$

for some l_m, where

$$(\langle T_h, I_h, F_h\rangle) = (\langle T_{r_{l_{m-1}l_m}}, I_{r_{l_{m-1}l_m}}, F_{r_{l_{m-1}l_m}}\rangle) = (\langle T_{r_{l_al_{a+1}}}, I_{r_{l_al_{a+1}}}, F_{r_{l_al_{a+1}}}\rangle) \wedge\ldots\wedge (\langle T_{r_{l_{b-1}l_b}}, I_{r_{l_{b-1}l_b}}, F_{r_{l_{b-1}l_b}}\rangle).$$

Since it is clear that $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{(n)} \geq (\langle T_g, I_g, F_g\rangle)$
for $m \leq c$, suppose that $m > c$.
If $\langle T_{r_{l_al_m}}, I_{r_{l_al_m}}, F_{r_{l_al_m}}\rangle \leq \bigvee_{k=1}^{n} (\langle T_{r_{l_ak}}, I_{r_{l_ak}}, F_{r_{l_ak}}\rangle) \wedge (\langle T_{p_{kl_m}}, I_{p_{kl_m}}, F_{p_{kl_m}}\rangle)$,
then $(\langle T_g, I_g, F_g\rangle) \leq (\langle T_h, I_h, F_h\rangle) \leq (\langle T_{r_{l_ak_1}}, I_{r_{l_ak_1}}, F_{r_{l_ak_1}}\rangle) \wedge (\langle T_{p_{k_1l_m}}, I_{p_{k_1l_m}}, F_{p_{k_1l_m}}\rangle)$
$\geq (\langle T_h, I_h, F_h\rangle)$, which contradicts the fact that

$$(\langle T_h, I_h, F_h\rangle) = (\langle T_{s_{l_{m-1}l_m}}, I_{s_{l_{m-1}l_m}}, F_{s_{l_{m-1}l_m}}\rangle) > (\langle 0,0,1\rangle).$$

Hence $(\langle T_{s_{l_al_m}}, I_{s_{l_al_m}}, F_{s_{l_al_m}}\rangle) \geq (\langle T_g, I_g, F_g\rangle)$, so that

$$(\langle T_{s_{il_a}}, I_{s_{il_a}}, F_{s_{il_m}}\rangle)^{(a)} \wedge (\langle T_{s_{l_sl_m}}, I_{s_{l_al_m}}, F_{s_{l_al_m}}\rangle) \wedge (\langle T_{s_{l_ml_m}}, I_{s_{l_ml_m}}, F_{s_{l_ml_m}}\rangle)^{(m-a-2)}$$

$\wedge (\langle T_{s_{l_mj}}, I_{s_{l_mj}}, F_{r_{l_mj}}\rangle)^{(n+1-m)} \geq (\langle T_g, I_g, F_g\rangle)$.

Example 4.2. $\mathcal{R} = \begin{bmatrix} \langle 0.7,0.6,0.4\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0.2,0.1,0.8\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \end{bmatrix}$

$$\mathcal{P} = \begin{bmatrix} \langle 0.4,0.3,0.6\rangle & \langle 0,0,1\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.5,0.4,0.5\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 0.2,0.1,0.8\rangle & \langle 0.2,0.1,0.8\rangle \end{bmatrix}$$

$$\mathcal{R}^2 = \begin{bmatrix} \langle 0.7,0.6,0.4\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0.2,0.1,0.8\rangle & \langle 0.2,0.1,0.8\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \end{bmatrix} \leq \mathcal{R},$$

which means that $\mathcal{R}$ is transitive.

Next we compute $\mathcal{S} = \mathcal{R} \ominus (\mathcal{R} \times \mathcal{P}), \mathcal{S}^2$, and $\mathcal{S}^3$. We have

$$\mathcal{R} \times \mathcal{P} = \begin{bmatrix} \langle 0.4,0.3,0.6\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.2,0.1,0.8\rangle & \langle 0.2,0.1,0.8\rangle & \langle 0.2,0.1,0.8\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \end{bmatrix} \leq \mathcal{R},$$

$$\mathcal{S} = \mathcal{R} \ominus (\mathcal{R} \times \mathcal{P}) = \begin{bmatrix} \langle 0.7,0.6,0.4\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \end{bmatrix} \leq \mathcal{R},$$

$$\mathcal{S}^2 = \begin{bmatrix} \langle 0.7,0.6,0.4\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \end{bmatrix} \leq \mathcal{R},$$

$$\mathcal{S}^3 = \begin{bmatrix} \langle 0.7,0.6,0.4\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \end{bmatrix} \leq \mathcal{S}^2,$$

Thus we have $\mathcal{S}^3 = \mathcal{S}^4$

By Theorem 4.1, we obtain the following two corollaries.

Corollary 4.3. *If* $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ *is* $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}a_{(n\times n)}$ *then*

$$(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \ominus (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle) \times (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^n$$
$$= (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \ominus (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle) \times (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{n+1}$$

for any $\mathcal{S}q\mathcal{N}e\mathcal{S}o\mathcal{M}a$ $\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle$.

Corollary 4.4. *If* $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ *is* $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}a_{(n\times n)}$ *then*

$$(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^n = (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{n+1}.$$

We now consider conditions under which an $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}a_{(n\times n)}$ $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ satisfies the relationship $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{n-1} = (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^n$, where $n \geq 2$.

Theorem 4.5. *Let* $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ *be* $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}as_{(n\times n)}$.

If $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \wedge (\langle T_{\mathcal{J}_{ij}}, I_{\mathcal{J}_{ij}}, F_{\mathcal{J}_{ij}}\rangle) \leq (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle) \leq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ and $\bigvee_{i=1}^{n} (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \vee (\langle T_{r_{ji}}, I_{r_{ji}}, F_{r_{ji}}\rangle) \leq (\langle T_{r_{jj}}, I_{r_{jj}}, F_{r_{jj}}\rangle)$ for some j,

then $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{n-1} = (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^n$.

Proof: (1) First we show that $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{n-1} \leq (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^n$. Suppose that $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{n-1} = (\langle T_c, I_c, F_c\rangle) > \langle 0,0,1\rangle$. Then there exist indices $k_1, k_2, \ldots, k_{n-2}$ $\ni$

$(\langle T_{p_{ik_1}}, I_{p_{ik_1}}, F_{p_{ik_1}}\rangle) \wedge (\langle T_{p_{k_1k_2}}, I_{p_{k_{1_2}}}, F_{p_{k_1k_2}}\rangle) \wedge \ldots \wedge (\langle T_{p_{k_{n-2}j}}, I_{p_{k_{n-2}j}}, F_{p_{k_{n-2}j}}\rangle) =$ $(\langle T_c, I_c, F_c\rangle)$, so that

$$(\langle T_{r_{ik_1}}, I_{r_{ik_1}}, F_{r_{ik_1}}\rangle) \wedge (\langle T_{r_{k_1k_2}}, I_{r_{k_1k_2}}, F_{r_{k_1k_2}}\rangle) \wedge \ldots \wedge (\langle T_{r_{k_{n-2}j}}, I_{r_{k_{n-2}j}}, F_{r_{k_{n-2}j}}\rangle) \geq (\langle T_c, I_c, F_c\rangle).$$

Let $k_0 = i$ and $k_{n-1} = j$.

(a). If $k_a = k_b$ for some a an

Then,sothat $\langle T_{r_{k_ak_b}}, I_{r_{k_ak_b}}, F_{r_{k_ak_b}}\rangle^{(b-a)} \geq (\langle T_c, I_c, F_c\rangle)$; $(\langle T_{r_{k_ak_b}}, I_{r_{k_ak_b}}, F_{r_{k_ak_b}}\rangle) \geq (\langle T_c, I_c, F_c\rangle)$,

$(\langle T_{p_{k_ak_b}}, I_{p_{k_ak_b}}, F_{p_{k_ak_b}}\rangle) \geq (\langle T_c, I_c, F_c\rangle)$.

Thus $(\langle T_{p_{ik_1}}, I_{p_{ik_1}}, F_{p_{ik_1}}\rangle) \wedge (\langle T_{p_{k_{1_2}}}, I_{p_{k_1k_2}}, F_{p_{k_1k_2}}\rangle) \wedge \ldots \wedge (\langle T_{p_{k_{a-1}k_a}}, I_{p_{k_{a-1}k_a}}, F_{p_{k_{a-1}k_a}}\rangle)$

$$\wedge (\langle T_{p_{k_ak_a}}, I_{p_{k_ak_a}}, F_{p_{k_ak_a}}\rangle) \wedge (\langle T_{p_{k_ak_{a+1}}}, I_{p_{k_ak_{a+1}}}, F_{p_{k_ak_{a+1}}}\rangle) \wedge \ldots \wedge (\langle T_{p_{k_{n-2}j}}, I_{p_{k_{n-2}j}}, F_{p_{k_{n-2}j}}\rangle) \geq (\langle T_c, I_c, F_c\rangle).$$

Hence $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{(n)} \geq (\langle T_c, I_c, F_c\rangle)$.

(b). Suppose that $k_a \neq k_b \; \forall \; a \neq b$. By hypothesis

$$\bigvee_{l=1}^{n} (\langle T_{r_{lk_m}}, I_{r_{lk_m}}, F_{r_{lk_m}}\rangle) \vee (\langle T_{r_{k_ml}}, I_{r_{k_ml}}, F_{r_{k_ml}}\rangle) \leq (\langle T_{r_{k_mk_m}}, I_{r_{k_mk_m}}, F_{r_{k_mk_m}}\rangle)$$

for some m. Then $(\langle T_{r_{k_mk_m}}, I_{r_{k_mk_m}}, F_{r_{k_mk_m}}\rangle) \geq (\langle T_c, I_c, F_c\rangle)$;

Thus $(\langle T_{p_{ik_1}}, I_{p_{ik_1}}, F_{p_{ik_1}}\rangle) \wedge (\langle T_{p_{k_{1_2}}}, I_{p_{k_1k_2}}, F_{p_{k_1k_2}}\rangle) \wedge \ldots \wedge$ $(\langle T_{p_{k_{m-1}k_m}}, I_{p_{k_{m-1}k_m}}, F_{p_{k_{m-1}k_m}}\rangle)$

$\wedge (\langle T_{p_{k_mk_m}}, I_{p_{k_mk_m}}, F_{p_{k_mk_m}}\rangle) \wedge (\langle T_{p_{k_mk_{m+1}}}, I_{p_{k_mk_{m+1}}}, F_{p_{k_mk_{m+1}}}\rangle) \wedge \ldots \wedge$ $(\langle T_{p_{k_{n-2}j}}, I_{p_{k_{n-2}j}}, F_{p_{k_{n-2}j}}\rangle) \geq (\langle T_c, I_c, F_c\rangle)$. Hence $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{(n)} \geq (\langle T_c, I_c, F_c\rangle)$.

(2) Next we show that $(\langle T_{p_{k_{n-2}j}}, I_{p_{k_{n-2}j}}, F_{p_{k_{n-2}j}}\rangle)^n \leq (\langle T_{p_{k_{n-2}j}}, I_{p_{k_{n-2}j}}, F_{p_{k_{n-2}j}}\rangle)^{n-1}$.

Let $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{(n)} = (\langle T_c, I_c, F_c\rangle) > 0$. Then there exist indices $k_1, k_2, \ldots, k_{n-1}$

$$(\langle T_{p_{ik_1}}, I_{p_{ik_1}}, F_{ip_{k_1}}\rangle) \wedge (\langle T_{p_{k_1k_2}}, I_{p_{k_1k_2}}, F_{p_{k_1k_2}}\rangle) \wedge \ldots \wedge (\langle T_{p_{k_{n-1}j}}, I_{p_{k_{n-1}j}}, F_{p_{k_{n-1}j}}\rangle) = (\langle T_c, I_c, F_c\rangle).$$

Let $k_0 = i$ and $k_n = j$. Then $k_a = k_b$ for some a and b $b(b < a)$.

Thus $(\langle T_{p_{k_ak_a}}, I_{p_{k_ak_a}}, F_{p_{k_ak_a}}\rangle)^{b-a} \geq (\langle T_c, I_c, F_c\rangle)$, so that

$(\langle T_{r_{k_ak_a}}, I_{r_{k_ak_a}}, F_{r_{k_ak_a}}\rangle)^{b-a} \geq (\langle T_c, I_c, F_c\rangle)$, $(\langle T_{r_{k_ak_a}}, I_{r_{k_ak_a}}, F_{r_{k_ak_a}}\rangle) \geq (\langle T_c, I_c, F_c\rangle)$,

$(\langle T_{p_{k_ak_a}}, I_{p_{k_ak_a}}, F_{p_{k_ak_a}}\rangle) \geq (\langle T_c, I_c, F_c\rangle)$. Therefore

$$(\langle T_{p_{ik_1}}, I_{p_{ik_1}}, F_{p_{ik_1}}\rangle) \wedge (\langle T_{p_{k_1k_2}}, I_{p_{k_1k_2}}, F_{p_{k_1k_2}}\rangle) \wedge \ldots \wedge (\langle T_{p_{k_{a-1}k_a}}, I_{p_{k_{a-1}k_a}}, F_{p_{k_{a-1}k_a}}\rangle) \wedge (\langle T_{p_{k_ak_a}}, I_{p_{k_ak_a}}, F_{p_{k_ak_a}}\rangle)^{b-a-1} \wedge (\langle T_{p_{k_bk_{b+1}}}, I_{p_{k_bk_{b+1}}}, F_{p_{k_bk_{b+1}}}\rangle) \wedge \ldots \wedge$$

$(\langle T_{p_{k_{n-1}j}}, I_{p_{k_{n-1}j}}, F_{p_{k_{n-1}j}}\rangle) \geq (\langle T_c, I_c, F_c\rangle)$. Hence $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{n-1} \geq (\langle T_c, I_c, F_c\rangle)$.

Example 4.6. $\mathcal{R} = \begin{bmatrix} \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0.4,0.5,0.6\rangle & \langle 0.5,0.4,0.5\rangle \end{bmatrix}$,

and $\mathcal{P} = \begin{bmatrix} \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.4,0.3,0.6\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0.5,0.4,0.5\rangle \end{bmatrix}$,

$$\mathcal{R}^2 = \begin{bmatrix} \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0.4,0.5,0.6\rangle & \langle 0.5,0.4,0.5\rangle \end{bmatrix} \leq \mathcal{R}$$

$$\mathcal{R}^3 = \begin{bmatrix} \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.5,0.4,0.5\rangle \\ \langle 0,0,1\rangle & \langle 0.4,0.5,0.6\rangle & \langle 0.5,0.4,0.5\rangle \end{bmatrix} \leq \mathcal{R}^2$$

$$\mathcal{P}^2 = \begin{bmatrix} \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.4,0.3,0.6\rangle \\ \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.4,0.3,0.6\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0.5,0.4,0.5\rangle \end{bmatrix},$$

$$\mathcal{P}^3 = \begin{bmatrix} \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.4,0.3,0.6\rangle \\ \langle 0,0,1\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.4,0.3,0.6\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0.5,0.4,0.5\rangle \end{bmatrix} = \mathcal{P}^2.$$

Similarly we have the following theorem.

Theorem 4.7. *Let* $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ *be a* $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}as_{(n\times n)}$.
If $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \wedge (\langle T_{\mathcal{I}_{ij}}, I_{\mathcal{I}_{ij}}, F_{\mathcal{I}_{ij}}\rangle) \leq (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle) \leq (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$
and $\bigvee_{i=1}^{n} (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle) \vee (\langle T_{r_{ji}}, I_{r_{ji}}, F_{r_{ji}}\rangle) \leq (\langle T_{p_{jj}}, I_{p_{jj}}, F_{p_{jj}}\rangle)$ for some j,
then $(\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{n-1} = (\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)^{n}$. As a special case of Theorem 4.5 or Theorem 4.7, we obtain the following corollary.

Corollary 4.8. *Let* $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ *be a* $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}as_{(n\times n)}$, *and*
$\bigvee_{i=1}^{n} (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \vee (\langle T_{r_{ji}}, I_{r_{ji}}, F_{r_{ji}}\rangle) \leq (\langle T_{p_{jj}}, I_{p_{jj}}, F_{p_{jj}}\rangle)$ for some j,
then $(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{n-1} = (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)^{n}$.

5. Reduction of transitive neutrosophic soft matrices

We analyze the general reduction scheme of $\mathcal{N}e\mathcal{S}o\mathcal{M}as$, concerning a product of three $\mathcal{N}e\mathcal{S}o\mathcal{M}as$. If $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle$ is $\mathcal{N}e\mathcal{S}o\mathcal{M}a$, of order $(m \times n$ (file term), $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle$ is (term-term) $\mathcal{N}e\mathcal{S}o\mathcal{M}a_{n\times n}$ and $\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle$ is a (file-file) $\mathcal{N}e\mathcal{S}o\mathcal{M}a_{m\times m}$.

Theorem 5.1. If $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle$ and $\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle$ are max-max-min transitive neutrosophic soft matrix and $\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle_{n\times n}$ nilpotent neutrosophic soft matrix such that $\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle \leq \langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle$, then $(\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle) \circ (\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle //$
$(\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle, \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle)) \circ (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) = (\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle) \circ (\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle) \circ$
$(\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$ (1)
for any $\mathcal{N}e\mathcal{S}o\mathcal{M}a$, $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle$. Theorem 5.1, is extension of Theorem 1 of Hashimoto [11] in this part of all the $\mathcal{N}e\mathcal{S}o\mathcal{M}as$, are involved $\mathcal{B}oo\mathcal{N}e\mathcal{S}o\mathcal{M}a$. If $\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle_{m\times m}$ is on $\mathcal{I}d\mathcal{N}e\mathcal{S}o\mathcal{M}a$, (i.e., $\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle = \langle 0,0,1\rangle$
if $i \neq j$, $\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle = \langle 1,1,0\rangle$ if $i = j$), we conclude this result the below corollary.

Corollary 5.2. *If* $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle$ *is Max-max-min-*$\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}a$, *and*

$\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle_{n\times n}$ $\mathcal{N}i\mathcal{N}e\mathcal{S}o\mathcal{M}a$, such that $\langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle \leq \langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle$, then

$$(\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle / \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle) \circ (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) = (\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle) \circ (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle) \quad (2)$$

where $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle$ is any $\mathcal{R}e\mathcal{N}e\mathcal{S}o\mathcal{M}a$. Corollary 5.2 extends also, for $\mathcal{B}oo\mathcal{N}e\mathcal{S}o\mathcal{M}a$. By using the definition of the operation $\ominus$ " that values of the $\mathcal{N}e\mathcal{S}o\mathcal{M}a$ $(\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle // (\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle, \langle T_{p_{ij}}, I_{p_{ij}}, F_{p_{ij}}\rangle))$ are either "0" or "=" to the respective solutions of $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle$. From the equations (1) and (2), we find the description of the files by way of the terms no swap of the instruction of the system. $(\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle) \circ (\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle) \circ (\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle)$, i.e., some case of information contained in the $\mathcal{N}e\mathcal{S}o\mathcal{M}a$ $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle$ can be getback from $\langle T_{s_{ij}}, I_{s_{ij}}, F_{s_{ij}}\rangle$ (that represents a neutrosphic hierarchy of files) and $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle$ (that represents a neutrosophic hierarchy of terms).

Example 5.3. Reduce the description of the set $\{f_1, f_2, \dots, f_6\}$ of six files by means of the set $\{t_1, t_2, t_3, t_4\}$ of terms using the following data:

$$\mathcal{A} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array}\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} \langle 0.6,0.5,0.4\rangle & \langle 0.6,0.5,0.4\rangle & \langle 0.8,0.7,0.2\rangle & \langle 0.9,0.8,0.1\rangle \\ \langle 0.8,0.7,0.2\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0.9,0.8,0.1\rangle & \langle 0.8,0.7,0.2\rangle \\ \langle 0.6,0.5,0.4\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.6,0.5,0.4\rangle \\ \langle 0.2,0.1,0.8\rangle & \langle 0.8,0.7,0.2\rangle & \langle 1,1,0\rangle & \langle 0.8,0.7,0.2\rangle \\ \langle 0.6,0.5,0.4\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.4,0.5,0.6\rangle & \langle 1,1,0\rangle \\ \langle 0.6,0.5,0.4\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.9,0.8,0.1\rangle & \langle 0.6,0.5,0.4\rangle \end{pmatrix} \end{array}$$

$$\mathcal{R} = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array}\begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{pmatrix} \langle 1,1,0\rangle & \langle 0.2,0.1,0.8\rangle & \langle 0.2,0.1,0.8\rangle & \langle 0.2,0.1,0.8\rangle \\ \langle 0.2,0.1,0.8\rangle & \langle 1,1,0\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.2,0.1,0.8\rangle & \langle 0.3,0.2,0.7\rangle & \langle 1,1,0\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0.2,0.1,0.8\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.3,0.2,0.7\rangle & \langle 1,1,0\rangle \end{pmatrix} \end{array}$$

Moreover, we assume $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle$ to be a similarity $\mathcal{N}e\mathcal{S}o\mathcal{M}a$. (i.e., reflexive, symmetric, and max-max-min transitive), where $\langle T_{r_{ij}}, I_{r_{ij}}, F_{r_{ij}}\rangle$ denotes the degree up to which the terms t_i and t_j can be considered to be similar.
Now, let

$$\mathcal{P} = \begin{bmatrix} \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \\ \langle 0.2,0.1,0.8\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0.7,0.6,0.3\rangle \\ \langle 0.2,0.1,0.8\rangle & \langle 0.3,0.2,0.7\rangle & \langle 0,0,1\rangle & \langle 0.3,0.2,0.7\rangle \\ \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle & \langle 0,0,1\rangle \end{bmatrix} \leq \mathcal{R}$$

be the $\mathcal{N}i\mathcal{N}e\mathcal{S}o\mathcal{M}a$ by methods of which we reduce $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle$. In accordance to the Corollary 5.2, we get the following results (i.e., with more zero entries) representation of the file-term $\mathcal{N}e\mathcal{S}o\mathcal{M}a$, $\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}} \rangle$ without loosing any useful information for the retrieval process.

$$(\mathcal{A}/\mathcal{P}) = \begin{bmatrix} \langle 0.6,0.5,04\rangle & \langle 0.6,0.5,04\rangle & \langle 0.8,0.7,0.2\rangle & \langle 0.9,0.8,0.1\rangle \\ \langle 0.8,0.7,0.2\rangle & \langle 0.4,0.3,0.6\rangle & \langle 0.9,0.8,0.1\rangle & \langle 0.8,0.7,0.2\rangle \\ \langle 0.6,0.5,04\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0,0,1\rangle \\ \langle 0,0,1\rangle & \langle 0.8,0.7,0.2\rangle & \langle 1,1,0\rangle & \langle 0.8,0.7,0.2\rangle \\ \langle 0.6,0.5,04\rangle & \langle 0,0,1\rangle & \langle 0.4,0.5,0.6\rangle & \langle 1,1,0\rangle \\ \langle 0.6,0.5,04\rangle & \langle 0.7,0.6,0.3\rangle & \langle 0.9,0.8,0.1\rangle & \langle 0,0,1\rangle \end{bmatrix}.$$

6. Conclusion

In this paper, we generalize the well-known results on the convergence and reduction of $\mathcal{N}e\mathcal{S}o\mathcal{M}as$. These results are useful for examining the reduction of powers of $\mathcal{T}r\mathcal{N}e\mathcal{S}o\mathcal{M}as$, which represent reduced systems. They highlight the graph-theoretic properties of $\mathcal{N}e\mathcal{S}o\mathcal{M}as$. We also provide some examples to support the theoretical content of this paper. In future work, we will develop an algorithm using transitive NeSoMas to draw digraphs and find strongly connected graphs. Moreover, these results are applied to a multicriteria decision-making method. Of course, the results hold for Boolean matrices, whose properties are used in the study of Markov chains.

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