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A Generalized Quadrature Specifically Designed for Numerical Integration of Indefinite Integrals

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Abstract. We propose a generalized quadrature rule for approximating indefinite integrals by combining Steffensen's rule with the Anti-Gaussian rule. The convergence properties of this new method are thoroughly analyzed to ensure its reliability. Our error analysis highlights the improved accuracy of the generalized quadrature rule compared to its base methods. We test the rule on various example integrals to support these theoretical findings, demonstrating its effectiveness and precision. This approach significantly improves numerical integration, making it a valuable tool for solving indefinite integrals with greater accuracy and efficiency.

Keywords: Generalized quadrature rule, Steffensen's rule, truncation error

AMS Mathematics Subject Classification (2010): 65D30, 65D32

1. Introduction

Quadrature rules are vital in numerical analysis for approximating definite integrals. These rules simplify the process of numerical integration by providing effective approximation methods. The precision of a quadrature rule plays a crucial role in determining its accuracy. Higher precision typically indicates a more effective quadrature rule. Many mixed-type quadrature rules [4,7,8,9,10] have been developed to handle definite integrals, and researchers continue to improve their precision through innovative techniques.

In recent studies, Mohanty and Dash introduced a generalized approach [5,6] for creating higher precision quadrature rules. Their method combines multiple lower precision rules to achieve enhanced accuracy. They focused specifically on closed-type quadrature rules for definite integrals, marking a significant step forward in the field of numerical integration.

This paper builds upon Mohanty and Dash's foundational work by extending their approach to open-type quadrature rules. Our primary objective is to develop an open-type generalized quadrature rule with precision-5. To achieve this, we combine two lower-precision rules: the anti-Gaussian rule [1,2,3,11,12] and Steffensen's rule and the, both of which have a precision of 3. By amalgamating these rules, we introduce a novel technique for achieving higher precision in numerical integration.

The organization of this paper is structured as follows: Section 2: Preliminaries. Section 3: Development of the Generalized Quadrature Rule. Section 4: Analysis of Error. Section 5: Numerical Examples. Section 6: Conclusion.

2. Preliminaries

In this section, we present two basic quadrature rules available in the literature, which are used in the construction of the proposed generalized quadrature.

2.1. Anti-Gaussian rule

Laurie's concept [3] enables deriving an anti-Gaussian quadrature rule from the Gaussian 2-point rule, which integrates polynomials up to degree 3 exactly. The anti-Gaussian rule complements this by minimizing errors for higher-degree polynomials, specifically targeting those orthogonal to the ones integrated accurately by the Gaussian rule [1,8,11,12]. In this paper, we employ the following anti-Gaussian rule.

$$aGL(f) = \frac{1}{13}\left[5f\left(-\sqrt{\frac{13}{5}}\right) + 16f(0) + 5f\left(\sqrt{\frac{13}{5}}\right)\right] \tag{1}$$

Due to Taylor [1],

$$aGL(f) = 2\left[f(0) + \frac{1}{3!}f''(0) + \frac{13}{9\times 5!}f^{iv}(0) + \frac{169}{675\times 6!}f^{vi}(0) + + \frac{(13)^3}{3\times 8!\times(15)^3}f^{viii}(0) + \frac{2\times(13)^4}{10!\times(15)^5}f^{x}(0) + \cdots\right] \tag{2}$$

Lemma 2.1. If $f(x)$ is sufficiently differentiable on the interval $[-1,1]$, the truncation error for $aGL(f)$ is given by $EaGL(f) = -\frac{1}{135}f^{iv}(0) - \frac{1016}{7!\times 675}f^{vi}(0) - \frac{6432}{9!\times(15)^3}f^{viii}(0) + \cdots$.

Proof: Using Taylor's theorem, the exact value of the integral is:

$$I(f) = 2\left[f(0) + \frac{1}{3!}f''(0) + \frac{1}{5!}f^{iv}(0) + \frac{1}{7!}f^{vi}(0) + + \frac{1}{9!}f^{viii}(0) + \cdots\right] \tag{3}$$

The truncation error due to $aGL(f)$ is given by

$$EaGL(f) = I(f) - aGL(f) \tag{4}$$

Using equations (2) and (3) in equation (4), we obtain:

$$EaGL(f) = -\frac{1}{135}f^{iv}(0) - \frac{1016}{7!\times 675}f^{vi}(0) - \frac{6432}{9!\times(15)^3}f^{viii}(0) + \cdots. \tag{5}$$

2.2. Steffensen's quadrature rule

Steffensen's quadrature rules, a type of open Newton-Cotes rules, are useful for numerical integration when function values at endpoints are unknown or have singularities. They are especially effective in solving differential equations numerically under such conditions, providing an efficient alternative for complex boundary cases.

The *Steffensen's 4-point transformed rule* [2] is given by

$$\int_{-1}^{1} f(x)\,dx \approx ST(f) = \frac{1}{12}\left[11f\left(-\frac{3}{5}\right) + f\left(-\frac{1}{5}\right) + f\left(\frac{1}{5}\right) + f\left(\frac{3}{5}\right)\right] \quad (6)$$

Using Taylor's expansion on (6), we get

$$ST(f) = 2\left[f(0) + \frac{1}{3!}f''(0) + \frac{223}{375\times5!}f^{iv}(0) + \frac{401}{9375\times6!}f^{vi}(0) + + \frac{18043}{3\times9!\times5^8}f^{viii}(0) + \frac{32477}{3\times10!\times5^9}f^{x}(0) + \cdots\right] \quad (7)$$

Lemma 2.2. If $f(x)$ is sufficiently differentiable on the interval $[-1,1]$, the truncation error for $ST(f)$ is given by

$$EST(f) = \frac{38}{5625}f^{iv}(0) + \frac{13136}{7!\times9375}f^{vi}(0) + \frac{2018976}{9!\times3\times5^8}f^{viii}(0) + \frac{11004256}{11!\times3\times5^8}f^{x}(0) + \cdots.$$

Proof: We have $EST(f) = I(f) - ST(f)$ (8)

Using values from (3) and (7) on (8), we get

$$EST(f) = \frac{38}{5625}f^{iv}(0) + \frac{13136}{7!\times9375}f^{vi}(0) + \frac{2018976}{9!\times3\times5^8}f^{viii}(0) + \frac{11004256}{11!\times3\times5^8}f^{x}(0) + \cdots \quad (9)$$

Equation (9), establishes that the degree of precision of $ST(f)$ is three. □

3. Development of the generalized quadrature rule

In this section, we develop the proposed generalized quadrature using a generalized technique [5,6]. The approach is systematically outlined, emphasizing its foundational principles and methodology to enhance precision and applicability in numerical computations.

Generalized quadrature rule: Quadrature rule of higher precision formed out by using n-rules of lower precision, $n \in N, n \geq 2$ is known as a generalized quadrature rule [5,6].

Given SR_n as the generalized quadrature rule of higher precision formed by using quadrature rules $R_1, R_2, R_3, \ldots. R_n$ of lower precision satisfying SR-Conditions [5,6], we can express

$$SR_n = a_1R_1 + a_2R_2 + a_3R_3 + \cdots + a_nR_n;\ \sum_{i=1}^{n} a_i = 1 \quad (10)$$

where $a_1, a_2, a_3, \ldots a_4$ are n-number of rational coefficients that can be obtained by making the rule SR_n exact for all polynomials of degree up to $dR_n + 2$. The truncation error due to the rule (10) is given by $SR_n = a_1ER_1 + a_2ER_2 + a_3ER_3 + \cdots + a_nER_n;\ \sum_{i=1}^{n} a_i = 1$ (11)

Assuming this error vanishes for all polynomials of degree upto dSR_n, we get the values of $a_1, a_2, a_3, \ldots a_4$. Using the values of $a_i's$ on (10), we can obtain the desired generalized quadrature rule.

We construct a generalized quadrature of order 2 using two specific rules based on the generalized quadrature technique:

$R_1(f) = aGL(f)$: Anti-Gaussian 3-point quadrature rule.

$R_2(f) = ST(f)$: Steffensen's quadrature rule.

Both rules have precision 3, meaning $dR_1(f) = dR_2(f) = 3$

Therefore, $R_1(f)$ and $R_2(f)$ satisfy the SR conditions [5,6]. Theorem 3.1 provides the formulation of the proposed generalized quadrature rule.

Theorem 3.1. If $f(x)$ is sufficiently differentiable on the interval $[-1,1]$, the Generalized quadrature rule $SM_{gs}(f)$ is given by $SM_{gs}(f) = \frac{114}{239}GL(f) + \frac{125}{239}ST(f)$ and truncation error due to the rule $ESM_{gs}(f) = \frac{114}{239}EaGL(f) + \frac{125}{239}EST(f)$.

Proof: The generalized quadrature rule $SM_{gs}(f)$ constructed using $R_1(f), R_2(f)$ is expressed as follows:

$$SM_{gs}(f) = [a_1R_1(f) + a_2R_2(f)],\ a_1 + a_2 = 1 \tag{12}$$

The truncation error due to $SM_{gs}(f)$ is given by

$$ESM_{gs}(f) = [a_1ER_1 + a_2ER_2] \tag{13}$$

Using (5) and (9) on (13), we have

$$ESM_{gs}(f) = a_1\left\{-\frac{1}{135}f^{iv}(0) - \frac{1016}{7!\times 675}f^{vi}(0) - \frac{6432}{9!\times(15)^3}f^{viii}(0) + \frac{131033}{11!\times(15)^5}f^{x}(0) + \cdots\right\} + a_2\left\{\frac{38}{5625}f^{iv}(0) + \frac{13136}{7!\times 9375}f^{vi}(0) + \frac{2018976}{9!\times 3\times 5^8}f^{viii}(0) + \frac{11004256}{11!\times 3\times 5^8}f^{x}(0) + \cdots\right\} + \tag{14}$$

We choose a_1 and a_2 in such a way that the rule $SM_{gs}(f)$ becomes exact for all polynomial of degree upto 5. From the error term,

we have $\frac{a_1}{135} - \frac{38}{5625}a_2 = 0$ (15)

On solving (12) and (15), we get $a_1 = \frac{114}{239}$ and $a_2 = \frac{125}{239}$.

Using the value of a_1 , a_2 on (14) and (15), we get the desire result.

Corollary 3.1. If $f(x)$ is sufficiently differentiable in the interval [-1, 1], the truncation error due to the $SM_{gs}(f)$ is given by

$$ESM_{gs}(f) = \frac{32}{7!\times 2151}f^{vi}(0) - \frac{17824}{9!\times 5^5\times 3}f^{viii}(0) + \frac{322011182}{11!\times 5^6\times 3^4}f^{x}(x) + \cdots$$

Proof: Putting the value of a_1, a_2 on (14), we get the result. □

4. Error analysis

From Sections 2 and 3, we obtain an error comparison between the constituent rules and the constructed rule. This comparison is detailed in the following theorem, which highlights the differences in accuracy and precision, providing a clear understanding of the advantages of the proposed generalised quadrature rule.

Theorem 4.1. The truncation error associated with the rule $SM_{gs}(f)$ is minimal compared to the errors of the base rules

Proof: From Lemma 2.1 and corollary to the Theorem 3.1

$$|ESM_{gs}(f)| \leq |EaGL(f)|.$$

From Lemma 2.2 and corollary to the Theorem 3.1

$$|ESM_{gs}(f)| \leq |EST| \qquad \square$$

We compare the precision-5 rule with the original lower-precision rules, showcasing its improved accuracy through theoretical validation and practical examples provided in Section 5.

5. Applications and numerical verification

This section examines the practical use of the new quadrature rule for numerical integration in open-type test integrals, focusing on indefinite integrals. To show its superiority over the constituent rules, five open-form test integrals are analyzed. The results are shown in Table 1, and errors are compared in Table 2.

Table 1: The values from five test integrals are calculated using the constituent rules and the $SM_{gs}(f)$ rule.

Integral	$aGL(f)$	$ST(f)$	$SM_{gs}(f)$
$I_1 = \int_1^\infty \frac{1}{xe^x} dx$	0.219383642	0.219384251	0.21938396051464435
$I_2 = \int_0^\infty \frac{\sin x}{xe^x} dx$	0.78539947	0.785397034	0.785398195941422594l4226
$I_3 = \int_0^\infty \frac{\sqrt{x}}{e^x} dx$	0.886227119	0.886226632	0.886226698 0.8862268642928870292887
$I_4 = \int_0^\infty \frac{\log x}{e^x} dx$	– 0.577215756	– 0.577215394	– 0.57721556669456066945607
$I_5 = \int_0^\infty \frac{\cos x\, dx}{e^x} dx$	0.496029	0.517651239	0.50733770240585774

Table 2: Comparison of the absolute values of truncation errors for different rules and the constructed rule.

Integral	Exact Value	$\lvert EaGL(f)\rvert$	$\lvert EST(f)\rvert$	$\lvert ESM_{gs}(f)\rvert$
I_1	0.219383934395520	1.936×10^{-7} 2.9239552×10^{-7}	3.1660448×10^{-7}	2.6119124×10^{-8}
I_2	0.7853981634	1.3066×10^{-6}	1.1294×10^{-6}	3.25414226×10^{-8}
I_3	0.886226925452758	1.9355×10^{-7}	2.9345×10^{-7}	6.11598×10^{-8}
I_4	–0.5772156649	1.011×10^{-7}	2.609×10^{-7}	8.8230543×10^{-8}
I_5	0.5	$0.05.0653\times 10^{-2}$	1.765124×10^{-2}	7.337×10^{-3}

5. Conclusion

The theorems and tables show that the constructed rule performs better than the base rules, both in theory and practice, by greatly reducing errors and improving accuracy in numerical integration. Introducing a new open-type quadrature rule with precision-5 is a big step forward in the field. Combining the anti-Gauss 3-point rule and Steffensen's 4-point rule provides a simple way to achieve higher precision, which could be useful in many numerical analysis tasks.

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