Annals of Pure and Applied Mathematics
Vol. 30, No. 2, 2024, 79-85
ISSN: 2279-087X (P), 2279-0888(online)
Published on 27 November 2024
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v30n2a01950

Annals of Pure and Applied Mathematics

Harmonic E-Banhatti Index of Graphs

V.R.Kulli

Department of Mathematics
Gulbarga University, Gulbarga 585 106, India
E-mail: vrkulli@gmail.com

Received 28 September 2024; accepted 25 November 2024

Abstract. In this study, we introduce the harmonic E-Banhatti index and its corresponding polynomial of a graph. Furthermore, we compute the harmonic E-Banhatti index for some standard graphs, wheel graphs and friendship graphs. Also, some mathematical properties of the harmonic E-Banhatti index are obtained.

Keywords: harmonic E-Banhatti index, harmonic E-Banhatti polynomial, graph

AMS Mathematics Subject Classification (2010): 05*C*10, 05*C*69

1. Introduction

In this paper, G denotes a finite, simple, connected graph, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u. The degree of an edge $e = uv$ in G is defined by $d_G(e)=d_G(u)+d_G(v)-2$. We refer [1], for other undefined notations and terminologies.

Graph indices have applications in various scientific and technological disciplines. For more information about graph indices, see [2].

Kulli [3] defined the Banhatti degree of a vertex u of a graph G as

$$B(u)=\frac{d_G(e)}{n-d_G(u)},$$

where $|V(G)|= n$ and the vertex u and edge e are incident in G.

The first and second E-Banhatti indices were introduced by Kulli in [3] and they are defined as

$$EB_1(G)=\sum_{uv\in E(G)}[B(u)+B(v)], \qquad EB_2(G)=\sum_{uv\in E(G)}B(u)B(v).$$

Recently, some E-Banhatti indices were studied in [4, 5, 6, 7, 8, 9].

The modified E-Banhatti Sombor index [10] of a graph G is

$$^{m}EBS(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{B(u)^2+B(v)^2}}.$$

The harmonic index [11] of a graph G is defined as

$$H(G)=\sum_{uv\in E(G)}\frac{2}{d_G(u)+d_G(v)}.$$

This index was studied, for example, in [12, 13, 14, 15].

We put forward a new topological index, defined as

$$HEB(G)=\sum_{uv\in E(G)}\frac{2}{B(u)+B(v)}$$

which we propose to be named as harmonic E-Banhatti index.

Considering the harmonic E-Banhatti index, we introduce the harmonic E-Banhatti polynomial of a graph G and defined it as

$$HEB(G,x)=\sum_{uv\in E(G)}x^{\frac{2}{B(u)+B(v)}}.$$

In this paper, we determine the E-Banhatti index of some standard graphs, wheel graphs and friendship graphs. Also some mathematical properties of harmonic E-Banhatti index are obtained.

2. Results for some standard graphs

Proposition 1. If G is an r-regular graph with n vertices and $r \geq 2$, then

$$HEB(G)=\frac{nr(n-r)}{4(r-1)}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$. Then $|E(G)| = \frac{nr}{2}$. For any edge $uv=e$ in G, $d_G(e)=2r-2$. Then

$$HEB(G)=\sum_{uv\in E(G)}\frac{2}{B(u)+B(v)}=\frac{nr}{2}\left(\frac{2}{\frac{2r-2}{n-r}+\frac{2r-2}{n-r}}\right)=\frac{nr(n-r)}{4(r-1)}.$$

Corollary 1.1. Let C_n be a cycle with $n\geq 3$ vertices. Then

$$HEB(C_n)=\frac{n(n-2)}{2}.$$

Corollary 1.2. Let K_n be a complete graph with $n\geq 3$ vertices. Then

$$HEB(K_n)=\frac{n(n-1)}{4(n-2)}.$$

Proposition 2. Let P_n be a path with $n\geq 3$ vertices. Then

$$HEB(P_n)=2\left(\frac{2}{\frac{1}{n-1}+\frac{2}{n-2}}\right)+(n-3)\left(\frac{2}{\frac{2}{n-2}+\frac{2}{n-2}}\right)$$

$$=\frac{4(n-1)(n-2)}{3n-4}+\frac{(n-3)(n-2)}{2}.$$

Proposition 3. Let $K_{m,n}$ be a complete bipartite graph with $1 \leq m\leq n$ *and* $n \geq 2$. Then

$$HEB(K_{m,n}) = \frac{2m^2n^2}{(m+n)(m+n-2)}.$$

Proof: Let $K_{m,n}$ be a complete bipartite graph with $m + n$ vertices and mn *edges* such that $|V_1| = m$, $|V_2| = n$, $V(K_{r,s}) = V_1 \cup V_2$ for $1 \leq m \leq n$, and $n \geq 2$. Every vertex of V_1 is incident with n edges and every vertex of V_2 is incident with m edges. Then
$d_G(e) = d_G(u) + d_G(v) - 2 = m + n - 2$.

$$HEB(G) = \sum_{uv \in E(G)} \frac{2}{B(u)+B(v)} = mn\left(\frac{2}{\frac{m+n-2}{m+n-n} + \frac{m+n-2}{m+n-m}}\right)$$

$$= \frac{2m^2n^2}{(m+n)(m+n-2)}.$$

Corollary 3.1. Let $K_{n,n}$ be a complete bipartite graph with $n \geq 2$. Then

$$HEB(K_{n,n}) = \frac{n^4}{2n(n-1)}.$$

Corollary 3.2. Let $K_{1,n}$ be a star with $n \geq 2$. Then $HEB(K_{1,n}) = \frac{2n^2}{n^2-1}$.

2. Mathematical properties

Theorem 1. Let G be a simple connected graph. Then

$$\sqrt{2}(^m EBS(G)) \geq HEB(G)$$

with equality if G is regular.

Proof: By the Jensen inequality, for a concave function *f(x)*,

$$f\left(\frac{1}{n}\sum x_i\right) \geq \frac{1}{n}\sum f(x_i)$$

with equality for a strictly concave function if $x_1 = x_2 = \ldots = x_n$.
Choosing $f(x) = \sqrt{x}$, we obtain

$$\sqrt{\frac{B(u)^2 + B(v)^2}{2}} \geq \frac{(B(u)+B(v))}{2}$$

thus

$$\frac{\sqrt{2}}{\sqrt{B(u)^2+B(v)^2}} \leq \frac{2}{(B(u)+B(v))}.$$

Hence
$$\sum_{uv \in E(G)} \frac{\sqrt{2}}{\sqrt{B(u)^2+B(v)^2}} \geq \sum_{uv \in E(G)} \frac{2}{B(u)+B(v)}.$$

Thus
$$\sqrt{2}(^m EBS(G)) \geq HEB(G)$$

with equality if G is regular.

Theorem 2. Let G be a simple connected graph. Then

$$HEB(G) < 2(^m EBS(G)).$$

Proof: It is known that for $1 \le x \le y$,

$$\sqrt{x^2 + y^2} < x + y$$

$$\frac{2}{x+y} < \frac{2}{\sqrt{x^2+y^2}}.$$

Setting $x=B(u)$ and $y=B(v)$, we get

$$\frac{2}{(B(u)+B(v))} < \frac{2}{\sqrt{B(u)^2+B(v)^2}}.$$

Thus $$\sum_{uv \in E(G)} \frac{2}{(B(u)+B(v))} < 2\sum_{uv \in E(G)} \frac{1}{\sqrt{B(u)^2+B(v)^2}}.$$

Hence $$HEB(G) < 2(^{m}EBS(G)).$$

3. Results for friendship graphs

A friendship graph F_4 is shown in Figure 1. A friendship graph F_n is a graph with $2n+1$ vertices and $3n$ edges.

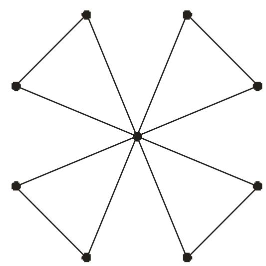

Figure 1: Friendship graph F_4

In F_n, there are two types of edges as follows:

$$E_1 = \{uv \in E(F_n) \mid d_{F_n}(u) = d_{F_n}(v) = 2\}, \qquad |E_1| = n.$$

$$E_2 = \{uv \in E(F_n) \mid d_{F_n}(u) = 2, d_{F_n}(v) = 2n\}, \quad |E_2| = 2n.$$

Therefore, in F_n, we obtain that $\{B(u), B(v): uv \in E(W_n)\}$ has two Banhatti edge set partitions.

$$BE_1 = \{uv \in E(F_n) \mid B(u) = B(v) = \frac{2}{2n-1}\}, \qquad |BE_1| = n.$$

$$BE_2 = \{uv \in E(F_n) \mid B(u) = \frac{2n}{2n-1}, B(v) = 2n\}, \quad |BE_2| = 2n.$$

Theorem 3. Let F_n be the friendship graph. Then

$$HEB(F_n) = \frac{(2n-1)(n^2+2)}{2n}.$$

Proof: We have

$$HEB(F_n)=\sum_{uv\in E(F_n)}\frac{2}{(B(u)+B(v))}=n\left(\frac{2}{\frac{2}{2n-1}+\frac{2}{2n-1}}\right)+2n\left(\frac{2}{\frac{2n}{2n-1}+2n}\right)$$

After simplification, we get

$$HEB(F_n)=\frac{(2n-1)(n^2+2)}{2n}.$$

Theorem 4. Let F_n be the friendship graph. Then

$$HEB(F_n,x)=nx^{\frac{2n-1}{2}}+2nx^{\frac{2n-1}{2n^2}}.$$

Proof: We have

$$HEB(F_n,x)=\sum_{uv\in E(F_n)}x^{\frac{2}{(B(u)+B(v))}}=nx^{\left(\frac{2}{\frac{2}{2n-1}+\frac{2}{2n-1}}\right)}+2nx^{\left(\frac{2}{\frac{2n}{2n-1}+2n}\right)}.$$

After simplification, we get

$$HEB(F_n,x)=nx^{\frac{2n-1}{2}}+2nx^{\frac{2n-1}{2n^2}}.$$

4. Results for wheel graphs

A wheel graph W_n is the join of C_n and K_1. Then W_n has n+1vertices and $2n$ edges. A graph W_n is presented in Figure 2.

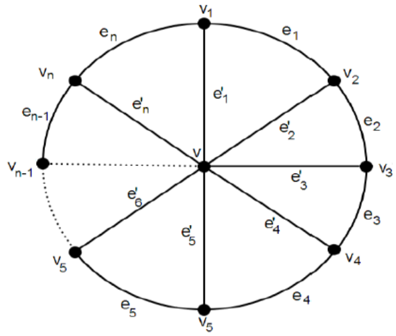

Figure 2: Wheel graph W_n

In W_n, there are two types of edges as follows:

$E_1 = \{uv \in E(W_n) \mid d(u) = d(v) = 3\}$, $|E_1| = n$.

$E_2 = \{uv \in E(W_n) \mid d(u) =3, d(v) = n\}$, $|E_2| = n$.

Therefore, in W_n, there are two types of Banhatti edges based on Banhatti degrees of end vertices of each edge follow:

$BE_1 = \{uv \in E(W_n) \mid B(u) = B(v) = \frac{4}{(n-2)}\}$, $|BE_1| = n$.

$BE_2 = \{uv \in E(W_n) \mid B(u) = \frac{n+1}{n-2}, B(v) = n+1\}$, $|BE_2| = n$.

Theorem 5. . Let W_n be the wheel graph. Then

$$HEB(W_n) = HEB(W_n) = \frac{n(n-2)(n^2+7)}{4n^2-1}.$$

Proof: We have

$$HEB(W_n) = \sum_{uv \in E(W_n)} \frac{2}{B(u)+B(v)} = n\left(\frac{2}{\frac{4}{n-2}+\frac{4}{n-2}}\right) + n\left(\frac{2}{\frac{n+1}{n-2}+(n+1)}\right)$$

After simplification, we get

$$HEB(W_n) = \frac{n(n-2)(n^2+7)}{4n^2-1}.$$

Theorem 6. Let W_n be the wheel graph. Then

$$HEB(W_n, x) = nx^{\frac{n-2}{4}} + nx^{\frac{2(n-2)}{n^2-1}}.$$

Proof: We have

$$HEB(W_n, x) = \sum_{uv \in E(W_n)} x^{\frac{2}{(B(u)+B(v))}} = nx^{\left(\frac{2}{\frac{4}{n-2}+\frac{4}{n-2}}\right)} + nx^{\left(\frac{2}{\frac{n+1}{n-2}+(n+1)}\right)}.$$

After simplification, we get

$$HEB(W_n, x) = nx^{\frac{n-2}{4}} + nx^{\frac{2(n-2)}{n^2-1}}.$$

5. Conclusion

We have introduced the harmonic E-Banhatti index and its corresponding exponential of a graph. Furthermore the harmonic E-Banhatti index and its exponential of some standard graphs, wheel graphs and friendship graphs are determined. Also, some mathematical properties of harmonic E-Banhatti index are obtained.

Acknowledgements. The authors thank the referee for their valuable suggestions and comments, which have significantly enhanced the quality of the original manuscript.

Conflict of interest. The authors declare that they have no conflict of interest.

Author's Contributions: All authors have equal contribution.

REFERENCES

1. V.R.Kulli, *College Graph Theory*, Vishwa International Publications, Gulbarga, India (2012).
2. V.R.Kulli, Graph indices, in *Hand Book of Research on Advanced Applications of Application Graph Theory in Modern Society*, M. Pal. S. Samanta and A. Pal, (eds.) IGI Global, USA (2019) 66-91.

3. V.R.Kulli, New direction in the theory of graph index in graphs, *International Journal of Engineering Sciences & Research Technology*, 11(12) (2022) 1-8.
4. V.R.Kulli, Hyper E-Banhatti indices of certain networks, *International Journal of Mathematical Archive*, 13(12) (2022) 1-10.
5. V.R.Kulli, Product connectivity E-Banhatti indices of certain nanotubes, *Annals of Pure and Applied Mathematics*, 27(1) (2023) 7-12.
6. V.R.Kulli, The (a,b)-KA E-Banhatti indices of graphs, *Journal of Mathematics and Informatics*, 23 (2022) 55-60.
7. V.R.Kulli, Computation of E-Banhatti Nirmala indices of tetrameric 1,3-adamantane, *Annals of Pure and Applied Mathematics*, 26(2) (2022) 119-124.
8. V.R.Kulli, Atom bond connectivity E-Banhatti indices, *International Journal of Mathematics and Computer Research*, 11(1) (2023) 3201-3208.
9. V.R.Kulli, First multiplicative K Banhatti index and coindex of graphs, *Annals of Pure and Applied Mathematics*, 11(2) (2016) 79-82.
10. V.R.Kulli, E-Banhatti Sombor indices, *International Journal of Mathematics and Computer Research*, 10(12) (2022) 2986-2994.
11. S.Fajtlowicz, On conjectures of Graffiti-II, *Congr. Numer.* 60 (1887) 187-197.
12. Y.Hu and X.Zhou, On the harmonic index of the unicyclic and bicyclic graphs, *Wseas Tran. Math.* 12 (2013) 716-726.
13. J.Liu, On the harmonic index of triangle free graphs, *Appl. Math.* 4 (2013) 1204-1206.
14. B.Shwetha Shetty, V.Lokesha and P.S.Ranjini, On the harmonic index of graph operations, *Tran. Combin.* 4 (2015) 5-14.
15. L.Zhong, The harmonic index of graphs, *Appl. Math. Lett.* 25 (2012) 561-566.