Annals of Pure and Applied Mathematics
Vol. 30, No. 1, 2024, 67-77
ISSN: 2279-087X (P), 2279-0888(online)
Published on 30 September 2024
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v30n1a06947

Annals of Pure and Applied Mathematics

Numerical Solution of One-Dimensional Differential Equations by Weighted Residual Method via Euler Wavelets

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Received 28 August 2024; accepted 29 September 2024

Abstract. Differential equations are essential in the field of mathematics, as they can describe a wide variety of real-world situations. One of the key benefits of numerical methods, in contrast to analytical approaches, is their straightforward application on contemporary computers, which facilitates faster solutions than those achieved through analytical methods. Moreover, Wavelet analysis has emerged as an exciting area within applied and computational numerical studies. The weighted residual methods represent a broader set of approaches that encompasses Galerkin's technique. This paper introduces the weighted residual method, specifically highlighting Galerkin's approach for solving one-dimensional differential equations, utilizing Euler wavelets as weight functions. The effectiveness and validity of the proposed method are illustrated through its application to several test problems.

Keywords: Euler wavelets; function approximation; weighted residual method; one-dimensional differential equations; boundary conditions

AMS Mathematics Subject Classification (2010): 65T60, 97N40, 30E25

1. Introduction

Differential equations are capable of representing almost all systems that experience change. For centuries, numerous mathematicians have explored the characteristics of these equations, leading to the development of various effective solution methods. Frequently, the systems represented by differential equations are either highly intricate or extensive in scale, rendering a purely analytical solution impractical. In such complex scenarios, computer simulations and numerical techniques prove to be helpful. In the literature, these equations are solved. Many researchers have attempted to obtain higher accuracy rapidly by using numerous methods. Solving such types of equations analytically is possible only in very rare cases [1]. Various methods are available in the literature concerning their numerical solution [2 – 4].

Wavelets represent a recent innovation in signal processing, facilitating the examination for local characteristics of complex signals across multiple time scales, particularly in regions that may exhibit non-stationary. Their versatility has led to numerous applications across diverse domains, including geophysics, astrophysics, telecommunications, imaging, and video compression. Wavelets serve as a fundamental basis for innovative techniques in both signal analysis and synthesis, addressing significant challenges such as data compression and denoising. The widespread adoption of wavelets within both academic and industrial sectors is noteworthy, primarily due to their ability to address a wide array of theoretical and practical issues [5].

In the field of applied mathematics, the Galerkin method is highly esteemed for its efficiency and practicality. Incorporating wavelets into the Galerkin method presents significant benefits compared to traditional finite difference and finite element techniques, leading to extensive uses in various domains of science and technology. Thus, the wavelet technique serves as a robust alternative to the finite element method to a certain degree. Furthermore, the wavelet method provides a valuable option for numerically solving differential equations [6–7].

This study presents the development of the weighted residual method utilizing Euler Wavelets (WRMEW) for addressing one-dimensional differential equations numerically. The approach involves representing the solution through Euler wavelets characterized by unknown coefficients.
Galerkin's approach and the characteristics of Euler wavelets allow us to identify the unknown coefficients, which in turn leads to the differential equations numerical solution. The paper is organized into five main sections. Section 2 focuses on Euler wavelets and their application in function approximation. In section 3, the weighted residual method that employs Euler wavelets is detailed. Section 4 presents a numerical illustration to support the concepts discussed. Finally, section 5 offers a discussion that encapsulates the conclusions drawn from the research findings.

2. Euler wavelets and function approximation

Euler wavelets and Function approximation

Euler wavelets are defined [8 – 9] as,

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}} Em(2^{k-1}t - n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0 & , \quad otherwise \end{cases} \tag{2.1}$$

with

$$Em\left(t\right) = \begin{cases} 1 & , \quad m = 0 \\ \dfrac{1}{\sqrt{\dfrac{2(-1)^{m-1}(m!)^2}{(2m)!}E_{2m+1}(0)}} E_m\left(t\right) , & m > 0 \end{cases} \tag{2.2}$$

Here, $n = 1,2,\ldots\ldots 2^{k-1}$, $\mathrm{m} = 0, 1, 2, 3, \ldots\ldots, \mathrm{M}-1$ & $k > 0$

For normality the coefficient is $\dfrac{1}{\sqrt{\dfrac{2(-1)^{m-1}(m!)^2}{(2m)!}E_{2m+1}(0)}}$,

$a = 2^{-(k-1)}$ is the dilation parameter and $b = (n-1)2^{-(k-1)}$ is the translation parameter.

The following generating functions can be used to define the well-known Euler polynomials $E_m(t)$ of order m. [13]

$$\frac{2e^{ts}}{e^{s} + 1} = \sum_{m=0}^{\infty} E_m(t)\frac{s^m}{m!} \quad \left(|s| < \pi\right) \tag{2.3}$$

In perticular, the rational numbers $E_m = 2^m E_m\left(\frac{1}{2}\right)$ are the familiar Euler numbers.

Additionally, the following relation can be used to create the first kind Euler polynomials for $m = 0, 1, 2, 3, \ldots.., N$

$$\sum_{m=0}^{N} \binom{m}{k} E_k(t) + E_m(t) = 2t^m \quad \& \quad \binom{m}{k} \text{ is a binomial coefficient.}$$

The first few fundamental polynomials are represented explicitly by

$$E_0(t) = 1,\ E_1(t) = t - \frac{1}{2},\ E_2(t) = t^2 - t,\ E_3(x) = t^3 - \frac{3}{2}t^2 + \frac{1}{4}, \ldots\ldots..$$

The following formula is satisfied by these polynomials

$$\int_0^1 E_m(t)E_n(t)\,dt = (-1)^{n-1}\frac{m!\,(n+1)!}{(m+n+1)!}E_{m+n+1}(0), \qquad m, n \geq 1 \tag{2.4}$$

Over the interval $[0, 1]$, Euler polynomials form a full basis.

If $t = 0$, the Euler polynomials are $E_0(0) = 1,\ E_1(0) = -\frac{1}{2},\ E_3(0) = \frac{1}{4}, \ldots\ldots$

For $k = 1$ & $M = 3$ in (2.1) and (2.2), then the Euler wavelets are given by

$\psi_{1,0}(t) = 1,$

$\psi_{1,1}(t) = 4t - 1,$

$\psi_{1,2}(t) = \sqrt{6}\left(4t^2 - 2t\right),$

$\psi_{1,3}(t) = \frac{4\sqrt{5\sqrt{17}}}{17}\left(8t^3 - 6t^2 + \frac{1}{4}\right)$ and so on.

Function approximation:

Suppose $u(t) \in L^2[0, 1)$ is expressions of Euler wavelets as:

$$u(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) \tag{2.5}$$

Truncating the above infinite series, we get

$$u(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) \tag{2.6}$$

3. Method of solution

Consider one-dimensional equation of the form,

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u = f(t) \tag{3.1}$$

and boundary conditions : $u(0) = a,\ u(1) = b$ (3.2)

Here $f(t)$ is a continuous function t and α & β are constants.

From Eq. (3.1), residual is written as $R(t) = \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u - f(t)$ (3.3)

The residual of the equation $R(t)$ is found here. The boundary conditions will be satisfied and if $R(t) = 0$ for the exact solution $u(t)$.

Here, $u(t)$ is the trial series solution of Eq. (3.1) in terms of modified Euler wavelets defined in [0, 1] and satisfies the boundary conditions. This involves the following unknown coefficients

$$u(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) \tag{3.4}$$

where $c_{n,m}$'s are unknown coefficients are need to be determined.

By selecting Euler wavelet polynomials of higher degree, the precision of the solution is superior.

Now, differentiate Eq. (3.4) twice w.r.t. t and in Eq. (3.4) put these values i.e. $u,\ \frac{\partial u}{\partial t},\ \frac{\partial^2 u}{\partial t^2}$.

To determine the values of $c_{n,m}$'s, by selecting the weight functions as assumed base elements and integrating the residual to zero together with the boundary values [10].

i.e. $\int_0^1 \psi_{1,m}(t)\, R(t)\, dt = 0,\ m = 0, 1, 2, \ldots\ldots$

From the above equation, the system of linear algebraic equations is derived from the equation above and by solving this system, the unknown coefficients are obtained. The numerical solution of Eq. (3.1) was then produced by substituting the unknown coefficients in Eq. (3.4).

To determine the correctness of WRMEW for one-dimensional equations, employ the error measure and it will be computed as

$E_{\max} = \max\left| u(t)_{numer} - u(t)_{exact} \right|$, $u(t)_{numer}$ and $u(t)_{exact}$ are respectively the numerical and exact solutions.

4. Numerical illustration

Test problem 4.1. The differential equation is of the form [11],

$$\frac{\partial^2 u}{\partial t^2} + u = -t, \quad 0 \le t \le 1 \tag{4.1}$$

$$\text{BCs:} \quad u(0) = 0, \quad u(1) = 0 \tag{4.2}$$

The method of solution for Eq. (4.1) is described in section 3 as follows:
The residual of Eq. (4.1) is given by:

$$R(t) = \frac{\partial^2 u}{\partial t^2} + u + t \tag{4.3}$$

Here, the weight function should be selected for Euler wavelet bases $w(t) = t(1-t)$ in order to satisfies the boundary conditions i.e. Eq. (4.2),

$$\psi_{1,0}(t) = \psi_{1,0}(t) \times t(1-t) = t(1-t)$$

$$\psi_{1,1}(t) = \psi_{1,1}(t) \times t(1-t) = (4t-1)\,t(1-t)$$

$$\psi_{1,2}(t) = \psi_{1,2}(t) \times t(1-t) = \sqrt{6}\left(4t^2 - 2t\right)t(1-t)$$

Considering that Eq. (4.1)'s trail solution for $k = 1$ & $m = 2$ and is provided by

$$u(t) = c_{1,0}\,\psi_{1,0}(t) + c_{1,1}\,\psi_{1,1}(t) + c_{1,2}\,\psi_{1,2}(t) \tag{4.4}$$

Then the Eq. (4.4) becomes

$$\begin{aligned} u(t) &= c_{1,0}\{t(1-t)\} + c_{1,1}\{(4t-1)\,t(1-t)\} + \\ &\quad c_{1,2}\left\{\sqrt{6}\left(4t^2 - 2t\right)t(1-t)\right\} \\ \Rightarrow u(t) &= c_{1,0}\left(t - t^2\right) + c_{1,1}\left(5t^2 - 4t^3 - t\right) + \\ &\quad c_{1,2}\sqrt{6}\left(6t^3 - 4t^4 - 2t^2\right) \end{aligned} \tag{4.5}$$

Differentiating Eq. (4.5) $w.r.t.\ t$ twice and substituting $u, \dfrac{\partial^2 u}{\partial t^2}$ into Eq. (4.3), residual of Eq. (4.1) is found. Using the weighted residual approach to go to the subsequent considerations if the weight functions in the trial solution are equal to the basis functions:

$$\int_0^1 \psi_{1,j}(t)\,R(t)\,dt = 0, \quad j = 0, 1, 2 \tag{4.6}$$

Put $j = 0, 1, 2$ in Eq. (4.6),

$$\text{i.e.} \quad \left.\begin{aligned} \int_0^1 \psi_{1,0}(t)\,R(t)\,dt &= 0 \\ \int_0^1 \psi_{1,1}(t)\,R(t)\,dt &= 0 \\ \int_0^1 \psi_{1,2}(t)\,R(t)\,dt &= 0 \end{aligned}\right\} \tag{4.7}$$

Eq. (4.7) gives a system of algebraic equations with unknown coefficients such as $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. By solving this system, then find the values for $c_{1,0} = 0.2329$, $c_{1,1} = 0.0461$ and $c_{1,2} = -0.0027$. Obtained the numerical solution on substituting $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ in Eq. (4.5). Table 1 gives the approximate solution, exact solution and absolute error, where, as figure 1, shows the approximate and exact solution of Eq. (4.1) $u(t) = \frac{\sin(t)}{\sin(1)} - t$.

Table 1: Comparison of exact, Ref [11] , WRMEW solution and the absolute errors of test problem 4.1

t	Ref [11]	WRMEW	Exact solution	Absolute error	
				Ref [11]	WRMEW
0.1	0.0186708	0.0185968	0.0186420	2.88e-05	4.50e-05
0.2	0.0361655	0.0360428	0.0360977	6.78e-05	5.50e-05
0.3	0.0512714	0.0511785	0.0511948	7.66e-05	1.60e-05
0.4	0.0628316	0.0627884	0.0627829	4.87e-05	5.50e-06
0.5	0.0697452	0.0697454	0.0697470	1.84e-06	1.60e-06
0.6	0.0709672	0.0710047	0.0710184	5.12e-05	1.40e-05
0.7	0.0655087	0.0655570	0.0655851	7.64e-05	2.80e-05
0.8	0.0524367	0.0524753	0.0525025	6.58e-05	2.70e-05
0.9	0.0308742	0.0308913	0.0309019	2.77e-05	1.10e-05

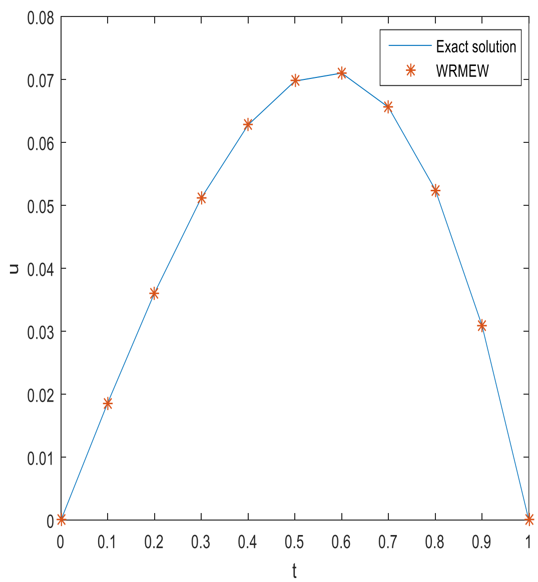

Figure 1: Graphical representation of WRMEW solution with the exact solution for test problem 4.1.

Test problem 4.2. Another differential equation of the form [12],

$$\frac{\partial^2 u}{\partial t^2} + \frac{16}{9}\pi^2 u = \frac{7}{9}\pi^2 \sin(\pi t), \; 0 \le t \le 1 \qquad (4.8)$$

$$\text{BCs}: \quad u(0) = 0, \; u(1) = 0 \qquad (4.9)$$

As per explained in section 3 and in test problem 4.1, obtained the values of $c_{1,0} = 3.5566$, $c_{1,1} = 0.4401$ and $c_{1,2} = -0.3594$. The numerical solution was then derived by substituting the values of $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ in Eq. (4.5). The numerical solution, the exact solution and absolute error of Eq. (4.8) $u(t) = \sin(\pi t)$ is presented in table 2, where figure 2 gives the graphical representation of the numerical and exact solution.

Table 2: Comparison of Ref [12], WRMEW solution with exact solution and the absolute errors for test problem 4.2

t	Ref [12]	WRMEW	Exact solution	Absolute error	
				Ref [11]	WRMEW
0.1	0.308930	0.3090056	0.309016	8.69e-05	1.00e-05
0.2	0.588656	0.5887781	0.588772	8.71e-04	6.10e-06
0.3	0.809599	0.8095397	0.809016	5.82e-04	5.20e-04
0.4	0.950632	0.9507637	0.951056	4.25e-04	2.90e-04
0.5	0.999072	0.9991750	1.000000	9.28e-04	8.20e-04
0.6	0.950687	0.9507496	0.951056	3.69e-04	3.10e-04
0.7	0.809697	0.8096150	0.809016	6.80e-04	6.00e-04
0.8	0.588766	0.5887500	0.587785	9.81e-04	9.60e-04
0.9	0.309113	0.3089845	0.309016	9.70e-05	3.20e-05

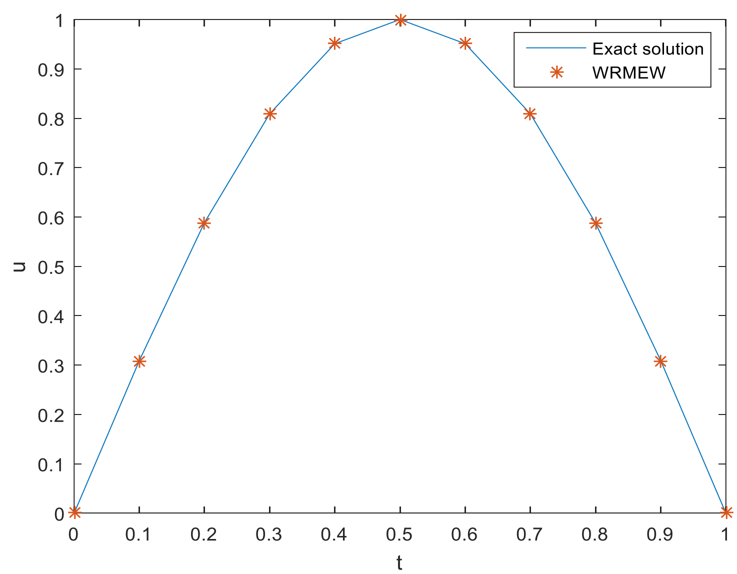

Figure 2: Graphical representation of the WRMEW solution with the exact solution for test problem 4.2.

Test problem 4.3. One more differential equation of the form [13],

$$\frac{\partial^2 u}{\partial t^2} - 4u = 4\cosh(1),\ 0 \le t \le 1 \tag{4.10}$$

$$\text{BCs:}\quad u(0)=0,\ u(1)=0 \tag{4.11}$$

Section 3 and the earlier problems are followed in order to obtain the values of the unknown coefficients i.e. $c_{1,0} = -2.2600$, $c_{1,1} = 0.0882$ and $c_{1,2} = -0.0720$. Enter the values of $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ in Eq. (4.5) then obtained numerical solution. Figure 3 shows a graphical representation of the numerical solution with the exact solution of Eq. (4.10) $u(t) = \cosh(2t-1) - \cosh(1)$. The numerical solution to the absolute errors is represented in Table 3.

Table 3: Comparison of WRMEW solution and absolute error with the exact solution for test problem 4.3.

t	WRMEW	Exact solution	Absolute error
0.1	-0.2056232	-0.2056457	2.20e-05
0.2	-0.3576501	-0.3576124	3.80e-05
0.3	-0.4620069	-0.4620083	1.40e-06
0.4	-0.5229269	-0.5230139	8.70e-05
0.5	-0.5429500	-0.5430806	1.30e-04
0.6	-0.5229233	-0.5230139	9.10e-05
0.7	-0.4620007	-0.4620083	7.60e-06
0.8	-0.3576430	-0.3576124	3.10e-05
0.9	-0.2056179	-0.2056457	2.80e-05

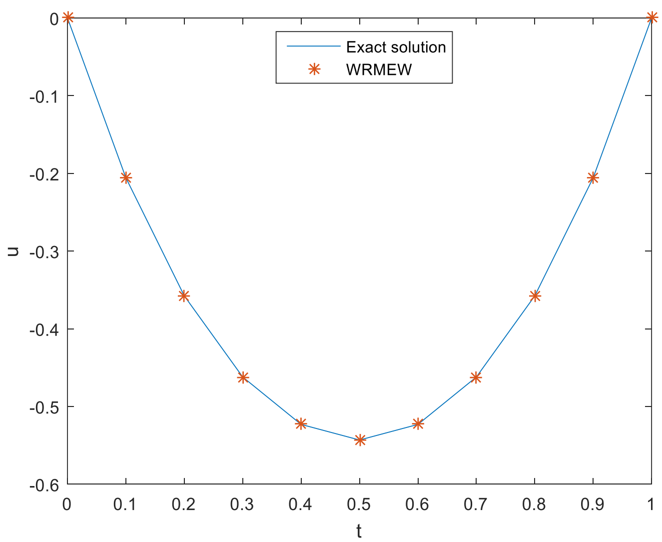

Figure 3: Graphical representation of the WRMEW solution with the exact solution for test problem 4.3.

Test problem 4.4. Finally, the non-linear differential equation of the form [14]

$$\frac{\partial^2 u}{\partial t^2} - u^2 = 2\pi^2 \cos(2\pi t) - \sin^4(2\pi t),\ 0 \le t \le 1 \qquad (4.12)$$

$$\text{BCs:} \qquad u(0)=0,\ u(1)=0 \qquad (4.13)$$

The table 4 represents the WRMEW solution and absolute error with the exact solution of Eq. (4.11) $u(t) = \sin^2(\pi t)$. The graphical representation of the numerical solution with the exact solution is given in figure 4, which was derived as described in section 3.

Table 4: Comparison of WRMEW solution and absolute error with the exact solution for test problem 4.4.

t	WRMEW	Exact solution	Absolute error
0.1	0.096578	0.0954920	1.09E-03
0.2	0.350889	0.3454920	5.40E-03
0.3	0.656662	0.6545082	2.15E-03
0.4	0.9057864	0.9045082	1.28E-03
0.5	0.9989985	1	1.00E-03
0.6	0.910045	0.9045082	5.54E-03
0.7	0.656598	0.6545082	2.09E-03
0.8	0.346989	0.3454920	1.50E-03
0.9	0.096834	0.0954920	1.34E-03

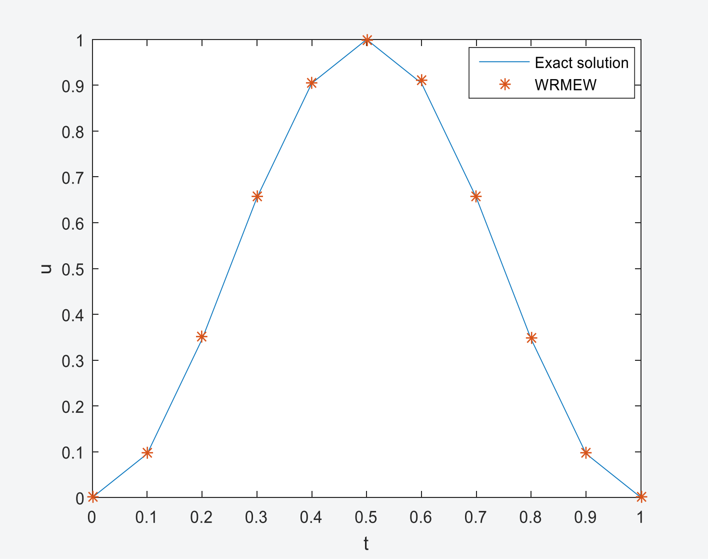

Figure 4: Graphical representation of the WRMEW solution with the exact solution for test problem 4.4.

5. Conclusion

This study introduces a numerical solution for one-dimensional differential equations using a weighted residual method that incorporates Euler wavelets. The findings demonstrate that this method yields superior numerical solutions compared to previously established techniques (Ref [11] and [12]). The results indicate a closer alignment with the exact solutions, highlighting the effectiveness of the proposed approach. Additionally, the absolute error associated with the weighted residual method utilizing Euler wavelets is significantly lower than that of the existing methods, underscoring its potential as a highly effective tool for solving one-dimensional differential equations.

Acknowledgements: We are grateful to the reviewers for their critical suggestions and corrections, which have significantly contributed to the improvement of this paper.

Conflict of interest. The author declares no conflicts of interest.

Author's Contributions: This work represents the sole contribution of the author.

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