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A Lower Bound of the First Zagreb Index and Some Hamiltonian Properties of Graphs

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Abstract. The first Zagreb index of a graph is defined as the sum of the squares of the degrees of vertices in the graph. In this note, we present a lower bound for the first Zagreb index of a graph. Using that lower bound, we present sufficient conditions for the Hamiltonian properties of a graph.

Keywords: The first Zagreb index, Hamiltonian graph, traceable graph

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G = (V(G), E(G)) be a graph with n vertices and e edges, The degree of a vertex v is denoted by $d_G(v)$. We use δ and Δ to denote the minimum degree and maximum degree of G, respectively. A set of vertices in a graph G is independent if the vertices in the set are pairwise nonadjacent. A maximum independent set in a graph G is an independent set of the largest possible size. The independence number, denoted β(G), of a graph G is the cardinality of a maximum independent set in G. For disjoint vertex subsets X and Y of V(G), we use E(X, Y) to denote the set of all the edges in E(G) such that one end vertex of each edge is in X and another end vertex of the edge is in Y. Namely, $E(X, Y) := \{ e : e = xy \in E, x \in X, y \in Y \}$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path.

The first Zagreb index of a graph was introduced by Gutman and Trinajstić [4] in 1972. For a graph G, its first Zagreb index, denoted $M_1(G)$, is defined as $\sum_{u \in V(G)}(d_G(u))^2$. Since its introduction, the first Zagreb index has been intensively investigated and a variety of results on the first Zagreb index have been obtained. The survey paper [3] and the references therein are good resources for the results. While investigating the first Zagreb index, researchers are often concerned about the bounds of it. In this note, we present a new lower bound for the first Zagreb of a graph. Using that lower bound, we present sufficient conditions for Hamiltonian and traceable graphs. The main results are as follows.

Theorem 1.1. Let G be a graph with n vertices and e edges. Then

$$M_1(G) \geq \beta\,(\delta^2 + \Delta^2) + 2\,\Delta\, e - n\,\Delta^2$$

with equality if and only if G is a bipartite graph with partition sets I and V – I such that $|I| = \beta$, $d(u) = \delta$ for each $u \in I$, and $d(v) = \Delta$ for each $v \in V - I$.

Using Theorem 1.1, we can prove the following corollaries.

Corollary 1.2. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges. If

$$M_1(G) \leq (k + 1)\,(\delta^2 + \Delta^2) + 2\,\Delta\, e - n\,\Delta^2,$$

then G is Hamiltonian or G is $K_{k,\, k+1}$.

Corollary 1.3. Let G be a k-connected ($k \geq 1$) graph with $n \geq 9$ vertices and e edges. If

$$M_1(G) \leq (k + 2)\,(\delta^2 + \Delta^2) + 2\,\Delta\, e - n\,\Delta^2,$$

then G is traceable or G is $K_{k,\, k+2}$.

2. Lemmas

We will use the following results as our lemmas. The first two are from [2].

Lemma 2.1. [2] Let G be a k-connected graph of order $n \geq 3$. If $\beta \leq k$, then G is Hamiltonian.

Lemma 2.2. [2] Let G be a k-connected graph of order n. If $\beta \leq k + 1$, then G is traceable.

Lemma 2.3 below is from [6].

Lemma 2.3. [6] Let G be a balanced bipartite graph of order 2n with bipartition (A, B). If $d(x) + d(y) \geq n + 1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then G is Hamiltonian.

Lemma 2.4 below is from [5].

Lemma 2.4. [5] Let G be a 2-connected bipartite graph with bipartition (A, B), where $|A| \geq |B|$. If each vertex in A has a degree of at least s and each vertex in B has a degree at least t, then G contains a cycle of length at least $2\min(|B|, s + t - 1, 2s - 2)$.

3. Proofs of the theorems

Proof of Theorem 1.1. Let G be a graph with n vertices and e edges.
Let $I := \{ u_1, u_2, ..., u_\beta \}$
be a maximum independent set in G. Then

$$\sum_{u \in I} d(u) = |E(I, V - I)| \leq \sum_{v \in V - I} d(v).$$

Since $\sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = 2e$, we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V - I} d(v).$$

Let v be any vertex in V – I. Clearly, $(\Delta - d(v))^2 \geq 0$ and $\Delta^2 + d^2(v) \geq 2 \Delta d(v)$. Thus

$$\sum_{v \in V - I} (\Delta^2 + d^2(v)) \geq \sum_{v \in V - I} (2 \Delta d(v)).$$

Therefore

$$(n - \beta) \Delta^2 + \sum_{v \in V - I} d^2(v) \geq 2 \Delta \sum_{v \in V - I} d(v) \geq 2 \Delta e.$$

Hence

$$\sum_{v \in V - I} d^2(v) \geq 2 \Delta e - (n - \beta) \Delta^2.$$

So

$$M_1(G) = \sum_{w \in V} d^2(w) = \sum_{u \in I} d^2(u) + \sum_{v \in V - I} d^2(v)$$

$$\geq \beta \delta^2 + 2 \Delta e - (n - \beta) \Delta^2 = \beta (\delta^2 + \Delta^2) + 2 \Delta e - n \Delta^2.$$

If

$$M_1(G) = \beta (\delta^2 + \Delta^2) + 2 \Delta e - n \Delta^2.$$

We, from the proofs above, have that $\sum_{v \in V - I} d(v) = e$ which implies that $\sum_{u \in I} d(u) = e$ and G is a bipartite graph with partition sets I and V – I such that $|I| = \beta$, $d(u) = \delta$ for each $u \in I$, and $d(v) = \Delta$ for each $v \in V - I$.

If G is a bipartite graph with partition sets I and V – I such that $|I| = \beta$, $d(u) = \delta$ for each $u \in I$, and $d(v) = \Delta$ for each $v \in V - I$, then $e = (n - \beta) \Delta$. A simple computation shows that

$$M_1(G) = \beta (\delta^2 + \Delta^2) + 2 \Delta e - n \Delta^2.$$

This completes the proof of Theorem 1.1. ■

Proof of Corollary 1.2. Let G be a k-connected ($k \geq 2$) graph with $n \geq 3$ vertices and e edges satisfying the conditions in Corollary 1.2. Suppose G is not Hamiltonian. Then Lemma 2.1 implies that $\beta \geq k + 1$. Also, we have that $n \geq 2 \delta + 1 \geq 2 k + 1$ otherwise $\delta \geq k \geq n/2$ and G is Hamiltonian. From the conditions in Corollary 1.2 and Theorem 1.1, we have

$$(k + 1) (\delta^2 + \Delta^2) + 2 \Delta e - n \Delta^2 \geq M_1(G)$$
$$\geq \beta (\delta^2 + \Delta^2) + 2 \Delta e - n \Delta^2 \geq (k + 1) (\delta^2 + \Delta^2) + 2 \Delta e - n \Delta^2.$$

Thus

$$(k + 1) (\delta^2 + \Delta^2) + 2 \Delta e - n \Delta^2 = M_1(G) = \beta (\delta^2 + \Delta^2) + 2 \Delta e - n \Delta^2.$$

From the proofs above and Theorem 1.1, we have that $\beta = k + 1$ and G is a bipartite graph with partition sets I and V – I such that $|I| = \beta$, $d(u) = \delta$ for each $u \in I$, and $d(v) = \Delta$ for each $v \in V - I$. Since V – I now is an independent set in G, we have $|V - I| \leq |I| = k + 1$. Thus

$n \leq 2k + 2$. Since $n \geq 2k + 1$, we have that $n = 2k + 2$ or $n = 2k + 1$. If $n = 2k + 2$, then Lemma 2.3 implies that G is Hamiltonian, a contradiction. If $n = 2k + 1$, then G is $K_{k, k+1}$.

This completes the proof of Corollary 1.2. ■

The proof of Corollary 1.3 is similar to the proof of Corollary 1.2. For the sake of completeness, we still present a full proof of Corollary 1.3 below.

Proof of Corollary 1.3. Let G be a k-connected ($k \geq 1$) graph with $n \geq 9$ vertices and e edges satisfying the conditions in Corollary 1.3. Suppose G is not traceable. Then Lemma 2.2 implies that $\beta \geq k + 2$. Also, we have that $n \geq 2\delta + 2 \geq 2k + 2$ otherwise $\delta \geq k \geq (n - 1)/2$ and G is traceable. From the conditions in Corollary 1.3 and Theorem 1.1, we have

$$(k + 2)(\delta^2 + \Delta^2) + 2\Delta e - n\Delta^2 \geq M_1(G)$$
$$\geq \beta(\delta^2 + \Delta^2) + 2\Delta e - n\Delta^2 \geq (k + 2)(\delta^2 + \Delta^2) + 2\Delta e - n\Delta^2.$$

Thus

$$(k + 2)(\delta^2 + \Delta^2) + 2\Delta e - n\Delta^2 = M_1(G) = \beta(\delta^2 + \Delta^2) + 2\Delta e - n\Delta^2.$$

From the proofs above and Theorem 1.1, we have that $\beta = k + 2$ and G is a bipartite graph with partition sets I and $V - I$ such that $|I| = \beta$, $d(u) = \delta$ for each $u \in I$, and $d(v) = \Delta$ for each $v \in V - I$. Since $V - I$ now is an independent set in G, we have $|V - I| \leq |I| = k + 2$. Thus $n \leq 2k + 4$. Since $n \geq 2k + 2$, we have that $n = 2k + 4$, $n = 2k + 3$, or $n = 2k + 1$. If $n = 2k + 4$, since $n \geq 9$, we have that $k \geq 3$. Therefore Lemma 2.3 implies that G is Hamiltonian and thereby G is traceable, a contradiction. If $n = 2k + 3$, since $n \geq 9$, we have that $k \geq 3$. Thus Lemma 2.4 implies that G has a cycle of length at least $(n - 1)$ and thereby G is traceable.

If $n = 2k + 2$, then G is $K_{k, k+2}$.

This completes the proof of Corollary 1.3. ■

4. Conclusion

In this note, we present a lower bound for the first Zagreb of a graph. Using that lower bound, we present sufficient conditions for Hamiltonian and traceable graphs.

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