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Extension of Occasionally Weak Compatible Maps in Fuzzy Metric Space

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Abstract. The primary aim of this paper is to develop and establish common fixed point theorems for seven self-mappings within a fuzzy metric space framework, leveraging the concept of occasionally weak compatibility. This research significantly broadens the scope of fixed point theory by extending and generalizing various existing results across different mathematical spaces.

Keywords: Fixed point, fixed point theorem, occasionally weak compatibility, fuzzy metric space and compatible maps

AMS Mathematics Subject Classification (2010): 46B85, 30L05

1. Introduction

The foundations of fuzzy set theory and fuzzy mathematics were laid down by Zadeh [16] in 1965 with the introduction of the notion of fuzzy sets. The theory of fuzzy sets has vast applications in applied sciences and engineering such as neural network theory, stability theory, mathematical programming, genetics, nervous systems, image processing, control theory, etc. to name a few. The theory of fixed points is one of the basic tools for handling physical formulations. This has led to the development and fuzzification of several concepts of analysis and topology. In 1975, Kramosil and Michalek [10] introduced the concept of a fuzzy metric space by generalizing the concept of a probabilistic metric space to the fuzzy situation. The concept of Kramosil and Michalek of a fuzzy metric space was later modified by George and Veeramani [6] in 1994. In 1988, Grabeic [7] followed the concept of Kramosil and Michalek [10] and obtained the fuzzy version of Banach's fixed point theorem. Using the notion of weak commuting property, Sessa [13] improved commutative conditions in fixed point theorems. Jungck [8] introduced the concept of compatibility in metric spaces. The concept of compatibility in fuzzy metric space was proposed by Mishra et al. [12]. In 1996, Jungck [9] introduced the concept of weakly compatible maps which was the generalization of the concept of compatible maps. Cho [5] and Singh and Chauhan [14] provided fixed point theorems in fuzzy metric space for four self maps using the concept of compatibility where two mappings needed to be continuous.

Many authors have discussed and studied extensively various results on coincidence existence and uniqueness of fixed and common fixed points by using the concept of weak commutativity compatibility non-compatibility and weak compatibility for single and set-valued maps satisfying certain contractive conditions in different spaces and they have been applied to diverse problems. Al-Thagafi and Shahzad [3] weakened the concept of compatibility by giving a new notion of occasionally weakly compatible maps which is more general among the commutativity concepts. Following their results, many authors like Alamgir Khan and Sumitra [1], Amit Govery and Mamta Singh [4], Manoj and Rathore [11], Ali et al. [2] and Sumitra D. [15] have studied and developed several common fixed point theorems in this framework. Despite the significant progress in the field of fuzzy metric spaces and fixed point theory, especially with the introduction of occasionally weakly compatible maps, there remains a gap in the generalization and extension of these results. Previous works, including those by Al-Thagafi, Shahzad, and others, have focused primarily on a limited number of mappings and contractive conditions. However, the study of common fixed points for multiple (up to seven) self-mappings in the fuzzy metric space, particularly within the framework of occasionally weak compatibility, has not been thoroughly explored. This creates an opportunity to extend existing results and develop new theorems that encompass more general cases and provide broader applicability across various mathematical spaces. Considering the contemplations given by different researchers, the primary aim of this paper is to develop and establish common fixed point theorems for seven self-mappings within a fuzzy metric space framework, leveraging the concept of occasionally weak compatibility.

2. Preliminaries

Definition 2.1. Let X be any set. A fuzzy set A in X is a function with domain in X and values in $[0,1]$.

Definition 2.2. A binary operation $*: [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t-$norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.3. The $3-$tuple $(X, \mathcal{M}, *)$ is called a fuzzy metric space if X is an arbitrary non$-$empty set, $*$ is a continuous $t-$norm and $\mathcal{M}$ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

(i) $\mathcal{M}(x, y, 0) > 0$;
(ii) $\mathcal{M}(x, y, t) = 1$ for all $t > 0$, iff $x = y$;
(iii) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
(iv) $\mathcal{M}(x, y, t) * \mathcal{M}(y, z, s) \leq \mathcal{M}(x, z, t + s)$;
(v) $\mathcal{M}(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 2.1. Let (X, d) be a metric space. Define $a * b = \min(a, b)$, and

$$\mathcal{M}(x, y, t) = \frac{t}{t + d(x, y)}$$

Induced by the metric d is often called the standard fuzzy metric.

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. A sequence $\{x_n\}$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be convergent to $x \in X$ if there exists $n_0 \in \mathbb{N}$ such that $\lim_{n \to \infty} \mathcal{M}(x_n, x, t) > 1 - \varepsilon$ for all $t > 0$ & $n \geq n_0$. A fuzzy metric space $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence in X converges to a point in X.

Lemma 2.1. $\mathcal{M}(x, y, .)$ is non-decreasing for all $x, y \in X$.
Proof: Suppose $\mathcal{M}(x, y, t) > M(x, y, s)$ for some $0 < t < s$.
Then $\mathcal{M}(x, y, t) * \mathcal{M}(y, y, s - t) \leq \mathcal{M}(x, y, s) < M(x, y, t)$.
Since $\mathcal{M}(y, y, s - t) = 1$,
Therefore, $\mathcal{M}(x, y, t) \leq \mathcal{M}(x, y, s) < M(x, y, t)$, this is a contradiction.
Thus, $\mathcal{M}(x, y, .)$ is non-decreasing for all $x, y \in X$.

Lemma 2.2. Let $(X, \mathcal{M}, *)$ be a fuzzy metric space then $\mathcal{M}$ is a continuous function on $X^2 \times (0, \infty)$ throughout in this paper $(X, \mathcal{M}, *)$ will denote the fuzzy metric space with the following condition $\lim_{n\to\infty} \mathcal{M}(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Lemma 2.3. If for all $x, y \in X, t > 0$ and $0 < k < 1$,
$\mathcal{M}(x, y, kt) \geq \mathcal{M}(x, y, t)$, then $x = y$.
Proof: Suppose that there exists $0 < k < 1$ such that
$\mathcal{M}(x, y, kt) \geq \mathcal{M}(x, y, t)$ for all $x, y \in X$ and $t > 0$.
Then $\mathcal{M}(x, y, t) \geq \mathcal{M}\left(x, y, \frac{t}{k}\right)$, and
$\mathcal{M}(x, y, t) \geq \mathcal{M}\left(x, y, \frac{t}{k^n}\right)$, for positive integer n.
Taking limit as $n \to \infty$ $\mathcal{M}(x, y, t) \geq 1$ and hence
$x = y$.

Definition 2.5. Two self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be weakly commuting if $\mathcal{M}(ABz, BAz, t) \geq \mathcal{M}(Az, Bz, t)$ for all $z \in X$ and $t > 0$.

Definition 2.6. Let A and B be mappings from a fuzzy metric space $(X, \mathcal{M}, *)$ into itself. Then the mappings are said to be compatible if
$\lim_{n \to \infty} \mathcal{M}(ABx_n, BAx_n, t) = 1$, for all $t > 0$,whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x$ for some $x \in X$.

Definition 2.7. If A and B are two self mappings of a fuzzy metric space $(X, \mathcal{M}, *)$, then a point $x \in X$ is called the coincidence point of A and B if and only if $Ax = Bx$.

Definition 2.8. Two self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be weakly compatible or coincidently commuting if they commute at their coincidence points, that is if $ABx = BAx$ whenever $Ax = Bx$ for some $x \in X$.

Definition 2.9. Two self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be occasionally weakly compatible if and only if there exists a point $x \in X$ which is the coincidence point of A and B at which A and B commute.

Definition 2.10. A pair (A, B) of self mappings of a fuzzy metric space $(X, \mathcal{M}, *)$ is said to be semi−compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} AB\, x_n = Bx$ whenever $\lim_{n\to\infty} A\, x_n = \lim_{n\to\infty} B\, x_n = x$ for some $x \in X$.

Lemma 2.4. Let X be a set, A and B be occasionally weakly compatible self maps on X of a fuzzy metric space $(X, \mathcal{M}, *)$. If A and B have unique points of coincidence that is $w = Ax = Bx$ for $x \in X$, then w is the unique common fixed point of A and B.

3. The main results

Theorem 3.1. Let $(X, \mathcal{M}, *)$ be a complete fuzzy metric space and let A, B, R, S, T, P and Q be self mappings of X. Let the pairs (P, ABR) and (Q, STR) be occasionally weakly compatible. If there exist $k \in (0,1)$ such that

$$\mathcal{M}(Px, Qy, kt) \geq min \left\{ \begin{array}{c} \mathcal{M}(ABRx, STRy, t), \mathcal{M}(Qy, ABRx, t), \mathcal{M}(Px, STRy, t), \mathcal{M}(ABRx, Px, t), \\ \dfrac{a\, \mathcal{M}(Px, Qy, t) + b\, \mathcal{M}(Px, STRy, t)}{a\, \mathcal{M}(ABRx, Qy, t) + b\, \mathcal{M}(ABRx, STRy, t)}, \\ \dfrac{c\, \mathcal{M}(ABRx, Qy, t) + d\, \mathcal{M}(ABRx, STRy, t)}{c\, \mathcal{M}(Qy, STRy, t) + d}, \\ \dfrac{e\, \mathcal{M}(ABRx, Px, t) + f\, \mathcal{M}(Qy, STRy, t)}{e + f}, \\ \dfrac{a\, \mathcal{M}(Px, STRy, t) + b\, \mathcal{M}(ABRx, Qy, t) + c\, \mathcal{M}(ABRx, STRy, t)}{a + b + c}, \\ \dfrac{d\, \mathcal{M}(Px, Px, t) + e\, \mathcal{M}(Qy, Qy, t) + f\, \mathcal{M}(Px, Qy, t)}{d + e + f\, \mathcal{M}(Px, Qy, t)} \end{array} \right\} \quad (1)$$

For all $x, y \in X$ and $t > 0$ where $a, b, c, d, e, f \geq 0$ with $a \,\&\, b$, $c \,\&\, d$ and $e \,\&\, f$ cannot be simultaneously 0, then there exist a unique point $w \in X$ such that $Pw = ABRw = w$ and a unique point $z \in X$ such that $Qz = STRz = z$. Moreover $w = z$, so that there is a unique common fixed point of A, B, R, S, T, P and Q.

Proof: Let the pairs (P, ABR) and (Q, STR) be occasionally weakly compatible, so there are points $x, y \in X$ such that $Px = ABRx$ and $Qy = STRy$.

We claim $Px = Qy$.

If not, by inequality (1)

$$\mathcal{M}(Px, Qy, kt)$$
$$\geq \min \left\{ \begin{array}{c} \mathcal{M}(ABRx, STRy, t), \mathcal{M}(Qy, ABRx, t), \mathcal{M}(Px, STRy, t), \mathcal{M}(ABRx, Px, t), \\ \dfrac{a\,\mathcal{M}(Px, Qy, t) + b\,\mathcal{M}(Px, STRy, t)}{a\,\mathcal{M}(ABRx, Qy, t) + b\,\mathcal{M}(ABRx, STRy, t)}, \\ \dfrac{c\,\mathcal{M}(ABRx, Qy, t) + d\,\mathcal{M}(ABRx, STRy, t)}{c\,\mathcal{M}(Qy, STRy, t) + d}, \\ \dfrac{e\,\mathcal{M}(ABRx, Px, t) + f\,\mathcal{M}(Qy, STRy, t)}{e + f}, \\ \dfrac{a\,\mathcal{M}(Px, STRy, t) + b\,\mathcal{M}(ABRx, Qy, t) + c\,\mathcal{M}(ABRx, STRy, t)}{a + b + c}, \\ \dfrac{d\,\mathcal{M}(Px, Px, t) + e\,\mathcal{M}(Qy, Qy, t) + f\,\mathcal{M}(Px, Qy, t)}{d + e + f\,\mathcal{M}(Px, Qy, t)} \end{array} \right\}$$

$$= \min \left\{ \begin{array}{c} \mathcal{M}(Px, Qy, t), \mathcal{M}(Qy, Px, t), \mathcal{M}(Px, Qy, t), \mathcal{M}(Px, Px, t), \\ \dfrac{a\,\mathcal{M}(Px, Qy, t) + b\,\mathcal{M}(Px, Qy, t)}{a\,\mathcal{M}(Px, Qy, t) + b\,\mathcal{M}(Px, Qy, t)}, \\ \dfrac{c\,\mathcal{M}(Px, Qy, t) + d\,\mathcal{M}(Px, Qy, t)}{c\,\mathcal{M}(Qy, Qy, t) + d}, \\ \dfrac{e\,\mathcal{M}(Px, Px, t) + f\,\mathcal{M}(Qy, Qy, t)}{e + f}, \\ \dfrac{a\,\mathcal{M}(Px, Qy, t) + b\,\mathcal{M}(Px, Qy, t) + c\,\mathcal{M}(Px, Qy, t)}{a + b + c}, \\ \dfrac{d\,\mathcal{M}(Px, Px, t) + e\,\mathcal{M}(Qy, Qy, t) + f\,\mathcal{M}(Px, Qy, t)}{d + e + f\,\mathcal{M}(Px, Qy, t)} \end{array} \right\}$$

$$= \min \left\{ \begin{array}{c} \mathcal{M}(Px, Qy, t), \mathcal{M}(Qy, Px, t), \mathcal{M}(Px, Qy, t), 1, \\ 1, \mathcal{M}(Px, Qy, t), 1, \mathcal{M}(Px, Qy, t), 1 \end{array} \right\}$$
$$= \mathcal{M}(Px, Qy, t)$$

Therefore $Px = Qy$ that is $Px = ABRx = Qy = STRy$. Suppose that there is another point z such that $Pz = ABRz$ then by inequality (1) we have $Pz = ABRz = Qz = STRz$. So $Px = Pz$ and $w = Px = ABRx$ is the unique point of coincidence of P and ABR. By lemma 2.4, w is the only common fixed point of P and ABR. Similarly there is a unique point $z \in X$ such that $z = Qz = STRz$. Assume that $w \neq z$.
We have, by inequality (1)

$$\mathcal{M}(w,z,kt) = \mathcal{M}(Pw,Qz,kt)$$

$$= min\left\{\begin{matrix} \mathcal{M}(ABRw,STRz,t), \mathcal{M}(Qz,ABRw,t), \mathcal{M}(Pw,STRz,t), \mathcal{M}(ABRw,Pw,t), \\ \dfrac{a\,\mathcal{M}(Pw,Qz,t) + \,b\,\mathcal{M}(Pw,STRz,t)}{a\,\mathcal{M}(ABRw,Qz,t) + \,b\,\mathcal{M}(ABRw,STRz,t)}, \\ \dfrac{c\,\mathcal{M}(ABRw,Qz,t) + d\,\mathcal{M}(ABRw,STRz,t)}{c\,\mathcal{M}\,(Qz,STRz,t) + \,d}, \\ \dfrac{e\,\mathcal{M}\,(ABRw,Pw,t) + \,f\,\mathcal{M}\,(Qz,STRz,t)}{e\,+\,f}, \\ \dfrac{a\,\mathcal{M}(Pw,STRz,t) + b\,\mathcal{M}(ABRw,Qz,t) + c\,\mathcal{M}(ABRw,STRz,t)}{a+b+c}, \\ \dfrac{d\,\mathcal{M}(Pw,Pw,t) + e\,\mathcal{M}(Qz,Qz,t) + f\,\mathcal{M}(Pw,Qz,t)}{d+e+f\,\mathcal{M}(Pw,Qz,t)} \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mathcal{M}(w,z,t), \mathcal{M}(z,w,t), \mathcal{M}(w,z,t), \mathcal{M}(w,w,t), \\ \dfrac{a\,\mathcal{M}(w,z,t) + \,b\,\mathcal{M}(w,z,t)}{a\,\mathcal{M}(w,z,t) + \,b\,\mathcal{M}(w,z,t)}, \\ \dfrac{c\,\mathcal{M}(w,z,t) + d\,\mathcal{M}(w,z,t)}{c\,\mathcal{M}\,(z,z,t) + \,d}, \\ \dfrac{e\,\mathcal{M}\,(w,w,t) + \,f\,\mathcal{M}\,(z,z,t)}{e\,+\,f}, \\ \dfrac{a\,\mathcal{M}(w,z,t) + b\,\mathcal{M}(w,z,t) + c\,\mathcal{M}(w,z,t)}{a+b+c}, \\ \dfrac{d\,\mathcal{M}(w,w,t) + e\,\mathcal{M}(z,z,t) + f\,\mathcal{M}(w,z,t)}{d+e+f\,\mathcal{M}(w,z,t)} \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mathcal{M}(w,z,t), \mathcal{M}(z,w,t), \mathcal{M}(w,z,t), 1, \\ 1\,, \mathcal{M}(w,z,t)\,, 1, \mathcal{M}(w,z,t), 1 \end{matrix}\right\}$$

$$= \mathcal{M}(w,z,t)$$

Therefore we have $z = w$, by lemma 2.4, z is a common fixed point of A, B, R, S, T, P and Q.

For uniqueness, let u be another common fixed point of A, B, R, S, T, P and Q. Then

$$\mathcal{M}(z,u,kt) = \mathcal{M}(Pz,Qu,kt)$$

$$= min\left\{\begin{matrix} \mathcal{M}(ABRz,STRu,t), \mathcal{M}(Qu,ABRz,t), \mathcal{M}(Pz,STRu,t), \mathcal{M}(ABRz,Pz,t), \\ \dfrac{a\,\mathcal{M}(Pz,Qu,t) + \ b\,\mathcal{M}(Pz,STRu,t)}{a\,\mathcal{M}(ABRz,Qu,t) + \ b\,\mathcal{M}(ABRz,STRu,t)}, \\ \dfrac{c\,\mathcal{M}(ABRz,Qu,t) + d\,\mathcal{M}(ABRz,STRu,t)}{c\,\mathcal{M}\ (Qu,STRu,t) + \ d}, \\ \dfrac{e\,\mathcal{M}\ (ABRz,Pz,t) + \ f\,\mathcal{M}\ (Qu,STRu,t)}{e\ +\ f}, \\ \dfrac{a\,\mathcal{M}(Pz,STRu,t) + b\,\mathcal{M}(ABRz,Qu,t) + c\,\mathcal{M}(ABRz,STRu,t)}{a+b+c}, \\ \dfrac{d\,\mathcal{M}(Pz,Pz,t) + e\,\mathcal{M}(Qu,Qu,t) + f\mathcal{M}\ (Pz,Qu,t)}{d+e+f\,\mathcal{M}(Pz,Qu,t)} \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mathcal{M}(z,u,t), \mathcal{M}(u,z,t), \mathcal{M}(z,u,t), \mathcal{M}(z,z,t), \\ \dfrac{a\,\mathcal{M}(z,u,t) + \ b\,\mathcal{M}(z,u,t)}{a\,\mathcal{M}(z,u,t) + \ b\,\mathcal{M}(z,u,t)}, \\ \dfrac{c\,\mathcal{M}(z,u,t) + d\,\mathcal{M}(z,u,t)}{c\,\mathcal{M}\ (u,u,t) + \ d}, \\ \dfrac{e\,\mathcal{M}\ (z,z,t) + \ f\,\mathcal{M}\ (u,u,t)}{e\ +\ f}, \\ \dfrac{a\,\mathcal{M}(z,u,t) + b\,\mathcal{M}(z,u,t) + c\,\mathcal{M}(z,u,t)}{a+b+c}, \\ \dfrac{d\,\mathcal{M}(z,z,t) + e\,\mathcal{M}(u,u,t) + f\,\mathcal{M}(z,u,t)}{d+e+f\,\mathcal{M}(z,u,t)} \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mathcal{M}(z,u,t), \mathcal{M}(u,z,t), \mathcal{M}(z,u,t), 1, \\ 1, \mathcal{M}(z,u,t), 1, \mathcal{M}(z,u,t), 1 \end{matrix}\right\}$$

$$= \mathcal{M}(z,u,t)$$

Therefore by lemma 2.4, we have $z = u$.

Theorem 3.2. Let $(X, \mathcal{M}, *)$ be a complete fuzzy metric space and let A, B, R, S, T, P and Q be self mappings of X. Let the pairs (P, ABR) and (Q, STR) be occasionally weakly compatible. If there exist $k \in (0,1)$ such that

$$\mathcal{M}(Px, Qy, kt) \geq min \left\{ \begin{array}{c} \mathcal{M}(ABRx, STRy, t), \mathcal{M}(Qy, ABRx, t), \mathcal{M}(Px, STRy, t), \mathcal{M}(ABRx, Px, t), \\ \dfrac{a\, \mathcal{M}(Px, Qy, t) + b\, \mathcal{M}(Px, STRy, t)}{a\, \mathcal{M}(ABRx, Qy, t) + b\, \mathcal{M}(ABRx, STRy, t)}, \\ \dfrac{c\, \mathcal{M}(ABRx, Qy, t) + d\, \mathcal{M}(ABRx, STRy, t)}{c\, \mathcal{M}\,(Qy, STRy, t) + d} \\ \dfrac{a\, \mathcal{M}(Px, STRy, t) + b\, \mathcal{M}(ABRx, Qy, t) + c\, \mathcal{M}(ABRx, STRy, t)}{a+b+c} \end{array} \right\} \quad (2)$$

For all $x, y \in X$ and $t > 0$ where $a, b, c, d \geq 0$ with a & b and c & d cannot be simultaneously 0, then there exist a unique point $w \in X$ such that $Pw = ABRw = w$ and a unique point $z \in X$ such that $Qz = STRz = z$. Moreover $w = z$, so that there is a unique common fixed point of A, B, R, S, T, P and Q.

Proof: Let the pairs (P, ABR) and (Q, STR) be occasionally weakly compatible, so there are points $x, y \in X$ such that $Px = ABRx$ and $Qy = STRy$.

We claim $Px = Qy$.

If not, by inequality (2)

$$\mathcal{M}(Px, Qy, kt) \geq min \left\{ \begin{array}{c} \mathcal{M}(ABRx, STRy, t), \mathcal{M}(Qy, ABRx, t), \mathcal{M}(Px, STRy, t), \mathcal{M}(ABRx, Px, t), \\ \dfrac{a\, \mathcal{M}(Px, Qy, t) + b\, \mathcal{M}(Px, STRy, t)}{a\, \mathcal{M}(ABRx, Qy, t) + b\, \mathcal{M}(ABRx, STRy, t)}, \\ \dfrac{c\, \mathcal{M}(ABRx, Qy, t) + d\, \mathcal{M}(ABRx, STRy, t)}{c\, \mathcal{M}\,(Qy, STRy, t) + d}, \\ \dfrac{a\, \mathcal{M}(Px, STRy, t) + b\, \mathcal{M}(ABRx, Qy, t) + c\, \mathcal{M}(ABRx, STRy, t)}{a+b+c} \end{array} \right\}$$

$$= min \left\{ \begin{array}{c} \mathcal{M}(Px, Qy, t), \mathcal{M}(Qy, Px, t), \mathcal{M}(Px, Qy, t), \mathcal{M}(Px, Px, t), \\ \dfrac{a\, \mathcal{M}(Px, Qy, t) + b\, \mathcal{M}(Px, Qy, t)}{a\, \mathcal{M}(Px, Qy, t) + b\, \mathcal{M}(Px, Qy, t)}, \\ \dfrac{c\, \mathcal{M}(Px, Qy, t) + d\, \mathcal{M}(Px, Qy, t)}{c\, \mathcal{M}\,(Qy, Qy, t) + d}, \\ \dfrac{a\, \mathcal{M}(Px, Qy, t) + b\, \mathcal{M}(Px, Qy, t) + c\, \mathcal{M}(Px, Qy, t)}{a+b+c} \end{array} \right\}$$

$$= min \left\{ \begin{array}{c} \mathcal{M}(Px, Qy, t), \mathcal{M}(Qy, Px, t), \mathcal{M}(Px, Qy, t), 1, \\ 1, \mathcal{M}(Px, Qy, t), \mathcal{M}(Px, Qy, t) \end{array} \right\}$$

$$= \mathcal{M}(Px, Qy, t)$$

Therefore $Px = Qy$ that is $Px = ABRx = Qy = STRy$. Suppose that there is another point z such that $Pz = ABRz$ then by inequality (2) we have $Pz = ABRz = Qz =$

$STRz$. So $Px = Pz$ and $w = Px = ABRx$ is the unique point of coincidence of P and ABR. By lemma 2.4, w is the only common fixed point of P and ABR. Similarly there is a unique point $z \in X$ such that
$z = Qz = STRz$. Assume that $w \neq z$.
We have, by inequality (2)

$$\mathcal{M}(w,z,kt) = \mathcal{M}(Pw,Qz,kt)$$

$$= min\left\{\begin{matrix} \mathcal{M}(ABRw,STRz,t), \mathcal{M}(Qz,ABRw,t), \mathcal{M}(Pw,STRz,t), \mathcal{M}(ABRw,Pw,t), \\ \dfrac{a\,\mathcal{M}(Pw,Qz,t) + \,b\,\mathcal{M}(Pw,STRz,t)}{a\,\mathcal{M}(ABRw,Qz,t) + \,b\,\mathcal{M}(ABRw,STRz,t)}, \\ \dfrac{c\,\mathcal{M}(ABRw,Qz,t) + d\,\mathcal{M}(ABRw,STRz,t)}{c\,\mathcal{M}\,(Qz,STRz,t) + \,d} \\ \dfrac{a\,\mathcal{M}(Pw,STRz,t) + b\,\mathcal{M}(ABRw,Qz,t) + c\,\mathcal{M}(ABRw,STRz,t)}{a+b+c} \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mathcal{M}(w,z,t), \mathcal{M}(z,w,t), \mathcal{M}(w,z,t), \mathcal{M}(w,w,t), \\ \dfrac{a\,\mathcal{M}(w,z,t) + \,b\,\mathcal{M}(w,z,t)}{a\,\mathcal{M}(w,z,t) + \,b\,\mathcal{M}(w,z,t)}, \\ \dfrac{c\,\mathcal{M}(w,z,t) + d\,\mathcal{M}(w,z,t)}{c\,\mathcal{M}\,(z,z,t) + \,d} \\ \dfrac{a\,\mathcal{M}(w,z,t) + b\,\mathcal{M}(w,z,t) + c\,\mathcal{M}(w,z,t)}{a+b+c} \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mathcal{M}(w,z,t), \mathcal{M}(z,w,t), \mathcal{M}(w,z,t), 1, \\ 1, \mathcal{M}(w,z,t), \mathcal{M}(w,z,t) \end{matrix}\right\}$$

$$= \mathcal{M}(w,z,t)$$

Therefore we have $z = w$, by lemma 2.4, z is a common fixed point of A, B, R, S, T, P and Q.
For uniqueness, let u be another common fixed point of A, B, R, S, T, P and Q. Then

$$\mathcal{M}(z,u,kt) = \mathcal{M}(Pz,Qu,kt)$$

$$\geq min\left\{\begin{matrix} \mathcal{M}(ABRz,STRu,t), \mathcal{M}(Qu,ABRz,t), \mathcal{M}(Pz,STRu,t), \mathcal{M}(ABRz,Pz,t), \\ \dfrac{a\,\mathcal{M}(Pz,Qu,t) + \,b\,\mathcal{M}(Pz,STRu,t)}{a\,\mathcal{M}(ABRz,Qu,t) + \,b\,\mathcal{M}(ABRz,STRu,t)}, \\ \dfrac{c\,\mathcal{M}(ABRz,Qu,t) + d\,\mathcal{M}(ABRz,STRu,t)}{c\,\mathcal{M}\,(Qu,STRu,t) + \,d} \\ \dfrac{a\,\mathcal{M}(Pz,STRu,t) + b\,\mathcal{M}(ABRz,Qu,t) + c\,\mathcal{M}(ABRz,STRu,t)}{a+b+c} \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mathcal{M}(z,u,t), \mathcal{M}(u,z,t), \mathcal{M}(z,u,t), \mathcal{M}(z,z,t), \\ \dfrac{a\,\mathcal{M}(z,u,t) + b\,\mathcal{M}(z,u,t)}{a\,\mathcal{M}(z,u,t) + b\,\mathcal{M}(z,u,t)}, \\ \dfrac{c\,\mathcal{M}(z,u,t) + d\,\mathcal{M}(z,u,t)}{c\,\mathcal{M}\,(u,u,t) + d} \\ \dfrac{a\,\mathcal{M}(z,u,t) + b\,\mathcal{M}(z,u,t) + c\,\mathcal{M}(z,u,t)}{a+b+c} \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mathcal{M}(z,u,t), \mathcal{M}(u,z,t), \mathcal{M}(z,u,t), 1, \\ 1, \mathcal{M}(z,u,t), \mathcal{M}(z,u,t) \end{matrix}\right\}$$

$$= \mathcal{M}(z,u,t)$$

Therefore by lemma 2.4, we have $z = u$.

4. Motivation and scope of future work

The motivation behind this research is to address the aforementioned gap in the literature by expanding the fixed point theory to more complex mappings within fuzzy metric spaces. By investigating seven self-mappings and leveraging occasionally weak compatibility, this study aims to broaden the theoretical foundation of fuzzy metric space analysis and open up new avenues for solving complex problems in applied mathematics. The implications of this research extend beyond theoretical mathematics, potentially influencing applications in fields such as control theory, image processing, and neural networks, where fuzzy systems play a crucial role.

Future research can focus on extending these results to other types of mathematical spaces and exploring additional generalizations of occasionally weakly compatible maps. Furthermore, applying these theorems to real-world problems, such as those in engineering or data science, could yield valuable insights. Investigating fixed point results for larger sets of self-mappings or under different contractive conditions could further advance the field, providing a more comprehensive understanding of fuzzy metric spaces.

5. Conclusion

In this paper we have successfully established common fixed point theorems for seven self-mappings within a fuzzy metric space framework, leveraging the concept of occasionally weak compatibility. This advancement is significant as it not only broadens the scope of fixed point theory but also extends and generalizes several existing results across various types of spaces. The implications of our findings are particularly valuable, providing a more comprehensive framework that can be applied in diverse mathematical contexts. This work, therefore, offers a robust foundation for other researchers who are exploring similar problems, potentially leading to further developments and applications within the field of fixed point theory.

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