Annals of Pure and Applied Mathematics
Vol. 2, No. 2, 2012, 194-199
ISSN: 2279-087X (P), 2279-0888(online)
Published on 28 December 2012
www.researchmathsci.org

Annals of
Pure and Applied Mathematics

Some Characterizations of 0-distributive Nearlattice

A.S.A.Noor[1], Md. Zaidur Rahman[2] and Md. Bazlar Rahman[2]

[1]Department of ECE
East West University, Dhaka
E mail: noor@ewubd.edu

[2]Department of Mathematics
Khulna University of Engineering & Technology
E mail: mzrahman1968@gmail.com

Received 13 December 2012; accepted 23 December 2012

Abstract. In this paper we studied different properties of 0-distributive nearlattices. Here we prove that for a filter A of S, $A^0 = \{x \in S \mid x \wedge a = 0, \text{ for some } a \in A\}$ is an ideal if and only if S is 0-distributive. Then we include several characterizations of a 0-distributive nearlattice using A^0 where A is a filter. Finally we show that S is 0-distributive if and only if for all $a, b, c \in S$,

$$(a \wedge (b \vee c))^{\perp} = (a \wedge b)^{\perp} \cap (a \wedge c)^{\perp} \text{ provided } b \vee c \text{ exists.}$$

***Keywords**: 0-distributive nearlattice, Maximal filter, Prime filter, Minimal prime down set, Minimal prime ideal.*

***AMS Mathematics Subject Classification (2010)**: 06A12, 06A99, 06B10*

1. Introduction

J.C. Varlet [5] has given the definition of a 0-distributive lattice to generalize the notion of pseudocomplemented lattices. By [5], a lattice L with 0 is called a 0-distributive lattice if for all $a, b, c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. Then many authors including [1] and [4] studied the 0-distributive properties in lattices and meet semilattices. Recently [6] have studied the 0-distribuitve property in a nearlattice.

A *nearlattice* is a meet semilattice together with the property that any two elements possessing a common upper bound have a supremum. This property is known as the *upper bound property.*

A nearlattice S is called *distributive* if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, provided $y \vee z$ exists in S. For detailed literature on nearlattices, we refer the reader to consult [2] and [3].

A nearlattice S with 0 is called 0-distributive if for all $x, y, z \in S$ with $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists imply $x \wedge (y \vee z) = 0$.

It can be easily proved that it has the following alternative definition:
S is 0-distributive if for all $x, y, z, t \in S$ with $x \wedge y = 0 = x \wedge z$ imply $x \wedge ((t \wedge y) \vee (t \wedge z)) = 0$; $(t \wedge y) \vee (t \wedge z)$ exists by the upper bound property of S. Of course, every distributive nearlattice S with 0 is 0-distributive. Figure 1 is an example of a non-modular nearlattice which is 0-distributive, while Figure 2 gives a modular nearlattice which is not 0-distributive.

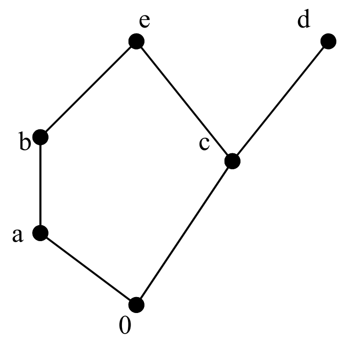

Figure 1

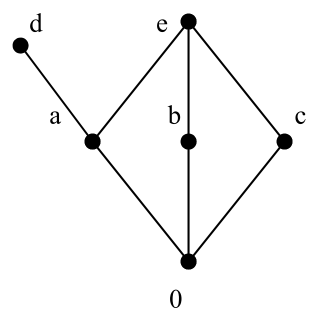

Figure 2

A subset I of a nearlattice S is called a *down set* if $x \in I$ and for $t \in S$ with $t \leq x$ imply $t \in I$. An *ideal(down set)* I in a nearlattice S is a non-empty subset of S such that it is a down set and whenever $a \vee b$ exists for $a, b \in I$, then $a \vee b \in I$. A proper ideal I in S is called a *prime ideal(down set)* if $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$. A non-empty subset F of S is called a filter if $t \geq a,\ a \in F$ implies $t \in F$ and if $a,\ b \in F$ then $a \wedge b \in F$. A proper filter F in S is called *prime* if $a \vee b$ exists and $a \vee b \in F$ implies either $a \in F$ or $b \in F$. In lemma 1, we prove that F is a filter of S if and only if S-F is a prime down set. Moreover, it is easy to show that a prime down set P is a prime ideal if and only if $S - P$ is a prime filter.

A proper filter M of a nearlattice S is called *maximal* if for any filter Q with $Q \supseteq M$ implies either $Q = M$ or $Q = S$. Dually, we define a *minimal prime ideal (down set)*

Let L be a lattice with 0. An element a^* is called the *pseudocomplement* of a if $a \wedge a^* = 0$ and if $a \wedge x = 0$ for some $x \in L$, then $x \leq a^*$. A lattice L with 0 and 1 is called *pseudocomplemented* if its every element has a pseudocomplement. Since a nearlattice S with 1 is a lattice, so pseudocomplementation is not possible in a general nearlattice. A nearlattice S with 0 is called sectionally

pseudocomplemented if the interval $[0,x]$ for each $x \in S$ is pseudocomplemented. For $A \subseteq S$, we denote $A^{\perp} = \{x \in S \mid x \wedge a = 0 \text{ for all } a \in A\}$. If S is distributive then clearly $A^{\perp}$ is an ideal of S.
Moreover, $A^{\perp} = \bigcap_{a \in A} \{\{a\}^{\perp}\}$. If A is an ideal, then obviously $A^{\perp}$ is the pseudocomplement of A in $I(S)$. Therefore, for a distributive nearlattice S with 0, I(S) is pseudocomplemented.

2. Some Results

Lemma 1. In a nearlattice S, F is a proper filter if and only if $S - F$ is a prime down set.
Proof. Let F be a proper filter. Let $x \in S - F$ and $t \leq x$. Then $x \notin F$ and so $t \notin F$ as F is a filter. Hence $t \in S - F$ and so $S - F$ is a down set.

Now let $a \wedge b \in S - F$ for some $a, b \in S$. Then $a \wedge b \notin F$. This implies either $a \notin F$ or $b \notin F$ and so either $a \in S - F$ or $b \in S - F$. Therefore $S - F$ is prime.

Conversely, suppose $S - F$ is a prime down set. Let $a \in F$ and $t \geq a$ $(t \in S)$. Then $a \notin S - F$ and so $t \notin S - F$ as it is a down set. Thus $t \in F$ and so F is an up set. Now let $a, b \in F$, then $a \notin S - F$ and $b \notin S - F$. Since $S - F$ is prime, so $a \wedge b \notin S - F$. This implies $a \wedge b \in F$, and so F is a filter. ●

Lemma 2. Every proper filter of a nearlattice with 0 is contained in a maximal filter.
Proof. Let F be a proper filter in S with 0. Let $\mathcal{F}$ be the set of all proper filters containing F. Then $\mathcal{F}$ is non-empty as $F \in \mathcal{F}$. Let C be a chain in $\mathcal{F}$ and let $M = \bigcup\{X \mid X \in C\}$. We claim that M is a filter with $F \subseteq M$. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as X is a filter. Therefore, $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. So $x, y \in Y$. Then $x \wedge y \in Y$ as Y is a filter. Hence $x \wedge y \in M$. Moreover M contains F. So M is maximum element of C. Then by Zorn's lemma $\mathcal{F}$ has a maximal element, say Q with $F \subseteq Q$.●

Use the same type of proof we have the following result

Lemma 3. *Every filter disjoint from an ideal I is contained in a maximal filter disjoint from I.* ●

Following result trivially follows from lemma 2 and lemma 1

Corollary 4. Every prime down set of a nearlattice contains a minimal prime down set.●

For a subset A of a nearlattice S, we define
$A^0 = \{x \in S \mid x \wedge a = 0 \text{ for some } a \in A\}$. It is easy to see that A^0 is a down set. Moreover, $\{a\}^{\perp} = \{a\}^0 = [a)^0$. By [7, Theorem 5] S is 0-distributive if and only if A^0 is an ideal for every filter A of S.

Theorem 5. Let S be a nearlattice with 0. Then the following conditions are equivalent.

(i) S is 0-distributive.

(ii) For a proper filter A , there exists a minimal prime ideal disjoint to A but containing A^0.

(iii) For a non-zero element $a \in S$, there is a minimal prime ideal containing $\{a\}^0$ but not containing a.

Proof. $(i) \Rightarrow (ii)$ Since A is a proper filter, so A^0 is an ideal by [7, Theorem 5]. Now $A \cap A^0 = \phi$. For if $x \in A \cap A^0$ then $x \in A$ and $x \wedge a = 0$ for some $a \in A$. This implies $0 \in A$, which is a contradiction as A is a proper. Then by lemma 3, there exists a maximal filter $M \supseteq A$ such that $M \cap A^0 = \phi$. Since S is 0-distributive , so by [6, Theorem 5], M is a prime filter. This implies $S - M$ is a minimal prime ideal containing A^0 and disjoint to A.

$(ii) \Rightarrow (iii)$ is trivial by considering the filter $[a)$.

$(iii) \Rightarrow (i)$ Suppose *(iii)* holds but S is not 0-distributive. Then there exists $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists but $a \wedge (b \vee c) \neq 0$. Then by *(iii)* , there exists a minimal prime ideal P such that $a \wedge (b \vee c) \notin P$ but $\{a \wedge (b \vee c)\}^0 \subseteq P$. Now $b \wedge [a \wedge (b \vee c)] = a \wedge b = 0$ and
$c \wedge [a \wedge (b \vee c)] = a \wedge c = 0$ imply $b, c \in \{a \wedge (b \vee c)\}^0 \subseteq P$. Since P is an ideal, so $b \vee c \in P$ and hence $a \wedge (b \vee c) \in P$ which gives a contradiction. Therefore S must be 0-distributive.●

Theorem 6. Let S be a nearlattice with 0. Then the following conditions are equivalent.

(i) S is 0-distributive.

(ii) If A is a an ideal and $\{A_i \mid i \in I\}$ is a family of ideals of S such that $A \cap A_i = (0]$ for all I, then $A \cap \left(\underset{i \in I}{\vee} A_i \right) = (0]$.

(iii) If $a_1, a_2, \cdots, a_n \in S$ such that $a \wedge a_1 = \cdots = a \wedge a_n = 0$, then $(a] \wedge ((a_1] \vee (a_2] \vee \cdots \vee (a_n]) = (0]$.

Proof: $(i) \Rightarrow (ii)$ By [2, lemma 1.2.1] , we know that $\underset{i\in I}{\vee} A_i = \overset{\infty}{\underset{n=0}{\cup}} B_n$, where $B_0 = \underset{i\in I}{\cup} A_i$ and $B_n = \{x \in S \mid x \le p \vee q \text{ for some } i, j \in B_{n-1}, i \vee j \text{ exists}\}$. Here clearly $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots\cdots$ and each B_n are down sets. Now $A \cap \left(\underset{i\in I}{\vee} A_i\right) = A \cap \left(\overset{\infty}{\underset{n=0}{\cup}} B_n\right) = \overset{\infty}{\underset{n=0}{\cup}} (A \cap B_n)$. Since $A \cap A_i = (0]$ for each i, so

$A \cap B_0 = A \cap \left(\underset{i\in I}{\cup} A_i\right) = \underset{i\in I}{\cup} (A \cap A_i) = (0]$. Now we use the method of induction. Suppose $A \cap B_{k-1} = (0]$. Then let $x \in A \cap B_k$. This implies $x \in A$ and $x \le r \vee s$ for some $r, s \in B_{k-1}$ and $r \vee s$ exists. Since $A \cap B_{k-1} = (0]$, so $x \wedge r = 0 = x \wedge s$. Then by the 0-distributively of S, $x \wedge (r \vee s) = 0$. That is, $x = 0$. Hence $A \cap B_k = (0]$. Therefore $A \cap B_n = (0]$ for all positive integer n, and so $A \cap \left(\underset{i\in I}{\vee} A_i\right) = \overset{\infty}{\underset{n=0}{\cup}} (A \cap B_n) = (0]$.

$(ii) \Rightarrow (iii)$ is trivial by considering $A = (a]$, $A_1 = (a_1], \cdots\cdots, A_n = (a_n]$.

$(iii) \Rightarrow (i)$ Let $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Then by (iii) , $(a] \wedge ((b] \vee (c]) = (0]$. This implies $(a \wedge (b \vee c)] = (0]$, and so $a \wedge (b \vee c) = 0$. Hence S is 0-distributive.●

Theorem 7. Let S be a nearlattice with 0. Then the following conditions are equivalent.

(i) S is 0-distributive.

(ii) For any filter A of S, A^0 is the intersection of all the minimal prime ideals disjoint from A .

(iii) For all $a, b, c \in S$, $\{a \wedge (b \vee c)\}^0 = (a \wedge b)^0 \cap (a \wedge c)^0$ provided $b \vee c$ exists.

Proof: $(i) \Rightarrow (ii)$ Let N be a minimal prime down set disjoint from A. If $x \in A^0$, then $x \wedge a = 0$ for some $a \in A$. Thus $x \wedge a \in N$. But $N \cap A = \phi$ implies $a \notin N$. So $x \in N$ as N is prime. Therefore, $A^0 \subseteq N$.

Now let $y \in S - A^0$. Then $a \wedge y \ne 0$ for all $a \in A$. Hence $A \vee [y) \ne S$. Then by lemma 2 , there exists a maximal filter $M \supseteq A \vee [y)$. Thus, $S - M$ is a minimal prime down set such that $(S - M) \cap A = \phi$ and $y \notin S - M$. Therefore, A^0 is the intersection of all minimal prime down sets disjoint from A. Since S is 0-

distributive, so by [6, theorem 9], all minimal prime down set are minimal prime ideals and this proves (ii).

(ii)$\Rightarrow$(iii) Let $A = [a) \vee ([b) \cap [c))$. Suppose Q is a minimal prime ideal disjoint from A. Then $Q \cap [a) = \phi$ and $Q \cap [b) \cap [c) = \phi$. Then either $Q \cap ([a) \vee [b)) = \phi$ or $Q \cap ([a) \vee [c)) = \phi$. If not suppose $x \in Q \cap ([a) \vee [b))$ and $y \in Q \cap ([a) \vee [c))$. Then $x, y \in Q$ and $x \geq a \wedge b$, $y \geq a \wedge c$. This implies $a \wedge b, a \wedge c \in Q$. Since $Q \cap A = \phi$, so $a \notin Q$. Thus $b, c \in Q$ as ideal. Hence $b \vee c \in Q$. Also $b \vee c \in [b) \cap [c)$, which contradicts the fact $Q \cap [b) \cap [c) = \phi$. Therefore either $Q \cap ([a) \vee [b)) = \phi$ or $Q \cap ([a) \vee [c)) = \phi$. This implies either $a \wedge b \notin Q$ or $a \wedge c \notin Q$. Now let, $x \in ([a) \vee [b))^0 \cap ([a) \vee [c))^0 = [a \wedge b)^0 \cap [a \wedge c)^0$ Then $x \wedge a \wedge b = 0$ and $x \wedge a \wedge c = 0$. This implies $x \in Q$ as Q is prime. Hence by (ii), $x \in A^0 = ([a) \vee ([b) \cap [c)))^0$. Therefore,
$([a) \vee ([b) \cap [c)))^0 \supseteq ([a) \vee [b))^0 \cap ([a) \vee [c))^0$. Since the reverse inclusion is obvious, so $([a) \vee ([b) \cap [c)))^0 = ([a) \vee [b))^0 \cap ([a) \vee [c))^0$. This implies

$[a \wedge (b \vee c))^0 = [a \wedge b)^0 \cap [a \wedge c)^0$ and so,

$\{a \wedge (b \vee c)\}^0 = (a \wedge b)^0 \cap (a \wedge c)^0$.

(iii)$\Rightarrow$(i) Suppose (iii) holds. Let $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Then $(a \wedge b)^0 = S = (a \wedge c)^0$. Hence by (iii) $(a \wedge (b \vee c))^0 = S$. It follows that $a \wedge (b \vee c) = 0$. Thus S is 0-distributive.●

REFERENCES

1. P.Balasubramani and P.V. Venkatanarasimhan, Characterizations of the 0-Distributive Lattice, *Indian J. Pure Appl. Math.*, 32(3) (2001), 315-324.
2. M. B. Rahman, A study on distributive nearlattices, Ph.D Thesis, Rajshahi University, Bangladesh (1994).
3. W.H. Cornish and A.S.A. Noor, Standard elements in a nearlattice, *Bull. Austral. Math. Soc.,* 26(2) (1982) , 185-213.
4. Y. S Pawar and N. K. Thakare, 0-Distributive semilattices, *Canad . Math. Bull.,* 21(4) (1978) , 469-475.
5. J. C. Varlet, A generalization of the notion of pseudo-complementedness, *Bull.Soc. Sci. Liege,* 37 (1968), 149-158.
6. Md. Zaidur Rahman, Md. Bazlar Rahman and A. S. A. Noor, 0-distributive Nearlattice, *Annals of Pure and App. Math.,* 2(2) (2012), 177-184.
7. Md. Zaidur Rahman, Md. Bazlar Rahman and A. S. A. Noor, Some properties of 0-distributive Nearlattice, *Annals of Pure and App. Math.*, 2(2) (2012), 185-193.