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0-distributive Nearlattice

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Abstract. J.C.Varlet gave the notion of 0-distributive lattices to generalize the concept of pseudocomplemented lattices. In this paper, the authors extended the concept for nearlattices. They include several characterizations of these nearlattices. They provide a separation theorem in a 0-distributive nearlattice S for a filter F and an annihilator $\{x\}^{\perp}$ $(x \in S)$. At the end they include a result on minimal prime ideals.

Keywords: 0-distributive nearlattice, Maximal filter, Prime filter, Minimal prime ideal.

AMS Mathematics Subject Classification (2010): 06A12, 06A99, 06B10

1. Introduction

J.C. Varlet [7] has given the definition of a 0-distributive lattice to generalize the notion of pseudocomplemented lattice. By [7], a lattice with 0 is called a 0-distributive lattice if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. In other words, a lattice with 0 is 0-distributive if and only if for each $a \in L$, the set of elements disjoint to a is an ideal of L. Of course, every distributive lattice with 0 is 0-distributive. Also, every pseudocomplemented lattice is 0-distributive. In fact, in a pseudocomplemented lattice L, the set of all elements disjoint to $a \in L$, is a principal ideal $(a^*]$. Then many authors including [1], [3] and [5] studied the 0-distributive and 0-modular properties in lattices and meet semilattices. In fact, [3] has referred the condition of 0-distributive nearlattice given in this paper as weakly 0-distributive semilattice in a general meet semilattice. In this paper we study the 0-distributive nearlattices.

A *nearlattice* is a meet semilattice together with the property that any two elements possessing a common upper bound have a supremum. This property is known as the *upper bound property.*

A nearlattice S is called *distributive* if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, provided $y \vee z$ exists in S. For detailed literature on nearlattices, we refer the reader to consult [2, 4, 6].

A nearlattice S with 0 is called 0-distributive if for all $x, y, z \in S$ with $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists imply $x \wedge (y \vee z) = 0$.

It can be easily proved that it has the following alternative definition:
S is 0-distributive if for all $x, y, z, t \in S$ with $x \wedge y = 0 = x \wedge z$ imply $x \wedge ((t \wedge y) \vee (t \wedge z)) = 0$; $(t \wedge y) \vee (t \wedge z)$ exists by the upper bound property of S. Of course, every distributive nearlattice S with 0 is 0-distributive. Figure 1 is an example of a non-modular nearlattice which is 0-distributive, while Figure 2 gives a modular nearlattice which is not 0-distributive.

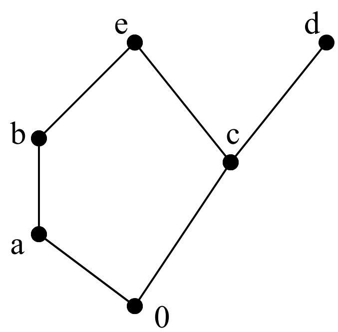

Figure 1

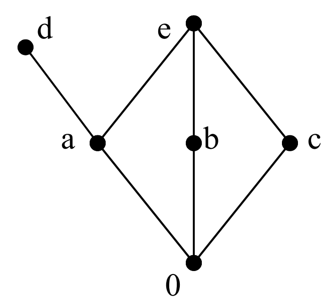

Figure 2

A subset I of a nearlattice S is called a *down set* if $x \in I$ and for $t \in S$ with $t \le x$ imply $t \in I$. An *ideal* I in a nearlattice S is a non-empty subset of S such that it is a down set and whenever $a \vee b$ exists for $a, b \in I$, then $a \vee b \in I$. A proper ideal I in S is called a *prime ideal* if $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$. A non-empty subset F of S is called a filter if $t \ge a$, $a \in F$ implies $t \in F$ and if $a, b \in F$ then $a \wedge b \in F$. A proper filter F in S is called *prime* if $a \vee b$ exists and $a \vee b \in F$ implies either $a \in F$ or $b \in F$. It is easy to prove that F is a filter of S if and only if S-F is a prime down set. Moreover, a prime down set P is a prime ideal if and only if $S - P$ is a prime filter.

A proper filter M of a nearlattice S is called *maximal* if for any filter Q with $Q \supseteq M$ implies either $Q = M$ or $Q = S$. Dually, we define a *minimal prime ideal (down set).*

Let L be a lattice with 0. An element a^* is called the *pseudocomplement* of a if $a \wedge a^* = 0$ and if $a \wedge x = 0$ for some $x \in L$, then $x \le a^*$. A lattice L with 0 and 1 is called *pseudocomplemented* if its every element has a pseudocomplement.

Since a nearlattice S with 1 is a lattice, so pseudocomplementation is not possible in a general nearlattice. A nearlattice S with 0 is called sectionally pseudocomplemented if the interval $[0,x]$ for each $x \in S$ is pseudocomplemented. For $A \subseteq S$, we denote $A^{\perp} = \{x \in S \mid x \wedge a = 0 \text{ for all } a \in A\}$. If S is distributive then clearly $A^{\perp}$ is an ideal of S.

Moreover, $A^{\perp} = \bigcap_{a \in A} \{\{a\}^{\perp}\}$. If A is an ideal, then obviously $A^{\perp}$ is the pseudocomplement of A in $I(S)$. Therefore, for a distributive nearlattice S with 0, I(S) is pseudocomplemented.

2. Some Results

Theorem 1. *If a nearlattice S with 0 is sectionally pseudocomplemented, then $I(S)$ is pseudocomplemented.*

Proof. Suppose S is sectionally pseudocomplemented. Let $I \in I(S)$. $I^{\perp} = \{x \in S \mid x \wedge i = 0 \text{ for all } i \in I\}$. Suppose $x \in I^{\perp}$ and $t \le x$. Then $x \wedge i = 0$ for all $i \in I$ and so $t \wedge i = 0$ for all $i \in I$. Hence $t \in I^{\perp}$. Now let $x, y \in I^{\perp}$ and $x \vee y$ exists. Let $r = x \vee y$. Then $0 \le x, y, r \wedge i \le r$ for all i, and $x \wedge (r \wedge i) = 0 = y \wedge (r \wedge i)$. Since $[0,r]$ is pseudocomplemented, $x, y \le (r \wedge i)^{+}$ for all $i \in I$, where $(r \wedge i)^{+}$ is the relative pseudocomplement of $r \wedge i$ in $[0,r]$. Then $x \vee y \le (r \wedge i)^{+}$, and so $r \wedge i \wedge (x \vee y) = 0$. That is $i \wedge (x \vee y) = 0$ for all $i \in I$. This implies $x \vee y \in I^{\perp}$. Therefore, $I^{\perp}$ is an ideal. Clearly $I^{\perp}$ is the pseudocomplement of I in $I(S)$. Hence $I(S)$ is pseudocomplemented.●

Following example (Figure 3) shows that $I(S)$ can be pseudocomplemented but S is not sectionally pseudocomplemented.

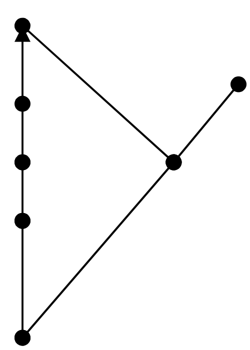

Figure 3

Again, Figure1 gives a non-distributive nearlattice S where $I(S)$ is pseudocomplemented.

Theorem 2. *If the intersection of all prime ideals of a nearlattice S with 0 is* $\{0\}$*, then S is 0-distributive.*

Proof. Let $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Let P be any prime ideal of S. If $a \in P$, then $a \wedge (b \vee c) \leq a$ implies that $a \wedge (b \vee c) \in P$. If $a \notin P$, then by the primeness of P, $b,c \in P$, and so $b \vee c \in P$. This implies $a \wedge (b \vee c) \in P$. Thus $a \wedge (b \vee c)$ is in every prime ideal P of S, and hence $a \wedge (b \vee c) = 0$, proving that S is 0-distributive. ●

By [6] we know that a nearlattice S is distributive if and only if $I(S)$ is distributive, which is also equivalent to that $D(S)$, the lattice of filters of S is distributive. Thus if S is a nearlattice with 0 such that $I(S)$ (similarly $D(S)$) is distributive, then S is 0-distributive.

Following lemma are needed for further development of the paper.

Lemma 3. *Every proper filter of a nearlattice with 0 is contained in a maximal filter.*

Proof. Let F be a proper filter in S with 0.Let $\mathcal{F}$ be the set of all proper filters containing F. Then $\mathcal{F}$ is non-empty as $F \in \mathcal{F}$. Let C be a chain in $\mathcal{F}$ and let $M = \bigcup\{X \mid X \in C\}$. We claim that M is a filter with $F \subseteq M$. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as X is a filter. Therefore, $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. So $x, y \in Y$. Then $x \wedge y \in Y$ as Y is a filter. Hence $x \wedge y \in M$. Moreover M contains F. So M is maximum element of C. Then by Zorn's lemma $\mathcal{F}$ has a maximal element, say Q with $F \subseteq Q$.●

Lemma 4. *Let S be a nearlattice with 0. A proper filter M in S is maximal if and only if for any element* $a \notin M$ *there exists an element* $b \in M$ *with* $a \wedge b = 0$.

Proof. Suppose M is maximal and $a \notin M$. Let $a \wedge b \neq 0$ for all $b \in M$. Consider $M_1 = \{y \in S \mid y \geq a \wedge b, \textit{for some } b \in M\}$. Clearly M_1 is a filter and is proper as $0 \notin M$. For every $b \in M$ we have $b \geq a \wedge b$ and so $b \in M_1$. Thus $M \subseteq M_1$. Also $a \notin M$ but $a \in M_1$. So $M \subset M_1$, which contradicts the maximality of M. Hence there must exist some $b \in M$ such that $a \wedge b = 0$.

Conversely, if the proper filter M is not maximal, then as $0 \in S$, there exists a maximal filter N such that $M \subset N$. For any element $a \in N - M$ there exists an element $b \in M$ such that $a \wedge b = 0$. Hence $a,b \in N$ imply $0 = a \wedge b \in N$, which is a contradiction. Thus M must be a maximal filter. ●

Following result gives several nice characterizations of 0-distributive nearlattice.

Theorem 5. *For a nearlattice S with 0, the following conditions are equivalent:*

(i) *S is 0-distributive.*

(ii) *$\{a\}^{\perp}$ is an ideal for all $a \in S$.*

(iii) *$A^{\perp}$ is an ideal for all $A \subseteq S$.*

(iv) *$I(S)$ is pseudocomplemented.*

(v) *$I(S)$ is 0-distributive.*

(vi) *Every maximal filter is prime.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ are trivial.

For any ideal I of S, $I^{\perp}$ is clearly the pseudocomplement of I in $I(S)$ if $I^{\perp} \in I(S)$, and so (iv) holds.

Since every pseudocomplemented lattice is 0-distributive, so $(iv) \Rightarrow (v)$.

$(v) \Rightarrow (vi)$ Let $I(S)$ be 0-distributive and F be a maximal filter. Suppose $f, g \notin F$ with $f \vee g$ exists.

By Lemma 4, there exist $a, b \in F$ such that $a \wedge f = 0 = b \wedge g$. Hence $(f] \wedge (a \wedge b] = (0]$ and $(g] \wedge (a \wedge b] = (0]$.

Then $(f \vee g] \wedge (a \wedge b] = ((f] \vee (g]) \wedge (a \wedge b] = (0]$, by 0-distributivity of $I(S)$. Hence $(f \vee g) \wedge (a \wedge b) = 0$. Since F is maximal, $0 \notin F$. Therefore $f \vee g \notin F$, and so F is prime.

$(vi) \Rightarrow (i)$ Let (vi) holds. Suppose $a, b, c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. If $a \wedge (b \vee c) \neq 0$, then by Lemma 3, $a \wedge (b \vee c) \in F$ for some maximal filter F of S. Then $a \in F$ and $b \vee c \in F$. As F is prime, by assumption, so either $a \in F$ and $b \in F$ or $c \in F$. That is, either $a \wedge b \in F$ or $a \wedge c \in F$. This implies $0 \in F$, which gives a contradiction and hence $a \wedge (b \vee c) = 0$. In other words, S is 0-distributive. ●

Corollary 6. *In a 0-distributive nearlattice, every proper filter is contained in a prime filter.*

Proof. This immediately follows by Lemma 3 and Theorem 5.●

Theorem 7. *In a 0-distributive nearlattice S, if $\{0\} \neq A$ is the intersection of all non-zero ideals of S, then $A^{\perp} = \{x \in S \mid \{x\}^{\perp} \neq \{0\}\}$.*

Proof. Let $x \in A^{\perp}$. Then $x \wedge a = 0$ for all $a \in A$. Since $A \neq \{0\}$, so $\{x\}^{\perp} \neq \{0\}$. Thus $x \in R.H.S$. That is $A^{\perp} \subseteq R.H.S$.

Conversely, let $x \in R.H.S$. Since S is 0-distributive, so $\{x\}^{\perp}$ is a non-zero ideal of S. Then $A \subseteq \{x\}^{\perp}$ and so $A^{\perp} \supseteq \{x\}^{\perp\perp}$. This implies $x \in A^{\perp}$,which completes the proof. ●

Theorem 8. *Let S be a nearlattice with 0. S is 0-distributive if and only if for any filter F disjoint with $\{x\}^{\perp}$ $(x \in S)$, there exist a prime filter containing F and disjoint with $\{x\}^{\perp}$.*

Proof. Let S be 0-distributive. Consider the set $\mathcal{F}$ of all filters of S containing F and disjoint with $\{x\}^{\perp}$. Clearly $\mathcal{F}$ is non-empty as $F \in \mathcal{F}$. Then using Zorn's lemma, there exists a maximal element Q in $\mathcal{F}$. Now we claim that $x \in Q$. If not, then $Q \vee [x) \supset Q$. So by the maximality of Q, $\{Q \vee [x)\} \cap \{x\}^{\perp} \neq \phi$. Then there exists $t \in Q \vee [x)$ and $t \in \{x\}^{\perp}$. Then $t \geq q \wedge x$ for some $q \in Q$ and $t \wedge x = 0$. Thus, $0 = t \wedge x \geq q \wedge x$, and so $q \wedge x = 0$. This implies $q \in \{x\}^{\perp}$, which contradicts the fact that $Q \cap \{x\}^{\perp} = \phi$. Therefore $x \in Q$. Finally, let $z \notin Q$. Then $\{Q \vee [z)\} \cap \{x\}^{\perp} \neq \phi$. Let $y \in \{Q \vee [z)\} \cap \{x\}^{\perp}$. Then $y \wedge x = 0$ and $y \geq q \wedge z$ for some $q \in Q$. Thus $0 = y \wedge x \geq q \wedge x \wedge z$, which implies $q \wedge x \wedge z = 0$. Now $x \in Q$ implies $q \wedge x \in Q$, and $z \wedge (q \wedge x) = 0$. Hence by Lemma 4, Q is a maximal filter of S and so by Theorem 5, Q is prime.

Conversely, let $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists. If $x \wedge (y \vee z) \neq 0$. Then $y \vee z \notin \{x\}^{\perp}$. Thus $[y \vee z) \cap \{x\}^{\perp} = \phi$. So, there exists a prime filter Q containing $[y \vee z)$ and disjoint with $\{x\}^{\perp}$. As $y, z \in \{x\}^{\perp}$, so $y, z \notin Q$. Thus $y \vee z \notin Q$, as Q is prime. This implies $[y \vee z) \not\subset Q$, a contradiction. Hence $x \wedge (y \vee z) = 0$ and so S is 0-distributive. ●

In [5] the authors have mentioned as a corollary to the above result that for distinct elements $a, b \in S$ for which $a \wedge b \neq 0$ are separated by a prime filter in a 0-distriburive semilattice , which is completely wrong. For example, Figure 1 is an example of a 0-distributive nearlattice, where a, b are distinct and $a \wedge b \neq 0$. But there does not exist any prime filter containing b but not containing a.

Now we give few more characterizations for 0-distributive nearlattices.

Theorem 9. *Let S be a nearlattice with 0 . Then the following conditions are equivalent:*

i) *S is 0-distributive.*
ii) *Every maximal filter of S is prime.*
iii) *Every minimal prime down set of S is a minimal prime ideal.*
iv) *Every proper filter of S is disjoint from a minimal prime ideal.*

v) *For each non-zero element $a \in S$, there is a minimal prime ideal not containing a .*

vi) *Each non-zero element $a \in S$ is contained in a prime filter.*

Proof. $(i) \Leftrightarrow (ii)$ follows from Theorem 5.

$(ii) \Leftrightarrow (iii)$. Let A be a minimal prime down set. Then S-A is a maximal filter. Then by (ii), S-A is a prime filter, and so A is an ideal. That is, A is a minimal prime ideal.
(iii) implies (ii) . Let F be a maximal filter of S . Then S-F is a minimal prime down set . Thus by (iii), S-F is a minimal prime ideal and so F is a prime filter.

(i) implies (iv). Let F be a proper filter of S. Then by Corollary 6, there is a prime filter $Q \supseteq F$. Then S-Q is a minimal prime ideal disjoint from F.

(iv) implies (v). Let $a \in S$ and $a \neq 0$. Then $[a)$is a proper filter. Then by (iv) there exists a minimal prime ideal A such that $A \cap [a) = \varphi$. Thus $a \notin A$.

(v) implies (vi). Let $a \in S$ and $a \neq 0$. Then by (v) there is a minimal prime ideal P such that $a \notin P$. Thus $a \in L - P$ and L-P is a prime filter.

(vi) implies (i). Let S be not 0-distributive. Then there exist $a, b, c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists but $a \wedge (b \vee c) \neq 0$.Then by (vi) there exists a prime filter Q such that $a \wedge (b \vee c) \in Q$. Let $F = [a \wedge (b \vee c))$. This is proper as $0 \notin F$ and $F \subseteq Q$. Now, $a \wedge (b \vee c) \in Q$ implies $a \in Q$ and $b \vee c \in Q$. Since $a \wedge b = 0 = a \wedge c$,
so $b, c \notin Q$ as $0 \notin Q$, but $b \vee c \in Q$, which contradicts that Q is prime. Hence $a \wedge (b \vee c) = 0$ and so S is 0-distributive. ●

We conclude the paper with the following result involving minimal prime ideals.

Theorem 10. *Let S be a 0-distributive nearlattice and $x \in S$. Then a prime ideal P containing $\{x\}^{\perp}$ is a minimal prime ideal containing $\{x\}^{\perp}$ if and only if for $p \in P$ there is $q \in S - P$ such that $p \wedge q \in \{x\}^{\perp}$.*

Proof. Let P be a prime ideal of S containing $\{x\}^{\perp}$ such that the given condition holds.
Let K be a prime ideal containing $\{x\}^{\perp}$ such that $K \subseteq P$. Let $p \in P$. Then there is $q \in S - P$ such that $p \wedge q \in \{x\}^{\perp}$. Hence $p \wedge q \in K$. Since K is prime and $q \notin K$, so $p \in K$. Thus ,$P \subseteq K$ and so K = P. Therefore, P must be a minimal prime ideal containing $\{x\}^{\perp}$.

Conversely, let P be a minimal prime ideal containing $\{x\}^{\perp}$. Let $p \in P$. Suppose for all $q \in S - P$, $p \wedge q \notin \{x\}^{\perp}$. Set $D = (S - P) \vee [p)$. We claim that $\{x\}^{\perp} \cap D = \varphi$. If not , let $y \in \{x\}^{\perp} \cap D$. Then $y \geq r \wedge p$ for some $r \in S - P$. Thus, $p \wedge r \leq y \in \{x\}^{\perp}$, which is a contradiction to the assumption. Then by Theorem 8, there exists a maximal (prime) filter $Q \supseteq D$ and disjoint with $\{x\}^{\perp}$. By the proof of Theorem 8, $x \in Q$. Let M = S-Q. Then M is prime ideal . Since $x \in Q$, so $x \notin M$. Let $t \in \{x\}^{\perp}$. Then $t \wedge x = 0 \in M$ implies $t \in M$ as M is prime. Thus $\{x\}^{\perp} \subseteq M$.

Now $M \cap D = \varphi$. Therefore, $M \cap (S - P) = \varphi$, and hence $M \subseteq P$. Also $M \neq P$, because $p \in D$ implies $p \notin M$ but $p \in P$. Hence M is a prime ideal containing $\{x\}^{\perp}$ which is properly contained in P . This gives a contradiction to the minimal property of P.

Therefore, the given condition holds. ●

REFERENCES

1. P.Balasubramani and P.V. Venkatanarasimhan, Characterizations of the 0-Distributive Lattice, *Indian J. Pure Appl. Math.*, 32(3) (2001), 315-324.
2. W.H. Cornish and A.S.A. Noor, Standard elements in a nearlattice, *Bull. Austral. Math. Soc.,* 26(2) (1982) , 185-213.
3. C. Jayaram, 0-modular semilattices, *Studia Sci. Math. Hung.*, 22(1987), 189-195.
4. A.S.A.Noor and M.B.Rahman, Separation properties in nearlattices, *The Rajshahi University Studies (Part B)*, 22 (1994), 181-188.
5. Y. S Pawar and N. K. Thakare, 0-Distributive semilattices, *Canad . Math. Bull.,* 21(4) (1978) , 469-475.
6. M. B. Rahman, A study on distributive nearlattices, Ph.D Thesis, Rajshahi University, Bangladesh (1994).
7. J. C. Varlet, A generalization of the notion of pseudo-complementedness, *Bull. Soc. Sci. Liege,* 37 (1968), 149-158.