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A Relation on Almost Distributive Lattices

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Abstract. In this paper we discuss some important results of tolerance relations on almost distributive lattices.

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1. Introduction

After Booles axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [4] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of a filters in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set PF(R) of all principal filters of R forms a distributive lattice. This enables us to extend many existing concepts from the class of distributive lattices to the class of ADLs. We introduced a relation on an ADL, which is reflexive and symmetric (i.e., Tolerance relation) on ADL. We proved that a tolerance relation is compatible. Finally, we proved that tolerance relation induced by filter if and only if it congruence relation on ADL.

2. Preliminaries

Definition 2.1.[4] An Almost Distributive Lattice with zero or simply ADL is an algebra $(R, \vee, \wedge, 0)$ of type (2, 2, 0) satisfying

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$

5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$
7. $x \vee 0 = x$, for any $x, y, z \in R$.

Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee$, $\wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL. If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on R.

Theorem 2.2. ([4]) *If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in R$, we have the following:*
(1) $a \vee b = a \Leftrightarrow a \wedge b = b$
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$
(3) $\wedge$ is associative in R
(4) $a \wedge b \wedge c = b \wedge a \wedge c$
(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(8) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ and $a \vee (b \wedge a) = a$
(9) $a \leq a \vee b$ and $a \wedge b \leq b$
(10) $a \wedge a = a$ and $a \vee a = a$
(11) $0 \vee a = a$ and $a \wedge 0 = 0$
(12) If $a \leq c$, $b \leq c$ then $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
(13) $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL R satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL R a distributive lattice.

Theorem 2.3. ([4]) *Let $(R, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*

(1) $(R, \vee, \wedge, 0)$ is a distributive lattice
(2) $a \vee b = b \vee a$, for all $a, b \in R$
(3) $a \wedge b = b \wedge a$, for all $a, b \in R$
(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, for all $a, b, c \in R$.

As usual, an element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$.

Theorem 2.4. ([4]) *Let R be an ADL and $m \in R$. Then the following are equivalent:*
(1) m is maximal with respect to $\leq$
(2) $m \vee a = m$, for all $a \in R$
(3) $m \wedge a = a$, for all $a \in R$
(4) $a \vee m$ is maximal, for all $a \in R$.

As in distributive lattices [1,2], a non-empty subset F of an ADL R is said to be a filter of R if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in R$.

For any subset S of R is the smallest filter containing S is given by

$[S)=\{x \mid x \vee (\wedge_{i=1}^{n} s_i) \mid s_i \in S, x \in R$ and $n \in \mathbb{Z}^+\}$. If $S=\{x\}$, we write $[x)$ instead of $[S)$.

3. Tolerance Relation on ADLs

We introduced the notion of tolerance relation on an ADL and proved some important properties on it. First, we give the following.

Definition 3.1. Let R be an ADL, $a \in R$ and F a filter of R. Define $a \wedge F = \{a \wedge f \mid f \in F\}$.

Theorem 3.2. *Let R be an ADL. Then for each filter F of R and each $a \in R$ the set $a \wedge F$ is a convex subADL of R.*

Proof: Let $x, y \in a \wedge F$. Then $x = a \wedge f_1$ and $y = a \wedge f_2$, where $f_1, f_2 \in F$.
Now $x \wedge y = (a \wedge f_1) \wedge (a \wedge f_2) = a \wedge f_1 \wedge f_2$.
Since $f_1 \wedge f_2 \in F$, we get $x \wedge y \in a \wedge F$.
Also, $x \vee y = (a \wedge f_1) \vee (a \wedge f_2) = a \wedge (f_1 \vee f_2)$ that implies $x \vee y \in a \wedge F$ (Since $f_1 \vee f_2 \in F$).
Therefore $a \wedge F$ is a subADL of R.
Let $x, y \in a \wedge F$ and $z \in R$ such that $x \leq z \leq y$.
Then $x = a \wedge f_1$, $y = a \wedge f_2$ where $f_1, f_2 \in F$.
Now we prove that $z \in a \wedge F$.
Take $f = (f_1 \vee z) \wedge f_2$. Then $f \in F$.
Now $a \wedge f = a \wedge (f_1 \vee z) \wedge f_2 = a \wedge (f_1 \vee z) \wedge a \wedge f_2 = [(a \wedge f_1) \vee (a \wedge z)] \wedge y$
$= [x \vee (a \wedge y \wedge z)] \wedge y = [x \vee (y \wedge z)] \wedge y = (x \vee z) \wedge y = z \wedge y = z$.
So that $z \in a \wedge F$.
Therefore $a \wedge F$ is a convex subADL of R. □

Lemma 3.3. *Let R be an ADL and F a filter of R. Then $S_F = \{a \wedge F \mid a \in R\}$ is covering of R.*

Proof: Let $a \in R$ and $x \in F$.
Now $a = a \wedge (a \vee x) \in a \wedge F$, since $a \vee x \in F$.
Therefore $a \in a \wedge F$. Hence S_F is covering of R. □

Definition 3.4. Let R be an ADL and F be a filter of R. The covering $S_F = \{a \wedge F \mid a \in R\}$ is called induced by F and tolerance relation T (S_F) induced by S_F is called tolerance on R induced by the filter of F.
For the sake of brevity, denote by $T_F = T(S_F)$.

Definition 3.5. Let F be a filter of R. Define $a \wedge F = \{a \wedge f \mid f \in F\}$.
A relation T_F on R defined by the rule $< x, y > \in T_F$ if and only if there exists $a \in R$ such that $x \in a \wedge F$ and $y \in a \wedge F$.
It is easy to verify that T_F is reflexive and symmetric relation on R.

Lemma 3.6. *Let R be an ADL, F a filter of R and T_F be the induced by F. If $< c, d > \in T_F$, then*
1. $< a \wedge d, a \wedge c > \in T_F$
2. $< d \wedge a, c \wedge a > \in T_F$
3. $< a \vee d, a \vee c > \in T_F$.
Proof: Let $< c, d > \in T_F$.
Then there exists $u \in R$ such that $c \in u \wedge F$, $d \in u \wedge F$.
That implies $c = u \wedge f_1$, $d = u \wedge f_2$, for some $f_1, f_2 \in F$.
1. Now $a \wedge d = a \wedge u \wedge f_2$
$a \wedge c = a \wedge u \wedge f_1$. That implies $< a \wedge d, a \wedge c > \in T_F$.
2. Similarly $< d \wedge a, c \wedge a > \in T_F$.
3. Now, $a \vee d = a \vee (u \wedge f_2) = (a \vee u) \wedge (a \vee f_2)$ (since $a \vee f_2 \in F$).

Also again, $a \vee c = a \vee (u \wedge f_1) = (a \vee u) \wedge (a \vee f_1)$ (since $a \vee f_1 \in F$).
Therefore $< a \vee d, a \vee c > \in T_F$.

Note that $< d \vee a, c \vee a > \notin T_F$ for each $a \in R$, since $d \vee a = (u \wedge f_2) \vee a$, there is no right distributivity of $\vee$ over $\wedge$ in an ADL.

Lemma 3.7. *Let R be an ADL, F a filter of R and T_F a relation on R. If $< a, b > \in T_F$, $< c, d > \in T_F$ then $< a \wedge c, b \wedge d > \in T_F$.*
Proof: Let $< a, b > \in T_F$, $< c, d > \in T_F$.
Then there exist $u, v \in R$ such that $a = u \wedge f_1$, $b = u \wedge f_2$
$c = v \wedge f_3$, $d = v \wedge f_4$, where $f_1, f_2, f_3, f_4 \in F$.
Now, $a \wedge c = u \wedge f_1 \wedge v \wedge f_3 = u \wedge v \wedge f_1 \wedge f_3$
$b \wedge d = u \wedge f_2 \wedge v \wedge f_4 = u \wedge v \wedge f_2 \wedge f_4$.
That implies $< a \wedge c, b \wedge d > \in T_F$.
Similarly, $< c \wedge a, d \wedge b > \in T_F$.
Now, $a \vee c = (u \wedge f_1) \vee (v \wedge f_3)$
$= [(u \wedge f_1) \vee v] \wedge [(u \wedge f_1) \vee f_3]$
$= [(u \vee v) \wedge (f_1 \vee v)] \wedge [(u \wedge f_1) \vee f_3]$
$= (u \vee v) \wedge (f_1 \vee v) \wedge [(u \wedge f_1) \vee f_3]$ (since $(f_1 \vee v) \wedge [(u \wedge f_1) \vee f_3] \in F$).
Now, $b \vee d = (u \wedge f_2) \vee (v \wedge f_4)$
$= (u \vee v) \wedge (f_2 \vee v) \wedge [(u \wedge f_2) \vee f_4]$ (since $(f_2 \vee v) \wedge [(u \wedge f_2) \vee f_4] \in F$).
Therefore $< a \vee c, b \vee d > \in T_F$.
Similarly, we get $< c \vee a, d \vee b > \in T_F$. □

Lemma 3.8. *Let R be an ADL, F a filter of R and T_F is relation on R. If $a, b \in R$ and $< a, b > \in T_F$ then $a = (a \vee b) \wedge f_1$, $b = (a \vee b) \wedge f_2$, for some $f_1, f_2 \in F$.*
Proof: Let $< a, b > \in T_F$.
Then there exists $u \in R$ such that $a, b \in u \wedge F$
that implies $a = u \wedge f_1$, $b = u \wedge f_2$, where $f_1, f_2 \in F$.
Now $(a \vee b) \wedge f_1 = [(u \wedge f_1) \vee (u \wedge f_2)] \wedge f_1 = u \wedge (f_1 \vee f_2) \wedge f_1 = u \wedge f_1 = a$.
Similarly, $b = (a \vee b) \wedge f_2$. Hence lemma.

□

We conclude this paper with the following result.

Theorem 3.9. *Let R be an ADL and F a filter of R. If T_F is a compatible on R then T_F is congruence relation on R.*
Proof: Clearly T_F is reflexive and symmetric
Let $< x, y > \in T_F$ and $< y, z > \in T_F$.
Then there exist $a, b \in R$ such that $x, y \in a \wedge F$ and $y, z \in b \wedge F$.
That implies $x = a \wedge f_1$, $y = a \wedge f_2$
and $y = b \wedge f_3$, $z = b \wedge f_4$, where $f_1, f_2, f_3, f_4 \in F$.

We prove that $< f_1, f_4 > \in T_F$.
Now $f_1 = (f_1 \vee f_4) \wedge f_1$
$f_4 = (f_1 \vee f_4) \wedge f_4$.
Therefore $< f_1, f_4 > \in T_F$.
Since $a \in a \wedge F$ and $x \in a \wedge F$, we get $< a, x > \in T_F$.
Similarly $< a, y >, < b, y >, < b, z > \in T_F$.
By symmetry, we get $< y, b >, < a, y > \in T_F$. Since T_F is compatibility,
$< y \vee a, b \vee y > \in T_F$.
Since T_F is compatibility, $< a, b > \in T_F$.
Since $< f_1, f_4 > \in T_F$, $< a \wedge f_1 , b \wedge f_4 > \in T_F$.
That implies $< x, z > \in T_F$.
So that T_F is transitive.
Therefore T_F is congruence on R. □

Corollary 3.10. *Let R be an ADL, F a filter of R and T_F is the relation induced by F . The following are equivalent*
1. T_F is a compatible relation R
2. T_F is transitive
3. T_F is a congruence relation on R.

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