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Automorphism Group of an Inverse Fuzzy Automaton

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Abstract. In this paper we define an inverse fuzzy automaton such that the corresponding transition monoid is an inverse monoid. We also study the structure of the automorphism group of an inverse fuzzy automaton and prove that it is equal to a subgroup of the symmetric inverse monoid of one one partial fuzzy transformations on Q.

Keywords: Inverse fuzzy automata, invertible fuzzy languages, inverse monoid

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1. Introduction

An automaton is a model of a sequential switching circuit with a finite number of states,with state changing when it is subjected to an input symbol. A sequential machine consits of two main structures, the transition structure and the output structure. Since our interest is in the transition structure of an automaton and its input semigroup, outputs are not considered here. Zadeh introduced the concept of fuzzy sets and W. G. Wee introduced fuzzy automata. The theory of inverse monoids were introduced independently by Wagner and Preston via the study of partial one-one transformations of a set. We define an inverse fuzzy automaton such that its transition monoid is an inverse monoid and study the structure of the automorphism group of an inverse fuzzy automaton.

2. Preliminaries

This section present basic definitions and results to be used in the sequel. A semigroup S is said to be an inverse semigroup if for every $a \in S$ there exist a unique $b \in S$

such that $aba = a$ and $bab = b$. We call b the inverse of a and denote by a^{-1}. If S has an identity then S is said to be an inverse monoid. Inverse monoids form a variety defined by the identities $aa^{-1}a = a, aa^{-1}bb^{-1} = bb^{-1}aa^{-1}$. Also $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$. For any element a of an inverse monoid, aa^{-1} is an idempotent and idempotents in an inverse monoid commute. An inverse monoid with a single idempotent is a group. As an anologous to Cayleys theorem for groups, Preston and Wagner proved that an inverse monoid I is isomorphic to a subinverse monoid of the monoid of all one-one partial transformations on I.

A fuzzy automaton on an alphabet X is a 5-tuple $M = (Q, X, \mu, i, \tau)$ where Q is a finite set of states , X is a finite set of input symbols and μ is a fuzzy subset of $Q \times X \times Q$ representing the transition mapping, i is a fuzzy subset of Q called initial state, τ is a fuzzy subset of Q called final state. (Q, X, μ) is called a fuzzy finite state machine. A fuzzy automaton can also be represented as a five tuple $\left(Q, X, \{T_u \mid u \in X\}, i, \tau\right)$ where $\{T_u \mid u \in X\}$ is the set of fuzzy transition matrices, $i = [i_1\ i_2 \ldots. i_n], i_k \in [0,1], \tau = [j_1\ j_2 \ldots. j_n]^T, j_k \in [0,1],$ for $k = 1,2,..., n.$

μ can be extended to the set $Q \times X^* \times Q$ by

$$\mu(q, \Lambda, p) = \begin{cases} 1, q = p \\ 0, q \neq p \end{cases}$$

$$\mu(q,u,p) = \bigvee_{q_i \in Q} \{\mu(q, x_1, q_1) \wedge \mu(q_1, x_2, q_2) \wedge \cdots \wedge \mu(q_{k-1}, x_k, p) \mid x_1 x_2 \ldots. x_k = u\}.$$

Let $M = (Q, X, \mu, i, \tau)$ be a fuzzy automaton. We say the triple (Q, X, μ) is the fuzzy finite state machine associated with M. For $p, q \in Q$, p is called an immediate successor of q if there exist an $a \in X$ such that $\mu(q, a, p) > 0$.

p is called a successor of q if there exist $x \in X^*$ such that $\mu^*(q, x, p) > 0$. Let $S(q)$ be the set of all successors of q. Let $T \subseteq Q$. The set of all successors of T, denoted by $S(T) = \cup\{S(q) \mid q \in T\}$. $N = (T, X, \nu)$ where $T \subseteq Q, \nu$ is a fuzzy subset of $T \times X \times T$ is a called a submachine of M if $\mu|_{T \times X \times T} = \nu$ and $S(T) \subseteq T$. N is said to be separated if $S(Q - T) \cap T = \phi$. M is said to be connected if M has no separated proper submachines.

Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be fuzzy finite state machines. A pair (α, β) of mappings $\alpha : Q_1 \to Q_2$ and $\beta : X_1 \to X_2$ is called a homomorphism, written $(\alpha, \beta) : M_1 \to M_2$ if

$\mu_1(q, x, p) \leq \mu_2(\alpha(q), \beta(x), \alpha(p)) \quad \forall q, p \in Q$ and $\forall x \in X_1$. (α, β) is called a strong homomorphism if

$$\mu_2(\alpha(q),\beta(x),\alpha(p)) = \vee\{\mu_1(q,x,t) \mid t \in Q_1, \alpha(t) = \alpha(p)\} \forall p,q \in Q_1$$

and $\forall x \in X_1$. A homomorphism is said to be an isomorphism if α and β are both one one and onto.

If $X_1 = X_2$ and β is the identity map, then we write $\alpha : M_1 \to M_2$ is a homomorphism.

If (α,β) is a strong homomorphism with α one-one, then $\mu_2(\alpha(q),\beta(x),\alpha(p) = \mu_1(q,x,p)$ $\forall q,p \in Q_1$ and $\forall x \in X_1$ (see., [5]).

Let $M = (Q,X,\mu)$ be fuzzy finite state machine. Consider the set of all strong homomorphisms $(\alpha,\beta): M \to M$ denoted by $END_X(M)$ and the set of all strong isomorphisms from $M \to M$ by $AUT_X(M)$. $END_X(M)$ form a monoid under the operation $(\alpha_1,\beta_1)\circ(\alpha_2,\beta_2) = (\alpha_1 \circ \alpha_2, \beta_1 \circ \beta_2)$ and $AUT_X(M)$ form a group where the inverse of (α,β) is (α^{-1},β^{-1}). Composition is associative and identity is the pair of identity maps on Q and X.

If β is the identity map on X, then we denote $END_X(M)$ as $End_X(M)$ and $AUT_X(M)$ as $Aut_X(M)$. Then $End_X(M)$ is a submonoid of $END_X(M)$ and $Aut_X(M)$ is a subgroup of $AUT_X(M)$. Define a congruence θ_M on X^* as $u\theta_M v$ iff $\mu(q,u,p) = \mu(q,v,p)$ $\forall q,p \in Q$. Then θ_M is a congruence and $T(M) \cong X^*/\theta_M$ (see [6]).

Considering the collection of all fuzzy finite state automata as a category F-AUT with objects are fuzzy automata over finite set of states and morphisms as the automata homomorphisms between them. Corresponds to every fuzzy automata $M = (Q,X,\mu)$ we get a finite monoid X^*/θ_M and every finite monoid is the transition monoid of the minimal fuzzy automaton recognizing some fuzzy language which is called the syntactic monoid of that fuzzy language (see [6]).

3. Inverse Fuzzy Automata and the Automorphism Group

Definition 3.1. *A connected fuzzy automaton $M = (Q,X,\mu)$ is said to be an inverse fuzzy automaton if $\forall a \in X^*$, there exist a unique $b \in X^*$ such that $\mu(q,aba,p) = \mu(q,a,p)$ and $\mu(q,bab,p) = \mu(q,b,p), \forall p,q \in Q$.*

Theorem 3.1. *A fuzzy automaton $A = (Q,X,\mu,i,\tau)$ is inverse if and only if X^*/θ_A is an inverse monoid.*

Proof. Suppose A is an inverse fuzzy automaton.

i.e. for each $a \in X^*$ there exist a unique $b \in X^*$ such that $\forall p,q \in Q, \mu(q,aba,p) = \mu(q,a,p)$ and $\mu(q,bab,p) = \mu(q,b,p)$ $\Leftrightarrow aba\theta_A a$ and $bab\theta_A b$ $\Leftrightarrow [aba] = [a]$ and $[bab] = [b]$.

Then, $[a][b][a] = [a]$ and $[b][a][b] = [b] \Leftrightarrow X^*/\theta_A$ is an inverse monoid.

Lemma 3.1. *If* $(\alpha,\beta)\in AUT_X(M)$ *then for any* $u,v\in X^*, u\theta_M v \Leftrightarrow \beta(u)\theta_M\beta(v)$.

Proof.

$u\theta_M v \Leftrightarrow \mu(q,u,p)=\mu(q,v,p), \forall q,p\in Q \Leftrightarrow \mu(\alpha(q),\beta(u),\alpha(p))=\mu(\alpha(q),$
$\beta(v),\alpha(p)), \forall \alpha(q),\alpha(p)\in Q \Leftrightarrow \mu(q,\beta(v),p), \forall q,p\in Q$,

since α is one-one $\Leftrightarrow \beta(u)\theta_M\beta(v)$.

Let M_1 and M_2 be two fuzzy automata and let (α,β) be a morphism between them. Let X_1^*/θ_{M_1} and X_2^*/θ_{M_2} be the corresponding transition monoids. Let $f_\beta : X_1^*/\theta_{M_1}$ to X_2^*/θ_{M_2} defined by $f_\beta[u]_{M_1}=[\beta(u)]_{M_2}, \forall u\in X_1^*$.

Theorem 3.2. *Let* $M_1=(Q_1,X_1,\mu_1)$ *and* $M_2=(Q_2,X_2,\mu_2)$ *be two objects in the category* $F-AUT$ *and let* $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in HOM(M_1,M_2)$
Then f_{β_1}, f_{β_2} *are semigroup morphisms and* $f_{\beta_1\beta_2}=f_{\beta_1}f_{\beta_2}$. *Thus the maps* f_{β_1} *and* $f_{\beta_2}\in HOM(X_1^*/\theta_{M_1}, X_2^*/\theta_{M_2})$.

Proof. f_{β_1} and f_{β_2} are well defined by the above lemma.

Let $[u],[v]\in X_1^*/\theta_{M_1}$, where $u,v\in X^*$.

Then $f_\beta([u][v])=f_\beta[uv]=[\beta(uv)]=[\beta(u)\beta(v)]=[\beta(u)][\beta(v)]=f_\beta[u]f_\beta[v]$.
So f_β is a semigroup morphism

$$f_{\beta_1\beta_2}[u]=[\beta_1\beta_2(u)]=f_{\beta_1}[\beta_2(u)]=f_{\beta_1}f_{\beta_2}[u].$$

We can define a covariant functor F between the category of fuzzy automata and the category of finite semigroups as $F(M)=X^*/\theta_M$ and $F(\alpha,\beta)=f_\beta$ for $(\alpha,\beta)\in HOM(M_1,M_2)$. The set of all inverse fuzzy automata form a full subcategory of $F-AUT$ and F as defined above is a functor from this subcategory to the category of finite inverse monoids which is a subcategory of finite monoids.

Definition 3.2. *A fuzzy automaton* $M=(Q,X,\mu)$ *is said to be faithful if for* $a,b\in X, \mu(q,a,p)=\mu(q,b,p)\forall p,q\in Q \Rightarrow a=b$.

Theorem 3.3. *Let* $M=(Q,X,\mu)$ *be a faithful fuzzy automata. Let* X^*/θ_M *be the transition monoid. Consider* $AUT(X^*/\theta_M)$ *of all automorphisms on* X^*/θ_M. *Let* $h: AUT_X(M)\to AUT(X^*/\theta_M)$ *be a map defined by* $h(\alpha,\beta)=f_\beta$. *Then* h *is a group homomorphism and* $Kerh=Aut_X(M)$.

Proof. $Ker\ h = \{(\alpha,\beta) : h(\alpha,\beta) = f_\beta\}$ where $f_\beta[u] = [u]$ for all $u \in X^*$.

$[\beta(u)] = [u] \Leftrightarrow \beta(u)\theta_M u \Leftrightarrow \mu(p,\beta(u),q) = \mu(p,u,q)\ \forall\ p,q \in Q, u \in X$

$\Leftrightarrow \beta(u) = u\ \forall u \in X \Leftrightarrow \beta$

is the identity map on X.

Thus $Ker\ h = Aut_X(M)$.

Corollary 1. *By homomorphism theorem for groups* $AUT_X(M)/Aut_X(M)$ *is isomorphic to a subgroup of* $AUT(X^*/\theta_M)$.

Consider the set of all one-one partial fuzzy transformations on Q which is the symmetric inverse monoid of fuzzy transformations denoted as FI_Q. We can consider FI_Q as a collection of fuzzy matrices of cardinality $|Q|$ with atmost one nonzero entry in each row and column. For each $\nu \in FI_Q$ there exist a unique inverse $\nu^{-1} \in FI_Q$ such that $\nu^{-1}(p,q) = \nu(q,p), \forall q \in Dom(\nu), p \in Q$. In matrix form it is the transpose of the fuzzy matrix corresponding to ν. The transition monoid X^*/θ_M of an inverse fuzzy automaton M is a subinverse monoid of FI_Q. Consider $N(X^*/\theta_M) = \{\nu \in FI_Q : \nu \circ (X^*/\theta_M) \circ \nu^{-1} = (X^*/\theta_M)\}$ and $C(X^*/\theta_M) = \{\nu \in FI_Q : \nu \circ T_a \circ \nu^{-1} = T_a\ \forall\ T_a \in X^*/\theta_M\}$ where the composition is the max-min composition of fuzzy matrices.

Lemma 3.2. *Let* $M = (Q,X,\mu)$ *be a faithful inverse fuzzy automaton. Let* $\nu \in N(X^*/\theta_M)$. *Then for any* $T_a \in X^*/\theta_M$ *there exist a unique* $T_b \in X^*/\theta_M$ *such that* $\nu \circ T_b \circ \nu^{-1} = T_a$.

Proof. Since $\nu \in N(X^*/\theta_M) \Rightarrow \nu \circ (X^*/\theta_M) \circ \nu^{-1} = X^*/\theta_M$, for $T_a \in X^*/\theta_M$ there exist a $T_b \in X^*/\theta_M$ such that $\nu \circ T_b \circ \nu^{-1} = T_a$. To prove the uniqueness suppose there exist another $T_c \in X^*/\theta_M$ such that $\nu \circ T_c \circ \nu^{-1} = T_a$. Then (since ν is one-one, atmost one nonzero entry will be there in each row and column of the fuzzy matrix corresponding to ν)

$\Rightarrow \mu(q',b,q'') = \mu(q',c,q'')\ \forall\ q',q'' \in Q \Rightarrow \mu(\nu(p),b,$

$\nu(q)) = \mu(\nu(p),c,\nu(q))\ \forall\ \nu(p),\nu(q) \in Q \Rightarrow \mu(p,b,q) = \mu(p,c,q)$.

$\forall\ p,q \in Q \Rightarrow b = c$

Let $N^*(X^*/\theta_M) = \{\nu \in N(X^*/\theta_M) \mid Dom(\nu) = Q\}$ and $C^*(X^*/\theta_M) = \{\gamma \in C(X^*/\theta_M) \mid Dom(\gamma) = Q\}$.

Theorem 3.4. *Let* $M=(Q,X,\mu)$ *be a faithful inverse fuzzy automaton, then* $Aut_X(M)=C^*(X^*/\theta_M)$.

Proof. Let $(\alpha,\beta)\in Aut_X(M)$. Then β is the identity map on X and α is a one-one mapping from Q onto Q satisfying $\mu(\alpha(p),a,\alpha(q))=\mu(p,a,q)$. i.e., $T_a(\alpha(p),\alpha(q))=T_a(p,q)\forall p,q\in Q,a\in X$.

Consider $\alpha\circ T_a\circ\alpha^{-1}(p,q)$

$$=\bigvee_{q'\in Q}\bigvee_{q''\in Q}\alpha(p,q)\wedge T_a(q,q'')\wedge\alpha^{-1}(q'',q)$$

$$=\bigvee_{q'\in Q}\bigvee_{q''\in Q}\alpha(p,q)\wedge T_a(q,q'')\wedge\alpha(q,q'')=\alpha(p,q)\wedge T_a(q,q'')\wedge\alpha(q,q'')$$

$$=T_a(\alpha(p),\alpha(q))=T_a(p,q)since\alpha(p,q)=\alpha(q,q'')=1\Rightarrow\alpha\circ T_a\circ\alpha^{-1}$$

$$=T_a\Rightarrow\alpha\in C^*(X^*/\theta_M).$$

Conversely, let $\alpha\in C^*(X^*/\theta_M)$. Then $\alpha\circ T_a\circ\alpha^{-1}=T_a\ \forall\ T_a\in X^*/\theta_M\Rightarrow\alpha\circ T_a\circ\alpha^{-1}(p,q)=T_a(p,q)\ \forall\ T_a\in X^*/\theta_M$, $p,q\in Q\Rightarrow\bigvee_{q'\in Q}\bigvee_{q''\in Q}\alpha(p,q)\wedge T_a(q,q'')\wedge\alpha^{-1}(q'',q)=T_a(p,q)$

$\Rightarrow\alpha(p,q)\wedge T_a(q,q'')\wedge\alpha(q,q'')=T_a(p,q)$

for some $q''\in Q\Rightarrow T_a(\alpha(p),\alpha(q))=T_a(p,q)$ with $\alpha(p)=q$ and $\alpha(q)=q''\Rightarrow\mu(\alpha(p),a,\alpha(q))=\mu(p,a,q)\ \forall\ p,q\in Q$ and clearly α is one-one and onto. Thus $\alpha\in Aut_X(M)$.

Theorem 3.5. *Let* $M=(Q,X,\mu)$ *be a faithful inverse fuzzy automaton, then* $AUT_X(M)=N^*(X^*/\theta_M)$.

Proof. Let $(\alpha,\beta)\in AUT_X(M)$. Then $(\alpha^{-1},\beta^{-1})\in AUT_X(M)$.

i.e., $\mu(\alpha^{-1}(p),\beta^{-1}(a),\alpha^{-1}(q))=\mu(p,a,q)\ \forall\ p,q\in Q.$

Equivalently, $T_{\beta^{-1}(a)}(\alpha^{-1}(p),\alpha^{-1}(q))=T_a(p,q)\ \forall\ p,q\in Q$. Let $T_a\in X^*/\theta_M$ .. Since β is one-one $\alpha\circ X^*/\theta_M\circ\alpha^{-1}=X^*/\theta_M\Rightarrow\alpha\in N^*(X^*/\theta_M)$.

Conversely, let $\alpha\in N^*(X^*/\theta_M)$. Then by the lemma $\forall T_a\in X^*/\theta_M$ there exist a $T_b\in X^*/\theta_M$ such that $\alpha\circ T_b\circ\alpha^{-1}=T_a$. Define $\beta:X\to X$ as $\beta(a)=b$ with $\alpha\circ T_b\circ\alpha^{-1}=T_a$. Then β is a well defined bijection for,let $\beta(t)=\beta(u)$. Then $\alpha\circ T_c\circ\alpha^{-1}=T_t$ and $\alpha\circ T_d\circ\alpha^{-1}=T_u$ where $\beta(t)=c$ and $\beta(u)=d$. Then $T_c=T_d$ and so $T_t=T_u\Rightarrow t=u$. β is onto for any $b\in X,T_b\in X^*/\theta_M$ and by lemma there exist a unique $T_a\in X^*/\theta_M$ such that $\alpha\circ T_b\circ\alpha^{-1}=T_a$. i.e., there exist an $a \in X$ such that $\beta(a) = b$.

Now, (α, β) is a homomorphism for, let

$p, q \in Q, a \in X, \mu(\alpha(p), \beta(a), \alpha(q)) = \mu(\alpha(p), b, \alpha(q)$ with $\alpha \circ T_b \circ \alpha^{-1} = T_a$.

i.e., $T_{\beta(a)}(\alpha(p), \alpha(q)) = T_b(\alpha(p), \alpha(q))$ with

$$\alpha \circ T_b \circ \alpha(p,q) = T_a(p,q) \Rightarrow \bigvee_{q' \in Q} \bigvee_{q'' \in Q} \alpha(p,q') \wedge T_a(q',q'') \wedge \alpha^{-1}(q'',q)$$

$$= T_a(p,q) \Rightarrow \alpha(p,q') \wedge T_a(q',q'') \wedge \alpha(q,q'') = T_a(p,q)$$

$$\Rightarrow T_b(\alpha(p), \alpha(q) = T_a(p,q)$$

Thus $T_{\beta(a)}(\alpha(p), \alpha(q)) = T_a(p,q) \Rightarrow \mu(\alpha(p), \beta(a), \alpha(q)) = \mu(p,a,q)$.

Thus we have proved that $(\alpha, \beta) \in AUT_X(M)$.

Theorem 3.6. *$C^*(X^*/\theta_M)$ is a normal subgroup of $N^*(X^*/\theta_M)$ or equivalently $Aut_X(M)$ is a normal subgroup of $AUT_X(M)$.*

Proof. Let $\alpha \in N^*(X^*/\theta_M), \nu \in C^*(X^*/\theta_M)$. Let $T_a \in X^*/\theta_M$.

Since $\alpha \in N^*(X^*/\theta_M)$, there exist a unique $T_b \in X^*/\theta_M$ such that

$\alpha \circ T_b \circ \alpha^{-1} = T_b$ and $\nu \circ T_b \circ \nu^{-1} = T_b$.

Consider

$$\alpha \circ \nu \circ \alpha^{-1} \circ T_a \circ (\alpha \circ \nu \circ \alpha^{-1})^{-1} = \alpha \circ \nu \circ \alpha^{-1} \circ T_a \circ \alpha \circ \nu^{-1} \circ \alpha^{-1}$$

$$= \alpha \circ \nu \circ T_b \circ \nu^{-1} \circ \alpha^{-1} = \alpha \circ T_b \circ \alpha^{-1} = T_a \Rightarrow \alpha \circ \nu \circ \alpha^{-1} \in C^*(X^*/\theta_M).$$

Thus $C^*(X^*/\theta_M)$ is a normal subgroup of $N^*(X^*/\theta_M)$.

Corollary 2. *By the theorem 1.5 and the corollary 1, $N^*(X^*/\theta_M)/C^*(X^*/\theta_M)$ is a subgroup of $AUT(X^*/\theta_M)$.*

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