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Some Properties of 0-distributive Meet Semilattices

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Abstract. J.C.Varlet introduced the concept of 0-distributive lattices to generalize the notion of pseudo complemented lattices. A lattice L with 0 is called a 0-distributive lattice if for all $a,b,c \in L$, $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. Of course every distributive lattice with 0 is 0-distributive. Also every pseudo complemented lattice is 0-distributive.Recently, Chakorborty and Talukder extended this concept for directed above meet semi lattices. A meet semi lattice S is called *directed above* if for all $a,b \in S$, there exists $c \in S$ such that $c \geq a,b$. Again Y. Rav has extended the concept of 0-distributivity by introducing the notion of *semi prime ideals* in a lattice. Recently, Noor and Begum have studied the semi prime ideals in a directed above meet semi lattice. In this paper we have included several characterizations and properties of 0-distributive meet semi lattices.. We proved that for a meet sub semi lattice A of S, $A^0 = \{x \in S : x \wedge a = 0 \ \textit{for some}\ a \in A\}$ is a semi prime ideal of S if and only if S is 0-distributive. Using different equivalent conditions of 0-distributive meet semi lattices we have given a 'Separation theorem' for α- ideals..

***Keywords**:* 0-distributive meet semi lattice, Semi prime ideal, Prime ideal, Maximal ideal, α-ideal.

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1.Introduction

J.C.Varlet [7] first introduced the concept of 0-distributive lattices.Then many authors including [1,2,5] studied them for lattices and semilattices. By [2], a meet semilattice S with 0 is called 0-distributive if for all $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge d = 0$ for some $d \geq b,c$. We also know that a

0-distributive meet semilattice is directed above. A meet semi lattice S is called *directed above* if for all $a,b \in S$, there exists $c \in S$ such that $c \geq a,b$.

A non-empty subset I of a directed above meet semilattice S is called a down set if for $x \in I$ and $y \leq x$ $(y \in S)$ imply $y \in I$. Down set *I* is called an ideal if for $x, y \in I$, there exists $z \geq x, y$ such that $z \in I$.

A non-empty subset *F* of *S* is called an upset if $x \in F$ and $y \geq x$ $(y \in S)$ imply $y \in F$. An upset F of S is called a filter if for all $x, y \in F$, $x \wedge y \in F$. An ideal (down set) P is called a prime ideal (down set) if $a \wedge b \in P$ implies either $a \in P$ or $b \in P$. A filter *Q* of S is called prime if $S - Q$ is a prime ideal.

A filter F of S is called a maximal filter if $F \neq S$ and it is not contained by any other proper filter of S. A prime down set P is called a minimal prime down set if it does not contain any other prime down set of S.

Following Lemmas in lattices are due to [1] and [5], and also hold for meet semi lattices by [2].

Lemma 1. *A proper subset F of a meet semilattice S is maximal if and only if S-F is a minimal prime down set.* □

Lemma 2. *Let F be a proper filter of a meet semilattice S with 0. Then there exists a maximal filter containing F.* □

Following result is due to [4,Lemma 5]

Lemma 3. *Let F be a filter and I be an ideal of a directed above meet semilattice S, such that* $F \cap I = \varphi$. *Then F is a maximal filter disjoint from I if and only if for each* $a \notin F$, *there exists* $b \in F$ *such that* $a \wedge b \in I$. □

Let S be a meet semilattice with 0. For a non-empty subset *A* of *S*, we define $A^{\perp} = \{x \in S \mid x \wedge a = 0 \quad \textit{for all } a \in A\}$.This is clearly a down set, but we can not prove that this is an ideal even in a distributive meet semilattice. If *L* is a lattice with 0, then it is well known that *L* is 0-distributive if and only if *I (L)* is 0-distributive.Unfortunately, we can not prove or disprove that when S is a 0-distributive meet semi lattice, then I(S) is
0-distributive. But if *I(S)* is 0-distributive, then it is easy to prove that *S* is also 0-distributive. We define $A^0 = \{x \in S \mid x \wedge a = 0 \textit{ for some } a \in A\}$.This is obviously a down set. Moreover, $A \subseteq B$ implies $A^0 \subseteq B^0$. For any $a \in S$, it easy to check that $(a)^{\perp} = (a)^0 = [a)^0$.

Following result is due to [2].

Theorem 4. *Let S be a directed above meet semilattice with 0. Then the following conditions are equivalent.*

(i) *S is 0-distributive*

(ii) *For each* $a \in S$, $(a)^{\perp} = (a)^{0} = [a)^{0}$ *is an ideal.*

(iii) *Every maximal filter of S is prime.* □

Since in a 0-distributive meet semilattice *S*, for each $a \in S$, $(a)^{\perp}$ is an ideal, so we prefer to denote it by $(a]^{*}$. Y Rav [6] have generalized the concept of 0-distributive lattices and introduced the notion of semi prime ideals in lattices. In a very recent paper [4] have extended the concept in a directed above meet semi lattice. In a directed above meet semilattice *S*, an ideal *J* is called a semi prime ideal if for all $x, y, z \in S$, $x \wedge y \in J$, $x \wedge z \in J$ imply $x \wedge d \in J$ for some $d \geq y, z$. In a distributive semilattice, every ideal is semi prime. Moreover, the semilattice itself is obviously a semi prime ideal. Also, every prime ideal of *S* is semi prime.

Theorem 5. *For any meet sub semilattice A of a directed above meet semi lattice S with 0,* A^{0} *is a semi prime ideal of S if and only if S is 0-distributive.*

Proof: Suppose S is 0-distributive. We already know that A^{0} is a down set, Now let $x, y \in A^{0}$. Then $x \wedge a = 0 = y \wedge b$ for some $a, b \in A$. Then $x \wedge a \wedge b = 0 = y \wedge a \wedge b$. Since *S* is 0-distributive, so $(a \wedge b) \wedge d = 0$ for some $d \geq x, y$. Now $a \wedge b \in A$ implies $d \in A^{0}$, and so A^{0} is an ideal. Finally let $x \wedge y \in A^{0}$, and $x \wedge z \in A^{0}$. Then $x \wedge y \wedge a_1 = 0 = x \wedge z \wedge b_1$ for some $a_1, b_1 \in A$. Thus $x \wedge a_1 \wedge b_1 \wedge y = 0 = x \wedge a_1 \wedge b_1 \wedge z$. Then by the 0-distributive property, $x \wedge a_1 \wedge b_1 \wedge d_1 = 0$ for some $d_1 \geq y, z$. Thus $x \wedge d_1 \in A^{0}$ as $a_1 \wedge b_1 \in A$. Therefore A^{0} is semi prime. Conversely, if A^{0} is a semi prime ideal for every meet sub semilattice *A* of *S*, then in particular $(a)^{0}$ is an ideal. Thus *S* is 0-distributive by Theorem 4. □

Following characterization of semi prime ideals is due to [4].

Theorem 6. *Let S be a directed above meet Semilattice with 0 and J be an ideal of S.*
Then the following conditions are equivalent.

(i) *J is semi prime*

(ii) *Every maximal filter disjoint to J is prime.* □

Thus we have the following separation theorem.

Theorem 7. *Let S be a 0-distributive meet semi lattice and A be a meet subsemilattice of S. Then for a filter F disjoint from* A^0*, there exists a prime ideal containing* A^0 *and disjoint from F.* □

Lemma 8. *Let A and B be filters of a directed above meet semilattice S with 0, such that* $A \cap B^0 = \varphi$*. Then there exists a minimal prime down set containing* B^0 *and disjoint from A.*

Proof: Observe that $0 \notin A \vee B$. For if $0 \in A \vee B$, Then $0 \geq a \wedge b$ for some $a \in A$, $b \in B$. That is, $a \wedge b = 0$, which implies $a \in B^0$ gives a contradiction. It follows that $A \vee B$ is a proper filter of *S*. Then by Lemma 2, $A \vee B \subseteq M$ for some maximal filter *M*. If $x \in M \cap B^0$, Then $x \in M$ and $x \wedge b_1 = 0$ for some $b_1 \in B \subseteq M$. This
implies $O \in M$ which is a contradiction as *M* is maximal. Thus, $M \cap B^0 = \varphi$. Then by Lemma 1, $S - M$ is a minimal prime down set containing B^0. Moreover, $(S - M) \cap A = \varphi$. □

Lemma 9. *Let A be a filter of a directed above meet semilattice S with 0. Then* A^0 *is the intersection of all the minimal prime down sets disjoint from A.*

Proof: Let N be any minimal prime down set disjoint from *A*. If $x \in A^0$, then $x \wedge a = 0$ for some $a \in A$ and so $x \in N$ as *N* is prime.

Conversely, let $y \in S - A^0$. Then $y \wedge a \neq 0$ for all $a \in A$. Hence $A \vee [y)$ is a proper filter of *S*. Then by Lemma 2, $A \vee [y) \subseteq M$ for some maximal filter *M*. Thus by Lemma 1, $S - M$ is a minimal prime down set. Clearly $(S - M) \cap A = \varphi$ and
$y \notin S - M$. □

Now we include some characterization of 0-distributive meet semilattices.

Theorem 10. *Let S be a directed above meet semilattice with 0. Then the following statements are equivalent.*

(i) *S is 0-distributive.*

(ii) *For each* $a \in S$*,* $(a)^0$ *is a semi prime ideal.*

(iii) *For any three filters A, B, C of S,*
$$(A \vee (B \cap C))^0 = (A \vee B)^0 \cap (A \vee C)^0$$

(iv) *For all* $a, b, c \in S$*,*
$$([a) \vee ([b) \cap [c)))^0 = ([a) \vee [b))^0 \cap ([a) \vee [c))^0$$

(v) *For all* $a,b,c \in S$, $(a \wedge d)^0 = (a \wedge b)^0 \cap (a \wedge c)^0$ *for some* $d \geq b,c$.

Proof: (i) $\Leftrightarrow$ (ii). Follows by theorem 4.
(i) $\Rightarrow$ (iii). Let $x \in (A \vee B)^0 \cap (A \vee C)^0$. Then $x \in (A \vee B)^0$ and $x \in (A \vee C)^0$. Thus $x \wedge f = 0 = x \wedge g$ for some $f \in A \vee B$ and $g \in A \vee C$. Then $f \geq a_1 \wedge b$, and $g \geq a_2 \wedge c$ for some $a_1, a_2 \in A$, $b \in B$, $c \in C$. This implies $x \wedge a_1 \wedge b = 0 = x \wedge a_2 \wedge c$ and so $x \wedge a_1 \wedge a_2 \wedge b = 0 = x \wedge a_1 \wedge a_2 \wedge c$. Since S is 0-distributive, so $x \wedge a_1 \wedge a_2 \wedge d = 0$ for some $d \geq b,c$. Now $a_1 \wedge a_2 \in A$ and $d \in B \cap C$. Therefore, $((a_1 \wedge a_2) \wedge d \in A \vee (B \cap C)$ and so $x \in (A \vee (B \cap C))^0$. The reverse inclusion is trivial as $A \vee (B \cap C) \subseteq A \vee B, A \vee C$. Hence (iii) holds.
(iii) $\Rightarrow$ (iv) is trivial by considering $A = [a)$, $B = [b)$ and $C = [c)$ in (iii).
(iv) $\Rightarrow$ (v). Let (iv) holds. Suppose $x \in (a \wedge b)^0 \cap (a \wedge c)^0$. Then by (iv) $x \in ([a) \wedge [b))^0 \cap ([a) \wedge [c))^0 = ([a) \vee ([b) \cap [c)))^0$. This implies $x \wedge f = 0$ for some $f \in [a) \vee ([b) \cap [c))$. Then $f \geq a \wedge d$ for some $d \in [b) \cap [c)$. That is, $f \geq a \wedge d$ for some $d \geq b,c$. It follows that $x \wedge a \wedge d = 0$ and so $x \in (a \wedge d)^0$.On the other hand, $[a) \vee [d) \subseteq [a) \vee [b)$ and $[a) \vee [d) \subseteq [a) \vee [c)$ implies that $(a \wedge d)^0 \subseteq (a \wedge b)^0 \cap (a \wedge c)^0$. Therefore (v) holds.
(v) $\Rightarrow$ (i). Suppose (v) holds. Let $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$. Then $a \wedge (a \wedge b) = 0 = a \wedge (a \wedge c)$ implies $a \in (a \wedge b)^0 \cap (a \wedge c)^0 = (a \wedge d)^0$ for some $d \geq b,c$. Thus, $a \wedge (a \wedge d) = 0$ for some $d \geq b,c$. That is $a \wedge d = 0$ for some $d \geq b,c$. and so S is 0 –distributive. □

Now we include few more characterizations of 0-distributive semilattices.

Theorem 11. *Let S be a directed above meet semi lattice with 0. Then the following are equivalent.*

(i) *S is 0-distributive.*

(ii) *For any three filters A, B, C of L.*

$$((A \cap B) \vee (A \cap C))^0 = A^0 \cap (B \vee C)^0$$

(iii) *For any two filters A, B of S,* $(A \cap B)^0 = A^0 \cap B^0$

(iv) *For all* $a,b \in S$, $(a)^0 \cap (b)^0 = (d)^0$ *for some* $d \geq b,c$.

(v) *For all* $a,b \in S$, $(a]^* \cap (b]^* = (d]^*$ *for some* $d \geq b,c$.

Proof: (i) $\Rightarrow$ (ii). Suppose S is 0-distributive, Since $(A \cap B) \vee (A \cap C) \subseteq A$ and $B \vee C$, so $((A \cap B) \vee (A \cap C))^0 \subseteq A^0 \cap (B \vee C)^0$. Now suppose $x \in A^0 \cap (B \vee C)^0$ Then $x \in A^0$ and $x \in (B \vee C)^0$. Thus $x \wedge a = 0$ for some $x \in A$ and $x \wedge d = 0$ for some $d \in B \vee C$. Now $d \in B \vee C$ implies $d \geq b \wedge c$ for some $b \in B$, $c \in C$. Hence $x \wedge a = 0 = x \wedge b \wedge c$. Then $x \wedge c \wedge a = 0 = x \wedge c \wedge b$. Since S is 0-distributive, so $x \wedge c \wedge d_1 = 0$ for some $d_1 \geq a, b$. Then $d_1 \in A \cap B$. Now $x \wedge a = 0$ implies $x \wedge d_1 \wedge a = 0 = x \wedge d_1 \wedge c$. Again by the 0-distributivity, $x \wedge d_1 \wedge d_2 = 0$ for some $d_2 \geq a, c$ that is $d_2 \in A \cap C$. Therefore, $x \in ((A \cap B) \vee (A \cap C))^0$ and so (ii) holds.

(ii) $\Rightarrow$ (iii) is trivial by considering $B = C$ in (iii).

(iii) $\Rightarrow$ (iv),. Choose $A = [a)$ and $B = [b)$ in (iii).

Now for all $d \geq a, b$, $[a) \supseteq [d)$ and $[b) \supseteq [d)$ and so $[d)^0 \subseteq (a)^0 \cap (b)^0$. Also by (iii), $(a)^0 \cap (b)^0 = ([a) \cap [b))^0$. Thus, $x \in (a)^0 \cap (b)^0$. implies $x \wedge d_1 = 0$ for some $d_1 \geq a, b$. That is, $x \in (d_1)^0$ for some $d_1 \geq a, b$. Thus (iv) holds.

(iv) $\Leftrightarrow$ (v) is obvious.

(v) $\Rightarrow$ (i).Suppose (v) holds and for $a, b, c \in S$, $a \wedge b = 0 = a \wedge c$. Then $a \in (b]^* \cap (c]^* = (d]^*$ for some $d \geq b, c$. Therefore, $a \wedge d = 0$ and so S is 0-distributive. □

An ideal I in a directed above meet semilattice S with 0 is called an α-ideal if for each $x \in I$, $\{x\}^{\perp\perp} \subseteq I$.

Proposition 12. *If I is an α-ideal of a 0-distributive meet semilattice S, Then* $I = \{y \in S \mid (y] \subseteq \{x\}^{\perp\perp}$ *for some* $x \in I\}$.

Proof: Let $y \in R.H.S$. Then $(y] \subseteq \{x\}^{\perp\perp} \subseteq I$. This implies $y \in I$. Conversely, let $y \in I$. Since S is 0-distributive, so by theorem 4, $(y]^{\perp}$ is an ideal and $(y] \cap (y]^{\perp} = (0]$. Thus, $(y] \subseteq (y]^{\perp\perp}$, which implies $y \in R.H.S.$ □

Prime separation theorem for α-ideals in 0-distributive lattices was given in [3]. Now we include a prime separation theorem on α-ideals for 0-distributive meet semilattices.

Theorem 13. *Let F be a filter and I be an α-ideal of a directed above meet semilattice S with 0, such that* $I \cap F = \varphi$. *If I (S) is 0-distributive, then there exists a prime α-ideal P containing I such that* $P \cap F = \varphi$.

Proof: By lemma 2, there exists a maximal filter *M* containing *F* and disjoint to *I*. Thus $P = S - M$ is a minimal prime down set containing *I* and disjoint to *M*. Now let $p, q \in S - M$. Then by lemma 3, there exist $a, b \in M$ such that $a \wedge p \in I$ and $b \wedge q \in I$. Then by proposition 12, $(a \wedge p] \subseteq (x]^{\perp\perp}$ and $(b \wedge q] \subseteq (y]^{\perp\perp}$ for some $x, y \in I$. Thus $(a \wedge p] \wedge (x]^{\perp} = (0] = (b \wedge q] \wedge (y]^{\perp}$ .This implies $(a \wedge b] \wedge (x]^{\perp} \wedge (y]^{\perp} \wedge (p] = (0] = (a \wedge b] \wedge (x]^{\perp} \wedge (y]^{\perp} \wedge (q]$, Now as *I* is an ideal, so there exists $d_1 \geq x, y$ such that $d_1 \in I$. Again by Theorem 11 (v), $(x]^{\perp} \wedge (y]^{\perp} = (d_2]^{\perp}$ for some $d_2 \geq x, y$. Then $d = d_1 \wedge d_2 \in I$, and so $(d]^{\perp} \subseteq (x]^{\perp} \wedge (y]^{\perp} = (d_2]^{\perp} \subseteq (d]^{\perp}$.Thus $(x]^{\perp} \wedge (y]^{\perp} = (d]^{\perp}$ for some $d \in I$, $d \geq x, y$. Then we have $(a \wedge b] \wedge (d]^{\perp} \wedge (p] = (0] = (a \wedge b] \wedge (d]^{\perp} \wedge (q]$. Since $I(S)$ is 0-distributive, so $(a \wedge b] \wedge (d]^{\perp} \wedge ((p] \wedge (q]) = (0]$. Then $(a \wedge b] \wedge (d]^{\perp} \wedge (t] = (0]$ for some $t \geq p, q$. Thus $(a \wedge b \wedge t] \subseteq (d]^{\perp\perp} \subseteq I \subseteq S - M$ .But $a \wedge b \in M$ implies $t \in S - M$ as $S - M$ is prime. Therefore $P = S - M$ is an ideal. Now let $x \in P$. If $x \in I$, Then $(x]^{\perp\perp} \subseteq I \subseteq P$ as *I* is an α-ideal. Finally if $x \in P - I$. Then again by Lemma 3, there exists $a \in M$ such that $a \wedge x \in I$. Thus $(a]^{\perp\perp} \wedge (x]^{\perp\perp} \subseteq I \subseteq P$. Since $a \notin P$, so $(a]^{\perp\perp} \nsubseteq P$. Therefore, $(x]^{\perp\perp} \subseteq P$ as *P* is prime, and hence *P* is also an α-ideal. □

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