

Interval-valued Fuzzy Vector Space

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Received 2 December 2012; accepted 17 December 2012

Abstract. The interval-valued fuzzy vector space with respect to the interval-valued fuzzy algebra is defined and investigated a lot of interesting properties. Each result is illustrated by a suitable example.

Keywords: *Interval-valued fuzzy sets, interval-valued fuzzy vector space, subspace.*

AMS Mathematics Subject Classification (2010): 08A72, 15B15

1. Introduction

There is a growing interest in extensions of fuzzy set theory which can be model not only vague information, but also uncertainty. Fuzzy algebraic structures play an important role in mathematics with wide applications in many other branches such as theoretical physics, computer science, information sciences, coding theory, topological spaces, logic, set theory, group theory, real analysis, measure theory etc. It is well known that the membership value completely depends on the decision maker's habits, mentality etc. So some times it happens that the membership value can not be measure as a point, but it can be measured appropriately as an interval. Interval-valued fuzzy sets (IVFS's), first proposed by Sambuc [17] and Zadeh [22], as extensions of fuzzy sets with interval-valued membership functions, in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Also, for further evidence of their wide relevance, interval-valued fuzzy sets have equivalent framework of Atanossou's intuitionistic fuzzy sets [3] and of Gau and Buchrer's vague sets [5].

In 1977, Katsaras and Liu [10] formulated the concept of a fuzzy subspace of vector space. After that a lot of mathematicians [1, 2, 11, 13, 14, 16] are involved in extending the basic concepts and results from the crisp vector spaces to the fuzzy vector space. In this article, we present some new types of definitions, properties and theorem of interval-valued fuzzy vector space (IVFVS) with respect to interval-valued fuzzy algebra(IVFA).

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2. Preliminaries

The IVFSs is the extension of fuzzy sets with interval-valued membership functions. In this section, some basic notions of interval-valued fuzzy sets are introduced. The interval-valued fuzzy set defined by Zadeh [22] in the year 1975, which is defined below.

Definition 1. [22](**Interval-valued fuzzy set**). An interval-valued fuzzy set (IVFS) A on the universe $U \neq \phi$ is given by:

$$A = \{(u, A(u)) : u \in U\},$$

where $A(u) = [\underline{A}(u), \overline{A}(u)] \in L([0,1])$ being

$$L([0,1]) = \{x = [\underline{x}, \overline{x}] : [\underline{x}, \overline{x}] \in [0,1]^2 \text{ and } \underline{x} \leq \overline{x}\}.$$

Obviously, $A(u) = [\underline{A}(u), \overline{A}(u)]$ is the membership degree and $\underline{A}(u), \overline{A}(u)$ are the lower and the upper limits of the membership degree of $u \in U$.

Let I be the set of all real numbers lying between 0 and 1, i.e., $I = \{x : 0 \leq x \leq 1\}$.

Also, let $D[0,1]$ be the set of all subsets of the interval $[0,1]$ which can be written as $D = \{[a, b] : a \leq b; a, b \in I\}$.

The addition and multiplication between any two elements of D are defined in the following way.

Definition 2. Let $x = [a_1, b_1]$ and $y = [a_2, b_2]$ be any two elements of D . The addition (+) and multiplication (\cdot) between x and y are defined as

$$\begin{aligned} x + y &= [a_1, b_1] + [a_2, b_2] = [\max(a_1, a_2), \max(b_1, b_2)] = [a_1 \vee a_2, b_1 \vee b_2] \\ \text{and } x \cdot y &= [a_1, b_1] \cdot [a_2, b_2] = [\min(a_1, a_2), \min(b_1, b_2)] = [a_1 \wedge a_2, b_1 \wedge b_2]. \end{aligned}$$

In arithmetic operations (such as addition, multiplication, etc.) only the values of lower limit and upper limit of membership degree are needed. So from now we denote IVFS as

$$A = \{[\underline{u}, \overline{u}] : [\underline{u}, \overline{u}] \in D[0,1]\}$$

where $\underline{u}, \overline{u}$ are the lower and upper limits of the membership degree of $u \in U$.

Two special elements, viz. zero element denoted by \emptyset and unit element denoted by I are defined below.

Definition 3 (Zero element). The zero element of an IVFS is denoted by \emptyset and is defined by $\emptyset = [0,0]$.

Definition 4 (Unit element). The unit element of an IVFS is denoted by I and is defined by $I = [1,1]$.

The equality of two elements of an IVFS is defined below.

Definition 5. Let \mathbf{F} be an IVFS and $x, y \in \mathbf{F}$ where $x = [a_1, b_1]$ and $y = [a_2, b_2]$, then $x = y$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

The logical operators \leq and $<$ are given in the following definitions.

Definition 6. Let \mathbf{F} be an IVFS and $x, y \in \mathbf{F}$ where $x = [a_1, b_1]$ and $y = [a_2, b_2]$, then $x \leq y$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$.

Definition 7. Let \mathbf{F} be an IVFS and $x, y \in \mathbf{F}$ where $x = [a_1, b_1]$ and $y = [a_2, b_2]$, then $x < y$ if and only if $x \leq y$ and $x \neq y$.

3. Interval-valued fuzzy vector space

In order to develop the theory of interval-valued fuzzy vectors (IVFVs) we begin with the concept of interval-valued fuzzy algebra (IVFA). An IVFA is a mathematical system $(\mathbf{F}, +, \cdot)$ with two binary operations $+$ and \cdot defined on the set \mathbf{F} satisfying the following properties:

- Idempotent. $a + a = a$, $a \cdot a = a$
- Commutativity. $a + b = b + a$, $a \cdot b = b \cdot a$
- Associativity. $a + (b + c) = (a + b) + c$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Absorption. $a + (a \cdot b) = a$, $a \cdot (a + b) = a$
- Distributivity. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, $a + (b \cdot c) = (a + b) \cdot (a + c)$
- Universal bounds. $a + \emptyset = a$, $a + \mathbf{1} = \mathbf{1}$, $a \cdot \emptyset = \emptyset$, $a \cdot \mathbf{1} = a$

where $a = [\underline{a}, \bar{a}]$, $b = [\underline{b}, \bar{b}]$ and $c = [\underline{c}, \bar{c}] \in \mathbf{F}$

Katsaras and Liu [10] first introduced the concept of fuzzy vector space, after that many researchers investigated its properties and characteristics [1, 2, 11, 12, 13, 14, 16]. Using these results Shi and Huang [19] presented new definitions of fuzzy basis and fuzzy dimension for a fuzzy vector space. Here we present some basic definitions and properties of interval-valued fuzzy vector space (IVFVS) with respect to IVFA.

To the best of our knowledge, no work is available on IVFVS. We define an interval-valued fuzzy vector and IVFVS.

Definition 8 (Interval-valued fuzzy vector). An interval-valued fuzzy vector is an n -tuple of elements from an IVFA. That is, an IVFV is of the form (x_1, x_2, \dots, x_n) , where each element $x_i \in \mathbf{F}$, $i = 1, 2, \dots, n$.

Definition 9 (Interval-valued fuzzy vector space). An interval-valued fuzzy vector space (IVFVS) is a pair $(E, A(x))$ where E is a vector space in crisp sense and $A: E \rightarrow D[0, 1]$ with the property that for all $a, b \in \mathbf{F}$ and $x, y \in E$ we have

$$\underline{A}(ax + by) \geq \underline{A}(x) \wedge \underline{A}(y) \text{ and } \bar{A}(ax + by) \geq \bar{A}(x) \wedge \bar{A}(y).$$

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Example 1. Let V_n denote the set of all n -tuples (x_1, x_2, \dots, x_n) over F . An element of V_n is called an IVFV of dimension n . For $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in V_n , the operations addition (+) and multiplication (\cdot) are defined as

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in V_n$$

and $ax = (ax_1, ax_2, \dots, ax_n) \in V_n$ for $a \in F$.

The set V_n together with these operations of componentwise addition and scalar multiplication is an IVFVS over F , as the scalars are restricted to F .

Definition 10. Let $V^n = \{x^t \mid x \in V_n\}$ where x^t is the transpose of the vector x . For $u, v \in V^n$ and $a \in F$ we define $av = (av^t)^t$, $u + v = (u^t + v^t)^t$. Then V^n is an IVFVS. If we write an element of V_n as $1 \times n$ matrix, it is called a row vector. The element of V^n are column vectors.

For any result of V_n there exists a similar result on V^n . Thus V^n is isomorphic to V_n , so in general we denote the vector space as V .

Definition 11 (Subspace). A subspace of V_n is a subset W of V_n such that $\emptyset_n \in W$ and for $x, y \in W$, $x + y \in W$, where the zero element of IVFVS V_n is denoted by \emptyset_n and is defined as $\emptyset_n = ([0,0], [0,0], \dots, [0,0])$.

Example 2. Let $W = \{(\emptyset, \emptyset, \emptyset), (l, \emptyset, \emptyset), (\emptyset, l, \emptyset), (\emptyset, \emptyset, l), (l, l, \emptyset), (l, \emptyset, l), (\emptyset, l, l), (l, l, l)\}$. Then W is a subspace of V_3 .

Definition 12 (Linear combination). A linear combination of elements of the set of IVFVs S is a finite sum $\sum a_i x_i$ where $x_i \in S$ and $a_i \in F$. The set of all linear combinations of elements of S is called the span of S , denoted by $\langle S \rangle$.

Definition 13 (Spanning set). Let S and W be two subsets of V_n . If $\langle S \rangle = W$ then S is called a spanning set or set of generators for W . If W is a subspace of V_n then $\langle W \rangle = W$.

Definition 14 (Basis). A basis for a subspace W of V_n is a minimal (i.e., cardinality is minimum) spanning set for W .

Example 3. Let $W = \{(x, x) \mid x \in F\}$ be a subspace of $F \times F$. The singleton set

$S = \{(1,1)\}$ is the minimal spanning set for W , since every element $(x,x) = x(1,1)$ with $x \in \mathbf{F}$ is a linear combination of $(1,1)$ in S , S is a basis of W . The set $S_1 = \{(1, \emptyset), (\emptyset, 1)\}$ is also a spanning set of W since every element $(x,x) \in W$ can be written as $(x,x) = x(1, \emptyset) + x(\emptyset, 1)$ with $x \in \mathbf{F}$ is a linear combination of $(1, \emptyset)$ and $(\emptyset, 1)$ in S_1 , but S_1 is not a basis of W since it is not minimal. The singleton set $S_2 = \{(1, \emptyset)\}$ is not a spanning set of W .

Definition 15. A set S of vectors over an IVFA \mathbf{F} is linearly dependent if at least one element of S is a linear combination of other elements of S . The set S is linearly independent over \mathbf{F} if it is not linearly dependent.

Definition 16. Let X and Y be the set of IVFVs, $X \subseteq Y$ if and only if every element of X are lies in Y . $X \subset Y$ if and only if $X \subseteq Y$ but $X \neq Y$.

Proposition 1. Let X and Y be the set of IVFVs.

- The set consisting of the zero vector is linearly dependent.
- If $X \subset Y$ and if X is linearly dependent, then Y is also linearly dependent.
- If $X \subset Y$ and if Y is linearly independent, then X is also linearly independent.

Proof. (i) Let $S = \{a_1, a_2, \dots, a_{r-1}, \emptyset, a_{r+1}, \dots, a_n\}$, where a_i 's are IVFVs for all $i \in \{1, 2, \dots, r-1, r+1, \dots, n\}$ and \emptyset be the interval-valued fuzzy zero vector. It can be written as $\emptyset = \emptyset \cdot a_1 + \emptyset \cdot a_2 + \dots + \emptyset \cdot a_{r-1} + \emptyset \cdot a_{r+1} + \dots + \emptyset \cdot a_n$.

Hence \emptyset is the linear combination of the other vectors in S . Thus S is linearly dependent.

(ii) Since $X \subset Y$ we take X and Y as $X = \{a_1, a_2, \dots, a_r\}$ and $Y = \{a_1, a_2, \dots, a_r, a_{r+1}, \dots, a_n\}$ where $n > r$; $r, n \in \mathbf{N}$ (set of natural numbers) and a_i 's are all IVFVs for all $i \in \{1, 2, \dots, n\}$ i.e., $a_i \in \mathbf{V}$.

Also since X is linearly dependent, there exist a vector $a_j \in X$ such that

$$a_j = c_1 a_1 + c_2 a_2 + \dots + c_{j-1} a_{j-1} + c_{j+1} a_{j+1} + \dots + c_r a_r$$

where $c_i \in \mathbf{F}$ for all $i \in \{1, 2, \dots, n\}$, which implies that

$$a_j = c_1 a_1 + c_2 a_2 + \dots + c_{j-1} a_{j-1} + c_{j+1} a_{j+1} + \dots + c_r a_r + \emptyset a_{r+1} + \dots + \emptyset a_n.$$

Hence Y is linearly dependent.

(iii) Given that $X \subset Y$ and Y is linearly independent. If possible suppose that X is linearly dependent. Then by (ii) (Proposition 3) it follows that Y is linearly dependent. This contradicts, our assumption. Hence X is linearly independent.

Example 4. The set of vectors

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$\{([0.5,0.6],[1,1]); ([1,1],[0.6,0.7]); ([0.7,0.8],[0.8,0.9])\}$

is linearly dependent since

$$\begin{aligned} & [0.8,0.9]([0.5,0.6],[1,1])+[0.7,0.8]([1,1],[0.6,0.7]) \\ &= ([0.5,0.6],[0.8,0.9])+([0.7,0.8],[0.6,0.7]) \\ &= ([0.7,0.8],[0.8,0.9]). \end{aligned}$$

But the set of vectors $\{([0.5,0.6],[1,1]); ([1,1],[0.6,0.7])\}$ is linearly independent.

Definition 17 (Standard basis). A basis \mathbf{B} over the IVFA \mathbf{F} is a standard basis if and only if whenever $a_i = \sum c_{ij}a_j$ for $a_i, a_j \in \mathbf{B}$ and $c_{ij} \in \mathbf{F}$, $c_{ii}a_i = a_i$.

Example 5. The basis

$\{([0.4,0.5],[1,1],[0.4,0.5]); ([0,0],[1,1],[0.4,0.5]); ([0,0],[0.4,0.5],[1,1])\}$

is not a standard basis for

$$\begin{aligned} & ([0.4,0.5],[1,1],[0.4,0.5]) = [0.4,0.5]([0.4,0.5],[1,1],[0.4,0.5]) \\ & + [1,1]([0,0],[1,1],[0.4,0.5]) + [0.3,0.4]([0,0],[0.4,0.5],[1,1]). \end{aligned}$$

But $([0.4,0.5],[1,1],[0.4,0.5]) \neq [0.4,0.5]([0.4,0.5],[1,1],[0.4,0.5])$.

However the basis

$\{([0.4,0.5],[0.4,0.5],[0.4,0.5]); ([0,0],[1,1],[0.4,0.5]); ([0,0],[0.4,0.5],[1,1])\}$

is a standard basis for the same subspace.

Theorem 1. Any finitely generated subspace over \mathbf{F} has a unique standard basis.

Proof. If possible, let us assume that \mathbf{B} and \mathbf{B}' be two standard bases with $|\mathbf{B}|=|\mathbf{B}'|$. Since \mathbf{B}' is a basis, each element of \mathbf{B} can be expressed as a linear combination of the elements of \mathbf{B}' .

Therefore, each element b_i of \mathbf{B} must be multiple of some elements $b_{j'}$ of \mathbf{B}' . Note that the product of two interval-valued fuzzy elements the minimum value is taken as a result. Thus $b_i \leq b_{j'}$. Similarly, $b_{j'} \leq b_i$, also $|\mathbf{B}|=|\mathbf{B}'|$ follows that $b_i = b_{j'}$. Hence $\mathbf{B} = \mathbf{B}'$, i.e., the standard basis is unique.

Theorem 2. Over the interval-valued fuzzy algebra \mathbf{F} , any two bases for a finitely generated subspace of IVFVS have the same cardinality.

Proof. Let S be the set of IVFVs each of whose entries is equal to some entry of a vector of any finite basis \mathbf{B} . Then S is a finite set. Here two cases arises,

Case I: If \mathbf{B} is not a standard basis, then $b_i = \sum a_{ij}b_j$ for some $b_i \in \mathbf{B}$ and $a_{ij} \in \mathbf{F}$ with $b_i \neq a_{ii}b_i$. i.e., $b_i \neq \min\{a_{ii}, b_i\}$. Therefore $a_{ii}b_i < b_i$.

Let \mathbf{B}_1 be the set obtained from \mathbf{B} by replacing b_i by $a_{ii}b_i$. Then $|\mathbf{B}|=|\mathbf{B}_1|$ and $\langle \mathbf{B} \rangle = \langle \mathbf{B}_1 \rangle$. Also \mathbf{B}_1 is minimal set and all the vectors of \mathbf{B}_1 are all in S .

If B_1 becomes a standard basis, then B_1 is the require standard basis with the same cardinality as B . If B_1 is not standard basis, then repeat the process of replacing B_1 by a basis B_2 and proceed. Therefore after replacing bases of the form B by the form B_i , the process must be terminated after some finite number of steps, which happen only if we obtained a standard basis B_i with the same cardinality as B . This proves that for any finite basis, there exists a standard basis with the same cardinality.

Case II: Let B is a standard basis. Also, if possible let B_1 be a basis of S such that $|B| \neq |B_1|$, then either $|B_1| > |B|$ or $|B_1| < |B|$.

Now $|B_1| > |B|$ contradict the definition of basis that B_1 is a minimal spanning set of S .

Also if $|B_1| < |B|$ then there exist a standard basis, by using the first case, of the cardinality equal to B_1 , which contradict that B is an unique standard basis. Hence $|B| = |B_1|$.

Theorem 3. Let S be a finitely generated subspace of V_n and let $\{e_1, e_2, \dots, e_n\}$ be the standard basis for S . Then any vector $x \in S$ can be expressed uniquely as a linear combination of vectors of the standard basis.

Proof. Given $\{e_1, e_2, \dots, e_n\}$ is the standard basis for S . Let x be any vector of

$$V_n. \text{ Let } x = \sum_{j=1}^n c_j e_j \text{ where } c_j \in F.$$

In this expression, the coefficients c_j 's are not unique. If we write this in the matrix form as $x = (c_1, c_2, \dots, c_n) \cdot E$, where E is the matrix whose rows are the basis vectors, then $x = p \cdot E$ has a solution $p = (c_1, c_2, \dots, c_n)$. Also it can be shown that, this equation has a unique maximal solution (Theorem 1 of [20])

(p_1, p_2, \dots, p_n) (say). Then $x = \sum_{j=1}^n p_j e_j$ with $p_j \in F$ is the unique representation of the vector x .

Theorem 4. Let S be a vector space over F , be the linear span of the vectors x_1, x_2, \dots, x_n . If some x_i is a linear combination of $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, then the vectors $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ also spans S .

Proof. Let $W = \{x_1, x_2, \dots, x_n\}$ such that $S = \langle W \rangle$. Since x_i is a linear combination of $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$, then there exist c_j 's for

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$j \in \{1, 2, \dots, i-1, i+1, \dots, n\}$ and $c_j \in \mathbf{F}$ such that

$$x_i = \sum_{j=1, j \neq i}^n c_j x_j.$$

Since $S = \langle W \rangle$, any vector $y \in S$ can be expressed as

$$\begin{aligned} y &= d_1 x_1 + d_2 x_2 + \dots + d_{i-1} x_{i-1} + d_{i+1} x_{i+1} + \dots + d_n x_n \\ &= \sum_{j=1, j \neq i}^n d_j x_j + d_i x_i \\ &= \sum_{j=1, j \neq i}^n d_j x_j + d_i \sum_{j=1, j \neq i}^n c_j x_j \\ &= \sum_{j=1, j \neq i}^n c'_j x_j \end{aligned}$$

where $c'_j = d_j + d_i c_j$ for $j \in \{1, 2, \dots, i-1, i+1, \dots, n\}$ are elements in \mathbf{F} . Since y is arbitrary vector in S , we have $S = \langle W - \{x_i\} \rangle$. Thus the vectors $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ spans S .

Definition 18 (Dimension). *The dimension of the finitely generated subspace S of a vector space V_n over the IVFA \mathbf{F} denoted by $\dim(S)$ is defined to be the cardinality of the standard basis of S .*

Example 6. *The set $\{([1, 1], [0, 0], [0, 0]); ([0, 0], [1, 1], [0, 0]); ([0, 0], [0, 0], [1, 1])\}$ form the standard basis for V_3 . Thus $\dim(V_3) = 3$.*

Theorem 5. *Let S be a vector space over \mathbf{F} of dimension n and let x_1, x_2, \dots, x_m ($m < n$) be linearly independent vectors in S . Then there exist a basis for S containing x_1, x_2, \dots, x_m .*

Proof. Let y_1, y_2, \dots, y_n be the unique standard basis for S . Then the set $W = \{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\}$ is a linearly dependent subset of S . Therefore y_i for some $i \in \{1, 2, \dots, n\}$ is a linear combination of the vectors in $W - \{y_i\}$. Since S is a linear span of W . By Theorem 3, $W - \{y_i\}$ also spans S . If the set $W - \{y_i\}$ is minimal set, then we have a basis for S as required. Otherwise, we continue the process, up to m th iteration, until we get a basis containing x_1, x_2, \dots, x_m .

Theorem 6. Any set of $(n+1)$ vectors in V_n is linearly dependent.

Proof. If the set of $(n+1)$ vectors in V_n is linearly independent, then by Theorem 3 we can find a basis for V_n containing the set i.e., $(n+1)$ vectors. This contradicts that the dimension of V_n is n , and every basis for V_n must contain n vectors. Thus any set of $(n+1)$ vectors in V_n is linearly dependent.

4. Conclusion

In this article, some basic propositions and theorems of IVFVS are investigated. The IVFVS is the topic of the interval-valued fuzzy linear space (IVFLS) so, further our investigation should be motions to investigate the other topics of the IVFLS like as interval-valued fuzzy matrix (IVFM), determinant, its properties, for example, to find the adjoint, inverse, eigenvalue, eigenvector etc of IVFMs.

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