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Bi-amalgamated Rings with Weakly p-clean Properties

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Abstract. The concept of a clean ring, introduced by Nicholson in 1977, is defined as follows: for any x ∈ R, there exist e ∈ Id(R) and u ∈ U(R) such that x = e + u. In this paper, the concept of a weakly p-clean ring is introduced. A ring R is said to be weakly p-clean if each element, x, can be written as x = p + e or x = p − e, where p is a pure element and e is an idempotent element. The transfer of the notion of weakly p-clean rings to the amalgamation of rings and bi-amalgamation of rings along the ideal is studied. Additionally, some characterizations of weakly p-clean rings are provided.

Keywords: clean ring, p-clean ring, weakly clean ring, weakly p-clean ring.

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1. Introduction

Over the past years, there have been many research studies on variations of the clean properties. As defined by Nicholson [10] an element r in a ring R is clean, if it can be written as x = u + e, where u ∈ U(R), the group of units of R, and e ∈ Id(R), the set of idempotents of R. A ring R is clean if every element is clean. In [1], Ahn and Anderson defined a ring R as weakly clean if each element of R can be written as either the sum or difference of a unit and an idempotent. An element p in a ring R is called a pure element if there exists q in R such that p = pq [8], and the set of pure elements in R is written Pu(R) = {p ∈ R: p = pq, for some q ∈ R}. In [9], Mohammed et al. detailed the concept of a p-clean ring: an element c ∈ R is called p-clean if there exists e ∈ Id(R) and p ∈ Pu(R) such that c = e + p. We call the ring R p-clean if each element in R expresses itself as the sum of an idempotent element and a pure element.

Let R and S be two rings with unity; let J be an ideal of S; and let ϕ: R → S be a ring homomorphism. In [6], Anna et al. introduced and studied the new ring structure of the following subring of R × S:

$$R\bowtie^{\phi}J := \{(r, f(r) + j) \mid r \in R, j \in J\}$$

called the amalgamation of R with S along J with respect to ϕ. This new ring structure construction is a generalization of the amalgamated duplication of a ring along an ideal. Aruldoss et al. [2], Vijayanand and Selvaraj [15], Selvaganesh [14], and Selvaganesh and

Selvaraj [13] studied some ring properties and modules characterized via amalgamation construction.

Let ϕ: R → S and ψ : R → T be two commutative ring homomorphisms and let J and J′ be two ideals of S and T , respectively, such that $\phi^{-1}(J) = \psi^{-1}(J')$. Kabbaj et al. [7] introduced and studied the bi-amalgamation of R with (S, T) along (J, J′) with respect to (ϕ, ψ) which is the subring of S × T given by

$$R\bowtie^{\phi,\phi'}(J, J') := \{(\phi(r) + j, \phi'(r) + j') \mid r \in R, (j, j') \in (J, J')\}.$$

Aruldoss and Selvaraj [3, 4] and Vijayanand and Selvaraj [16] investigated some ring properties and modules characterized by bi-amalgamation construction.

The notion of a regular element was first introduced by von Neumann [17], where an element r ∈ R is called a regular if there exists s ∈ R such that r = rsr. A ring R is called a regular ring if each element in R is regular. Ashrafi and Nasibi [5] introduced the concept of r-clean ring, where the ring R is called r-clean ring if for each a ∈ R there exist e ∈ Id(R) and r ∈ Reg(R) such that a = e + r. Saravanan [11, 12] studied feebly r-clean rings, feebly r-clean ideals, and their properties.

The above studies have motivated the researchers to study the weakly p-clean rings properties in amalgamation of rings and bi-amalgamation of rings.

This paper aims at proposing the idea of a weakly p-clean ring and studying the transfer of the notion of weakly p-clean rings to the amalgamation of rings and bi-amalgamation of rings along an ideal. We also provide some weakly p-clean ring characterizations. We denote by U(R), Id(R), Nilp(R), and Pu(R) denote the set of unit elements, the set of idempotents, the set of nilpotent elements, and the set of all pure elements of R, respectively.

The paper is organized as follows: In Section 2, we introduce the concept of a weakly p-clean ring and study many properties of weakly p-clean rings. In Section 3, we study the transfer of the notion of feebly p-clean rings to the amalgamation of rings along an ideal. In Section 4, we study the transfer of the notion of feebly p-clean rings to the bi-amalgamation of rings along an ideal. Section 5 contains conclusions.

2. Weakly p-clean ring

This section covers the concept of a weakly p-clean ring, as well as some of its characteristics.

Definition 2.1. An element x ∈ R is called weakly p-clean if x = p + e or x = p – e, where p ∈ Pu(R) and e ∈ Id(R).

Definition 2.2. Let R be a ring. If each element of R is weakly p-clean, we call R a weakly p-clean ring.

Example 2.3. Every clean ring, every weakly clean ring, and every p-clean ring is a weakly p-clean ring.

Proposition 2.4. Every homomorphic image of a weakly p-clean ring is a weakly p-clean ring.

Proof. Every homomorphic image of a weakly p-clean ring is weakly p-clean since every homomorphic image of a pure element and every homomorphic image of an idempotent element are alike.

Theorem 2.5. Let $\{R_\alpha\}$ be a collection of rings. Then the direct product $R = \prod_\alpha R_\alpha$ is a weakly p-clean ring if and only if each R_α is a weakly p-clean ring.
Proof. By Proposition 2.4, R is a weakly p-clean ring, implying that each R_α is a weakly p-clean ring. Conversely, assume that each R_α is a weakly p-clean ring. Then, for each α, there exist $p_\alpha \in Pu(R_\alpha)$ and $e_\alpha \in Id(R_\alpha)$ such that $x_\alpha = p_\alpha + e_\alpha$ or $x_\alpha = p_\alpha - e_\alpha$. Let $x = (x_\alpha) \in \prod_\alpha R_\alpha$. Then $x = p + e$ or $x = p - e$, where $p = (p_\alpha) \in Pu(\prod_\alpha R_\alpha)$ and $e = (e_\alpha) \in Id(\prod_\alpha R_\alpha)$. Hence, $R = \prod_\alpha R_\alpha$ is a weakly p-clean ring.

Proposition 2.6. Every weakly clean ring is a weakly p-clean ring.
Proof. Let R be a weakly clean ring, and $x \in R$. Then $x = u + e$ or $x = u - e$, where $u \in U(R)$ and $e \in Id(R)$. It is only necessary to prove that u is a pure element. Since $u \in U(R)$, then $u = u.1$, and so u is a pure element. Thus, x is weakly p-clean. Therefore, R is a weakly p-clean ring.

The statement above is untrue in its converse form.

Example 2.7. The ring (Z, +, –) is weakly p-clean, but not weakly clean, because there are no unit elements in Z other than 1 and –1.

To prove the converse of the previous proposition, we can select a ring R such that $Id(R) = \{0, 1\}$ and has no zero divisors.

Theorem 2.8. Let R be a ring with no zero divisor, and $Id(R) = \{0, 1\}$. Then R is a weakly clean ring if and only if it is a weakly p-clean ring.
Proof. Each weakly clean ring is a weakly p-clean ring, according to Proposition 2.6. Conversely, suppose that R is a weakly p-clean ring. Let $x \in R$. Since x is weakly p-clean, there exists $p \in Pu(R)$ and $e \in Id(R)$ such that $x = p + e$ or $x = p - e$. Since $p \in Pu(R)$, there is a non-zero element $q \in R$ such that $p = pq$. Consider $q = sp$, then $p = psp$. Now, $(ps)^2 = (ps)(ps) = (psp)s = ps$, which implies that $ps \in Id(R)$, and hence, by hypothesis, either $ps = 0$ or $ps = 1$. If $ps = 0$, then $p = 0$ or $s = 0$, which is a contradiction; thus $ps = 1$. Yet, on the other hand, $(sp)^2 = (sp)(sp) = s(psp) = sp$, which implies that $sp \in Id(R)$, and hence, by hypothesis, either $sp = 0$ or $sp = 1$. If $sp = 0$, then $s = 0$ or $p = 0$, which is a contradiction; thus $sp = 1$. This implies that $p \in U(R)$. Then x is the sum of a unit and idempotent elements, and hence x is a weakly clean element. Therefore, R is a weakly clean ring.

Proposition 2.9. Every weakly r-clean ring is a weakly p-clean ring.
Proof. Let R be a weakly r-clean ring, and let $x \in R$. Then $x = r + e$ or $x = r - e$, where $r \in Reg(R)$ and $e \in Id(R)$. To prove that x is a weakly p-clean element in R, proving that x is a pure element suffices. Since $r \in Reg(R)$, then there is $s \in R$ such that $r = rsr$. Let $q = sr$,

then $q \in R$. Hence, $r = rq$. Thus, r is a pure element, which implies x is a weakly p-clean element. Therefore, R is a weakly p-clean ring.

The statement above is untrue in its converse form.

Example 2.10. The ring (Z, +, –) is weakly p-clean but not weakly r-clean because all the elements in Z are pure elements, but $x \neq \{1, -1\}$ in Z are not regular elements.

Proposition 2.11. Let R be a ring, with $x \in R$. Then x is weakly p-clean if and only if $-x$ is weakly p-clean.

Proof. Let R be a ring, and $x \in R$. Assume that x is weakly p-clean, then $x = p+e$ or $x = p - e$, where $p \in Pu(R)$ and $e \in Id(R)$. Now $-x = -(p + e) = -p - e$ or $-x = -(p - e) = -p + e$, we have to prove that $(-p) \in Pu(R)$. Since $p \in Pu(R)$, then there is $q \in R$ such that $p = pq$. Hence $-p = -(pq) = (-p)q$, thus $(-p) \in Pu(R)$. Therefore, $-x$ is weakly p-clean. Conversely, let $-x$ be weakly p-clean, then $-x = p + e$ or $-x = p - e$, where $p \in Pu(R)$ and $e \in Id(R)$. Now, $-x = -(p + e) = -p - e$ or $-x = -(p - e) = -p + e$, as an earlier proof, $(-p) \in Pu(R)$, which implies that x is a weakly p-clean.

Proposition 2.12. Let J be an ideal of a weakly p-clean ring R, then R/J is a weakly p-clean ring.

Proof. Let $x+J \in R/J$. Then $x \in R$. Since R is a weakly p-clean ring, $x = p + e$ or $x = p - e$. Hence, $x+J = p \pm e + J = p + J \pm e + J$. To prove x+J is a weakly p-clean element in R/J, we have to prove that p+J is a pure element in R/J and e+J is an idempotent element in R/J. Since $p \in Pu(R)$, there is $q \in R$ such that $p = pq$. Now, $p+J = pq+J = (p+J)(q+J)$, and so p+J is a pure element in R/J. Since $e \in Id(R)$, $e^2 = e$. Now $e+J = e^2+J = e.e+J = (e+J).(e+J) = (e+J)^2$ and so e + J is an idempotent element in R/J. Thus, x + J is weakly p-clean element in R/J. Hence, R/J is a weakly p-clean ring.

3. Weakly p-clean properties in amalgamated rings along ideals

In this section, we study the transfer of the notion of weakly p-clean rings to the amalgamation of rings along the ideal.

Proposition 3.1. Let $\phi: R \to S$ be a ring homomorphism, and J be an ideal of S. If $R \bowtie^{\phi} J$ is a weakly p-clean ring, then R and $\phi(R) + J$ are weakly p-clean rings.

Proof. Define $p_R : R \bowtie^{\phi} J \to R$ by $p_R(r, \phi(r) + k) = r$ and $p_S : R \bowtie^{\phi} J \to S$ by $p_S(r, \phi(r) + k) = \phi(r) + k$.Then $R \bowtie^{\phi} J /(\{0\} \times J) \cong R$ and $R \bowtie^{\phi} J /(\phi^{-1}(J) \times \{0\}) \cong \phi(R)+J$. According to proposition 2.4, R and $\phi(R) + J$ are weakly p-clean rings.

The converse of the above proposition is not true.

Proposition 3.2. Let $\phi: R \to S$ be a ring homomorphism, and J be an ideal of S. Assume that $(\phi(R) + J)/J$ is uniquely weakly p-clean and S is an Integral domain. Then $R \bowtie^{\phi} J$ is a weakly p-clean ring if and only if R and $\phi(R) + J$ are weakly p-clean rings.

Proof. According to proposition 3.1, $R \bowtie^{\phi} J$ is a weakly p-clean ring, which implies R and $\phi(R) + J$ are weakly p-clean rings. Conversely, assume that R and $\phi(R) + J$ are weakly p-clean rings. Since R is weakly p-clean, we can write $x = p + e$ or $x = p - e$ with $p \in Pu(R)$ and $e \in Id(R)$. Similarly, since $\phi(R) + J$ is weakly p-clean, $\phi(x) + j = \phi(p) + j_1$

+ $\phi(e) + j_2$ or $\phi(x) + j = \phi(p) + j_1 - \phi(e) + j_2$, where $\phi(p) + j_1$ and $\phi(e) + j_2$ are respectively pure and idempotent elements of $\phi(R) + J$. It is clear that $\overline{\phi(p_1)} = \overline{\phi(p_1) + j_1}$ (resp., $\overline{\phi(p)}$) and $\overline{\phi(e_1)} = \overline{\phi(e_1) + j_2}$ (resp., $\overline{\phi(e)}$) are respectively pure and idempotent elements of $(\phi(R) + J)/J$. Then, we have $\overline{\phi(x)} = \overline{\phi(p)} + \overline{\phi(e)} = \overline{\phi(p_1)} + \overline{\phi(e_1)}$ or $\overline{\phi(x)} = \overline{\phi(p)} - \overline{\phi(e)} = \overline{\phi(p_1)} - \overline{\phi(e_1)}$. Thus, $\overline{\phi(p)} = \overline{\phi(p_1)}$ and $\overline{\phi(e)} = \overline{\phi(e_1)}$ since $(\phi(R) + J)/J$ is a uniquely weakly p-clean ring. Consider $j_1', j_2' \in J$ such that $\phi(p_1) = \phi(p) + j_1'$ and $\phi(e_1) = \phi(e) + j_2'$. Then, we have $(x, \phi(x)+j) = (p, \phi(p)+j_1'+j_1)+(e, \phi(e)+j_2'+j_2)$ or $(x, \phi(x)+j) = (p, \phi(p)+j_1'+j_1) - (e, \phi(e)+j_2'+j_2)$. It is clear that $(e, \phi(e)+j_2'+j_2)$ is an idempotent element of $R \bowtie^{\phi} J$. Since $\phi(p)+j_1'+j_1$ is pure in $\phi(R)+J$, there exists an element $\phi(q_0)+j_0$ such that $\phi(p) + j_1' + j_1 = (\phi(p) + j_1' + j_1)(\phi(q_0) + j_0)$. Since $p \in Pu(R)$, there exists $q \in Pu(R)$ such that $p = pq$. Then, we have $\overline{\phi(p)\phi(q)} = \overline{\phi(p)} = \overline{\phi(p)\phi(q_0)}$. Since S is an integral domain, $\overline{\phi(q)} = \overline{\phi(q_0)}$. Then $\phi(q_0) = \phi(q) + j_0'$ and hence $\phi(q_0)+j_0 = \phi(q)+j_0'+j_0$. Therefore, $\phi(p)+j_1'+j_1 = (\phi(p)+j_1'+j_1)(\phi(q)+j_0'+j_0)$. Hence $(pq, (\phi(p)+j_1'+j_1)(\phi(q)+j_0'+j_0)) = (p, (\phi(p)+j_1'+j_1))(q, (\phi(q)+j_0'+j_0))$. Therefore, $(p, \phi(p) + j_1' + j_1)$ is a pure element in $R \bowtie^{\phi} J$ and hence $(x, \phi(x) + j)$ is a weakly p-clean element. Hence, $R \bowtie^{\phi} J$ is a weakly p-clean ring.

Remark 3.3. Let $\phi: R \to S$ be a ring homomorphism, and J be an ideal of S.

(1) If $S = J$, then $R \bowtie^{\phi} S$ is a weakly p-clean ring if and only if R and S are weakly p-clean ring since $R \bowtie^{\phi} S = R \times S$.

(2) If $\phi^{-1}(J) = 0$, then $R \bowtie^{\phi} J$ is weakly p-clean ring if and only if $\phi(A) + J$ is weakly p-clean ring, as follows from [[6], proposition 5.1(3)].

Corollary 3.4. Let R be a ring and I be an ideal such that R/I is a uniquely weakly p-clean ring. Then, $R \bowtie^{\phi} I$ is a weakly p-clean ring if and only if R is a weakly p-clean ring.

Theorem 3.5. Let $\phi: R \to S$ be a ring homomorphism and J an ideal of S such that $\phi(p) + j \in Pu(S)$ for each $p \in Pu(R)$ and $j \in J$. Then $R \bowtie^{\phi} J$ is weakly p-clean ring if and only if R is a weakly p-clean ring.

Proof. According to Proposition 3.1, $R \bowtie^{\phi} J$ is weakly p-clean, which implies R is weakly p-clean. Conversely, assume that R is weakly p-clean and $\phi(p) + j \in Pu(S)$ for each $p \in Pu(R)$ and $j \in J$. Since R is weakly p-clean, we can write $x = p + e$ or $x = p - e$ with $p \in Pu(R)$ and $e \in Id(R)$. Since $p \in Pu(R)$, then there exists $q \in Pu(R)$ such that $p = pq$. Therefore $(p, \phi(p) + j)(q, \phi(q)) = (pq, (\phi(p) + j)\phi(q)) = (pq, \phi(p)\phi(q) + j) = (pq, \phi(pq) + j) = (p, \phi(p) + j)$. Thus, $(p, \phi(p) + j)$ is pure in $R \bowtie^{\phi} J$. Also, $(e, \phi(e))$ is an idempotent element in $R \bowtie^{\phi} J$. Then, we have, $(x, \phi(x)+j) = (p, \phi(p)+j)+(e, \phi(e))$ or $(x, \phi(x)+j) = (p, \phi(p)+j) - (e, \phi(e))$ with $(p, \phi(p)+j)$ in $Pu(R \bowtie^{\phi} J)$ and $(e, \phi(e))$ in $Id(R \bowtie^{\phi} J)$. Hence $(x, \phi(x)+j)$ is weakly p-clean in $R \bowtie^{\phi} J$. Therefore, $R \bowtie^{\phi} J$ is a weakly p-clean ring.

Theorem 3.6. Let ϕ: R → S be a ring homomorphism, and J be an ideal of S. Set $\overline{R}$ = R/Nilp(R), $\overline{S}$ = S/Nilp(S). $\pi : S \to \overline{S}$, the canonical projection, and $\overline{J} = \pi(J)$. Consider a ring homomorphism $\overline{\phi} : \overline{R} \to \overline{S}$ defined by $\overline{\phi}\left(\overline{x}\right) = \overline{\phi(x)}$. Then R⋈$^{\phi}$J is weakly p-clean (resp., uniquely weakly p-clean) if and only if $\overline{R}$ ⋈$^{\phi}$ $\overline{J}$ is weakly p-clean (resp., uniquely weakly p-clean).

Proof. $\overline{\phi}$ is well defined. Consider the map χ: (R ⋈$^{\phi}$ J)/Nilp(R ⋈$^{\phi}$ J) → $\overline{R}$ ⋈$^{\phi}$ $\overline{J}$ defined by $\chi\left(\overline{(x, \phi(x) + j)}\right) = \left(\overline{x}, \overline{\varphi}\left(\overline{x}\right) + \overline{j}\right)$. To prove R⋈$^{\phi}$J is weakly p- clean (resp., uniquely weakly p-clean) if and only if $\overline{R}$ ⋈$^{\phi}$ $\overline{J}$ is weakly p-clean (resp., uniquely weakly p-clean), it is enough to prove that the above defined function χ is an isomorphism. If $\overline{(x, \phi(x) + j)} = \overline{(y, \phi(y) + j)}$, then (x − y, ϕ(x − y) + j − j′) ∈Nilp(R⋈$^{\phi}$J). Therefore, x − y ∈ Nilp(R) and j − j′ ∈ Nilp(S). Then x = y and j = j′. Hence, χ is well defined. We can easily check that χ is a ring homomorphism. Moreover, $\left(\overline{x}, \overline{\varphi}\left(\overline{x}\right) + \overline{j}\right)$ = (0, 0) implies that x ∈ Nilp(R) and j ∈ Nilp(S). Consequently, (x, ϕ(x) + j) ∈ Nilp (R⋈$^{\phi}$J). Hence, $\overline{(x, \phi(x) + j)}$ = (0, 0) and so χ is injective. Clearly, by the construction, χ is surjective and so χ is an isomorphism. Hence proved.

Proposition 3.7. Let ϕ: R → S be a ring homomorphism, and let (e) be an ideal of S generated by the idempotent element e of S. Then R⋈$^{\phi}$(e) is weakly p-clean if and only if R and ϕ(R) + (e) are weakly p-clean. In particular, if e is an element of R, then R⋈$^{\phi}$ (e) is weakly p-clean if and only if R is weakly p-clean.

Proof. According to Proposition 3.1, R⋈$^{\phi}$ (e) is weakly p-clean, which implies R and ϕ(R) + (e) are weakly p-clean. Conversely, assume that R and ϕ(R) + (e) are weakly p-clean. Let (x, ϕ(x) +se) be an element of R⋈$^{\phi}$ (e) with x ∈ R and s ∈ S. Since R is weakly p- clean, there exists a pure element p and an idempotent element v such that x = p+v or x = p − v. Furthermore, since ϕ(R) + (e) is weakly p-clean, there exists a pure element p′ and an idempotent element v′ such that ϕ(x) + se = p′ + v′ or ϕ(x) + se = p′ − v′. We have, (x, ϕ(x)+se) = (p+v, p′+v′) = (p, ϕ(p)+(p′ − ϕ(p))e) + (v, ϕ(v)+(v′ − ϕ(v))e) or (x, ϕ(x)+se) = (p−v, p′−v′) = (p, ϕ(p)+(p′ − ϕ(p))e) − (v, ϕ(v)+(v′ − ϕ(v))e). Now, [ϕ(p) + (p′ − ϕ(p))e][ϕ(q) + (q′ − ϕ(q))e] = [ϕ(p)(1 − e) + p′ e][ϕ(q)(1− e) + q′ e] = ϕ(pq)(1− e) + p′ q′ e = ϕ(p)(1− e) + p′ e = ϕ(p) + (p′ − ϕ(p))e. Also $[\phi(v)+(v' - \phi(v))e]^2 = [\phi(v)(1-e)+v'e]^2$ = ϕ(v)(1 − e)+v′e = ϕ(v)+(v′ − ϕ(v))e. Then (p, ϕ(p)+(p′ − ϕ(p))e) and (v, ϕ(v)+(v′−ϕ(v))e) are the pure and idempotent elements in R⋈$^{\phi}$(e) and hence (x, ϕ(x) + se) is a weakly p-clean element in R⋈$^{\phi}$(e). Hence, R⋈$^{\phi}$ (e) is weakly p-clean. Moreover, if R = S and ϕ = id_A, then R⋈$^{\phi}$ (e) = R⋈(e) and ϕ(R) + (e) = R. Then R is a weakly p-clean ring.

4. Weakly p-clean properties in bi-amalgamated rings along ideals

In this section, we study the transfer of the notion of weakly p-clean rings to the bi-amalgamation of rings along the ideal.

Proposition 4.1. [7] Let f: A→B and g: A→C be ring homomorphisms, J and J′ be ideals of B and C, respectively. Then we have the following canonical isomorphism:
$A\bowtie^{f,g}(J, J')/(\{0\}\times J') \cong f(A)+J$ and $A\bowtie^{f,g}(J, J')/(J\times\{0\}) \cong g(A)+J'$.

Theorem 4.2. Let $\phi : R \to S$ and $\psi : R \to T$ be two ring homomorphisms, let J and J′ be two ideals of S and T respectively, such that $\phi^{-1}(J) = \psi^{-1}(J')$. If $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring, then $\phi(R) + J$ and $\psi(R) + J'$ are weakly p-clean rings.

Proof. According to Proposition 2.4, a homomorphic image of a weakly p-clean ring is a weakly p-clean ring. Thus, by Proposition 4.1, we have the following isomorphism of rings $R\bowtie^{\phi,\psi}(J, J')/(\{0\}\times J') \cong \phi(R)+J$ and $R\bowtie^{\phi,\psi}(J, J')/(J\times\{0\}) \cong \psi(R)+J'$. Hence, $\phi(R) + J$ and $\psi(R) + J'$ are weakly p-clean rings.

Remark 4.3. Let $\phi: R\to S$ and $\psi: R \to T$ be two ring homomorphisms, and let J and J′ be two ideals of S and T respectively.

(1) If J = (0) and J′ = (0). Then $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring if and only if $\phi(R) + J$ and $\psi(R) + J'$ are weakly p-clean rings.

(2) If J = S and J′ = T. Then, if $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring, then so are S and T.

Proof. (1) According to Proposition 4.1, we have the following isomorphism of rings $R\bowtie^{\phi,\psi}(J, J')/(\{0\}\times J') \cong \phi(R)+J$ and $R\bowtie^{\phi,\psi}(J, J')/(J\times\{0\}) \cong \psi(R)+J'$. If J = (0) and J′ = (0). Then, the conclusion follows directly from the above isomorphisms.
(2) Assume that J = R and J′ = T. In this case, $\phi(R)+J = S$ and $\psi(R)+J' = T$ and so $R\bowtie^{\phi,\psi}(J, J') = S\times T$, and hence, by Theorem 4.2, S and T are weakly p-clean rings.

Theorem 4.4. Let $\phi: R \to S$ and $\psi: R \to T$ be two ring homomorphisms, and let J and J′ be two ideals of S and T, respectively, such that $\phi^{-1}(J) = \psi^{-1}(J') = I_0$. Assume that the following conditions hold:
(1) R is a weakly p-clean ring, and R/I_0 is a uniquely weakly p-clean ring.
(2) $\phi(R) + J$ and $\psi(R) + J'$ are weakly p-clean rings, and at most one of them is not a p-clean ring.
Then $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring.

Proof. Without losing generality, we assume $\phi(R) + J$ to be a weakly p-clean ring and $\psi(R) + J'$ to be a p-clean ring. Let $x \in R$ and $(j, j') \in J\times J'$, then there exist pure elements p and $\phi(p_1) + j_1$ of R and $\phi(R) + J$ respectively and idempotent elements e and $\phi(e_1) + j_2$ of R and $\phi(R) + J$ respectively such that $x = p + e$ or $x = p - e$ and $\phi(x) + j = (\phi(p_1) + j_1) + (\phi(e_1) + j_2)$ or $\phi(x) + j = (\phi(p_1) + j_1) - (\phi(e_1) + j_2)$. Therefore, $\phi(x) = \phi(p) + \phi(e)$ or $\phi(x) = \phi(p) - \phi(e)$ and $\phi(x) +j = (\phi(p_1) +j_1) +(\phi(e_1) +j_2)$ or $\phi(x) +j = (\phi(p_1) +j_1) - (\phi(e_1) +j_2)$. Then, in $(\phi(R) + J)/J$ we have: $\overline{\phi(x)}=\overline{\phi(p)}+\overline{\phi(e)}$ or $\overline{\phi(x)}=\overline{\phi(p)}-\overline{\phi(e)}$ and $\overline{\phi(x)+j}=\overline{\phi(x)}=\overline{\phi(p_1)}+\overline{\phi(e_1)}$ or $\overline{\phi(x)+j}=\overline{\phi(x)}=\overline{\phi(p_1)}-\overline{\phi(e_1)}$. Clearly, $\overline{\phi(p_1)}$ (resp., $\overline{\phi(p)}$)and $\overline{\phi(e_1)}$ (resp., $\overline{\phi(e)}$) are respectively pure and idempotent elements of $\phi(R) + J)/J$.

On the other hand, since $(\phi(R) + J)/J \cong R/I_0$ is a uniquely weakly p-clean ring, it is clear that $\overline{\phi(p_1)}=\overline{\phi(p)}$ and $\overline{\phi(e_1)}=\overline{\phi(e)}$ in $(\phi(R) + J)/J$. Therefore, there is (k_1, k_2)

$\in J \times J$ such that $\phi(p_1) = \phi(p) + k_1$ and $\phi(e_1) = \phi(e) + k_2$. Hence, $\phi(x) + j = (\phi(p) + k_1 + j_1) + (\phi(e) + k_2 + j_2)$ or $\phi(x)+j = (\phi(p)+k_1+j_1) - (\phi(e)+k_2+j_2)$. If $\phi(x)+j = (\phi(p)+k_1+j_1)+(\phi(e)+k_2 + j_2)$, then we write $\psi(x) + j' = (\psi(p_2) + j_1') + (\psi(e_2) + j_2')$, where $\psi(p_2) + j_1'$ is a pure element, and $\psi(e_2) + j_2'$ is an idempotent element of $\psi(R) + J'$ since $\psi(R) + J'$ is a p-clean ring. Thus, applying the same approach as the previous part, $\psi(x) + j' = (\psi(p) + k_1' + j_1') + (\psi(e) + k_2' + j_2')$ for some $(k_1', k_2') \in J' \times J'$ since $(\psi(R)+J')/J' \cong R/I_0$ is an uniquely weakly p-clean ring. This implies that $(\phi(x)+j, \psi(x)+j') = (\phi(p)+k_1+j_1, \psi(p)+k_1'+j_1') + (\phi(e)+k_2+j_2, \psi(e)+k_2'+j_2')$, where $(\phi(p)+k_1+j_1, \psi(p)+k_1'+j_1') = (\phi(p_1)+j_1, \psi(p_2)+j_2') \in Pu(R\bowtie^{\phi,\psi}(J, J'))$ and $(\phi(e) + k_2 + j_2, \psi(e) + k_2'+ j_2') = (\phi(e_1) + j_2, \psi(e_2) + j_2') \in Id(R\bowtie^{\phi,\psi}(J, J'))$.

In the remaining case, $\phi(x) +j = (\phi (p) +k_1 +j_1) - (\phi (e) + k_2 +j_2)$. Let $\psi(x) +j' = (\psi (p_2) +j_1') - (\psi (e_2) +j_2')$. Thus, $\psi(x)+j' = (\psi(p)+k_1' +j_1') - (\psi(e)+k_2'+j_2')$ and so $(\phi(x)+j, \psi(x)+j') = (\phi(p)+k_1+j_1, \psi(p)+k_1'+j_1') - (\phi(e)+k_2 +j_2, \psi(e)+k_2'+j_2')$. In all cases, $(\phi(x) +j, \psi(x) +j')$ is a sum or difference of pure and idempotent elements of $R\bowtie^{\phi,\psi}(J, J')$. Then $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring. Hence the proof.

Theorem 4.5. Let $\phi: R \to S$ and $\psi: R \to T$ be two ring homomorphisms, and let J and J′ be two proper ideals of S and T, respectively, such that $\phi^{-1}(J) = \psi^{-1}(J') = I_0$. Then the following statements hold:

(1) If $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring and $J\cap Id(S) = \{0\}$, then $\psi(R) + J'$ is a weakly p-clean ring and $J \subseteq Pu(S)$.

(2) If $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring and $J'\cap Id(T) =\{0\}$, then $\phi(R) + J$ is a weakly p-clean ring and $J' \subseteq Pu(T)$.

Proof. (1): By Theorem 4.2, it is enough to prove that $J \subseteq Pu(R)$ if $J \cap Id(R) = \{0\}$. Suppose that $J\cap Id(S) = \{0\}$ and $j \in J$. Without loss of generality, we may assume that $0 \neq j$. Then, there are a pure element $(\phi(p) + j_1, \phi(p) + j_1') \in Pu(R\bowtie^{\phi,\psi}(J, J'))$ and an idempotent element $(\phi(e) + j_2, \psi(e) + j_2') \in Id(R\bowtie^{\phi,\psi}(J, J'))$ such that $(j, 0) = (\phi(p) + j_1, \psi(p) + j_1') + (\phi(e) + j_2, \psi(e) + j_2')$ or $(j, 0) = (\phi(p)+j_1, \psi(p)+j_1') - (\phi(e)+j_2, \psi(e)+j_2')$. Therefore, $j = (\phi(p)+j_1) + (\phi(e)+j_2)$ or $j = (\phi(p) + j_1) - (\phi(e) + j_2)$ and $0 = (\psi(p) + j_1') + (\psi(e) + j_2')$ or $0 = (\psi(p) + j_1') - (\psi(e) + j_2')$. The fact that $(\phi(p) + j_1, \psi(p) + j_1')$ is a pure element of $R\bowtie^{\phi,\psi}(J, J')$ and $(\phi(e) + j_2, \psi(e) + j_2')$ is an idempotent element of $R\bowtie^{\phi,\psi}(J, J')$ respectively implies that $(\phi(p) +j_1, \phi(e) +j_2) \in Pu(\phi(R) +J) \times Id(\phi(R) +J)$ and $(\psi(p)+j_1', \psi(e)+j_2') \in Pu(\psi(R)+J') \times Id(\psi(R)+J')$. Moreover, since $0 = (\psi(p)+ j_1') + (\psi(e)+j_2')$ or $0 = (\psi(p)+j_1') - (\psi(e)+j_2')$, we have $(\psi(e)+j_2') = -(\psi(p)+j_1')$ or $(\psi(e) +j_2') = (\psi(p) +j_1')$. Thus, $(\psi(e) +j_2') \in Pu(\psi(R) +J') \cap Id(\psi(R) +J') = \{0, 1\}$.

Case (1): If $(\psi(e)+j_2') \in Pu(\psi (R)+J')\cap Id(\psi (R)+J') = \{0\}$, then $\psi(e)+j_2' = \psi(p) + j_1' = 0$. Then $(p,e) \in \psi^{-1}(J') \times \psi^{-1}(J') = \phi^{-1}(J) \times \phi^{-1}(J)$ and so $(\phi(p), \phi(e)) \in J\times J$. Consequently, $\phi(e) +j_2 \in J\cap Id(\phi(R)+J) \subseteq J\cap Id(S) = \{0\}$and thus $\phi(e) + j2 = 0$. Hence, $j = \phi (p) + j_1 \in Pu(\phi(R) + J) \subseteq Pu(S)$.

Case (2): If $(\psi (e) + j_2') \in Pu(\psi (R) + J') \cap Id(\psi (R) + J') = \{1\}$, then $\psi (e) + j_2'= \psi(p) + j_1'= 1$. If $\psi (e) + j_2'= 1$, we have $\psi (e) = 1 - j_2' \notin J'$. Suppose $1 - j_2' \in J'$, which is a contradiction to the hypothesis.

(2): It is easy to prove that $J' \subseteq Pu(T)$ if $J'\cap Id(T) = \{0\}$. This could be attained by using the same approach as the previous one, by replacing J for J′ and S with T.

Theorem 4.6. Let $\phi: R \to S$ and $\psi: R \to T$ be two ring homomorphisms, and let J and J′ be two ideals of S and T, respectively, such that $\phi^{-1}(J) = \psi^{-1}(J')$. Assume that $J \times J' \subseteq Pu(S) \times Pu(S)$. If R is a weakly p-clean ring, then $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring.

Proof. Let $x \in R$ and $(j, j') \in J \times J'$. Since R is a weakly p-clean ring, we have $x = p+e$ or $x = p - e$. Then $(\phi(x)+j, \psi(x)+j') = (\phi(p)+j, \psi(p)+j')+(\phi(e), \psi(e))$ or $(\phi(x) + j, \psi(x) + j') = (\phi(p) + j, \psi(p) + j') - (\phi(e), \psi(e))$. Since $J\times J' \subseteq Pu(S)\times Pu(S)$, we have $(\phi(p)+j, \psi(p)+j')$ is a pure element of $R\bowtie^{\phi,\psi}(J, J')$ and $(\phi(e), \psi(e))$ is an idempotent element of $R\bowtie^{\phi,\psi}(J, J')$ because $e \in Id(R)$. Then $(\phi(x) +j, \psi(x) +j')$ is a weakly p-clean in $R\bowtie^{\phi,\psi}(J, J')$, and hence $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring.

Theorem 4.7. Let $\phi: R \to S$ and $\psi: R \to T$ be two ring homomorphisms, and let J and J′ be two ideals of S and T, respectively, such that $\phi^{-1}(J) = \psi^{-1}(J')$. Assume that $J \times J' \subseteq Id(S) \times Id(S)$. If R is a weakly p-clean ring, then $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring.

Proof. Let $x \in R$ and $(j, j') \in J \times J'$. Since R is a weakly p-clean ring, we have $x = p + e$ or $x = p - e$. Then $(\phi(x) + j, \psi(x) + j') = (\phi(p), \psi(p)) + (\phi(e) + j, \psi(e) + j')$ or $(\phi(x) + j, \psi(x) + j') = (\phi(p), \psi(p)) - (\phi(e) + j, \psi(e) + j')$. Since $J\times J' \subseteq Id(S) \times Id(S)$, we have $(\phi(e) + j, \psi(e) + j')$ is an idempotent element of $R\bowtie^{\phi,\psi}(J, J')$ and $(\phi(p), \psi(p))$ is a pure element of $R\bowtie^{\phi,\psi}(J, J')$ because $p \in Pu(R)$. Then $(\phi(x) + j, \psi(x) + j')$ is a weakly p-clean in $R\bowtie^{\phi,\psi}(J, J')$, and hence $R\bowtie^{\phi,\psi}(J, J')$ is a weakly p-clean ring.

5. Conclusion

In this study, we introduce the notion of weakly p-clean rings and extends it to the amalgamation and bi-amalgamation of rings along an ideal. Specifically, the study deals with the conditions that must be satisfied for amalgamation and bi-amalgamation to be considered as weakly p-clean rings. We also provide additional characterizations of weakly p-clean rings. This study will assist in further investigating the properties of other ring structures, such as the A + XB [X] and A + XB [[X]] constructions.

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