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Feebly *r*-clean Ideal and Feebly *-r-clean Ideal

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Abstract. In this article, we introduce the concept of feebly r-clean ideal and feebly $*$-r-clean ideal. An ideal I of a ring R is called a feebly r-clean ideal if for every $x \in I$, there exists a regular element $r \in R$ and orthogonal idempotents e, f of R such that $x = r + e - f$. An ideal I of a ring R is called feebly $*$-r-clean ideal if for every $x \in I$, there exist a regular element $r \in R$ and two orthogonal projection p, q of R such that $r = n + p - q$. Further we discuss some interesting properties of feebly r-clean ideal, feebly $*$-r-clean ideal and their relation with feebly r-clean ring and feebly $*$-r-clean ring respectively have been discussed.

Keywords: r-Clean rings, feebly clean ring, feebly r-clean rings, power series rings.

AMS Mathematics Subject Classification (2010): 13B30, 13H05

1. Introduction and preliminaries

Throughtout this paper, all rings are assumed to be associative with identity. As defined by Nicholson [5], an element x in a ring R is clean, if there exist a unit $u \in R$ and an idempotent $e \in R$ such that $x = u + e$. R is clean ring, if each of its element is clean. H. Nitin Arora and S. kundu [1] defined feebly clean if every element $x \in R$, there exist unit $u \in R$ and there exists orthogonal idempotenets $e, f \in R$ such that $x = u + e - f$. Recall that, an element r of a ring R is a regular (Von Neumann), if there exists $y \in R$ such that $r = ryr$. Ashrafi and Nasibi [3] defined, an element x of a ring R is r-clean if each of its element is r-clean. We call a ring R is feebly r-clean if for every x in R such that $x = r + e - f$, where u is a unit in R and e, f are orthogonal elements in R. Chen and M. Chen [4] defined, an ideal I of a ring R to be clean ideal if for every $x \in I$, there exist a unit $u \in R$ and an idempotent $e \in R$ such that $x = u + e$.

In this paper we introduce the concept of feebly r-clean ideal and feebly $*$-r-clean ideal. Recall that, a ring R is $*$-ring if there exists an operation $*: R \to R$ such that $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$, for all $x, y \in R$. An element p of a $*$-ring is projection if $p^2 = p = p^*$. L.Vas [6] defined, A $*$-ring R is called a $*$-clean ring if for every element of R is the sum of a unit u and a projection. We defined an element x in a $*$-ring R is feebly $*$-clean if $x = u + p - q$ where u is a unit u in R and p, q are orthogonal projections in R and an element x in a $*$-ring R is feebly $*$-r-clean if $x = r + p - q$ where r is a regular and p, q are orthogonal projections in R. We define, an ideal I of a ring R is feebly r-clean ideal if for every $x \in I$, there exist a regualr element $r \in R$ and orthogonal idempotents $e, f \in R$ such that $x = r + e - f$ and an ideal I of a ring R is feebly $*$-r-clean ideal if for every $x \in I$, there exist a regular element $r \in R$ and orthogonal projections $p, q \in R$ such that $x = r + p - q$.

Further, Let R be a commutative ring and M be a R-module, Then the idealization of R and M is the ring $R(M)$ with underlying set $R \times M$ under coordinatewise addition given by $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and multiplication given by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ for all $m_1, m_2 \in M$ and $r_1, r_2 \in R$. Also If I is an ideal of R then for any submodule M' of M, $I(M') = \{(i, m'): i \in I, m' \in M\}$ is an ideal of $R(M)$. A Morita context denoted by (R, S, M, N, ψ, ϕ) consists of two rings R and S, two bimodules M_B^A and M_A^B, a pair of bimodule homomorphims $\psi: N \otimes M \to R$ and $\psi: M \otimes N \to S$ which satisfies the following associativity: $\psi(n \otimes m)n' = n\phi(m \otimes n_,)$ and $\phi(m \otimes n)m' = m\psi(n \otimes m_,)$, for any $m,m' \in M$ and $n, n' \in N$. These conditions ensure that the set of matrices $\begin{pmatrix} r & n \\ m & s \end{pmatrix}$, where $r \in R$, $s \in S$, $m \in M$ and $n \in N$, forms a ring denoted by T, called the ring of the context. Further we investigate the properties of feebly r-clean ideal and feebly $*$-r-clean ideal.

For a ring R, the set of regular elements, the set of units, the set of jacobson radicals, the set of idempoptents and set of projections are denoted by $Reg(R)$, $U(R)$, $J(R)$, $Id(R)$ and $P(R)$ respectively.

2. Feebly r-clean ideal

Some basic definitions and terminologies are presented here.

Definition 2.1. An ideal I of a ring R is called feebly r-clean ideal of R, if for every $x \in I$, there exist a regular $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $x = r + e - f$.

Proposition 2.2. Every homomorphic image of feebly r-clean ideal of a ring is feebly r-clean ideal.

Theorem 2.3. Let $\{R_i\}$ be a family of rings and I_i's are ideals of R_i. Then the ideal $I = \prod I_i$ of $R = \prod R_i$ is feebly r-clean ideal if and only if each I_i is feebly r-clean ideal of $\{R_i\}$.

Proof: ($\Rightarrow$) This is immediate since the homomorphic image of a regular (resp., idempotent) is a regular (resp., idempotent).

($\Leftarrow$) Suppose each I_i is feebly r-clean ideal of R_i. Let $x = (x_i) \in \prod I_i$. For each i, there exist unit $r_i \in Reg(R_i)$ and orthogonal idempotents $e_i, f_i \in Id(R_i)$ such that $x_i = r_i + e_i - f_i$. Then $x = r + e - f$ where $r = (r_i) \in Reg(\prod R_i)$ and $e = (e_i)$, $f = (f_i)$ are orthogonal idempotents in $\prod R_i$. Hence $\prod I_i$ is feebly r-clean ideal.

Proposition 2.4. Let R be a ring with no zero divisor. Then I is feebly clean ideal if and only if I is feebly r-clean ideal.

Proof: ($\Rightarrow$) Suppose I is a feebly clean ring. For $a \in R$, then there exist $u \in U(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $a = u + e - f$. Since $u \in Reg(R)$, hence I is feebly r-clean ideal.

(⇐) Let I be a feebly r-clean ideal. For $x \in I$, there exist $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $x = r + e - f$. Let $r(\neq 0) \in Reg(R)$, then there exists $y \in R$ such that $r = ryr$, which implies $r(1 - yr) = 0$, thus r is a unit, so $r \in Reg(R)$. Therefore, I is a feebly r-clean ring.

Lemma 2.5. Let R be a ring. If every proper ideal of a ring R is feebly r- clean ideal then the ring R is feebly r-clean ring.
Proof: Since every unit of a ring R is feebly r-clean, so take $r \in R \backslash U(R)$. Then the ideal $I =< r >$ is proper ideal of R. Hence r is feebly r-clean in R

Corollary 2.6. R is feebly r-clean if and only if every proper ideal of R is feebly r-clean.

Lemma 2.7. If I is S-feebly r-clean ideal of R then $J(R) \subseteq I$.
Proof: Let $r \in J(R)$, then there exist $e, f \in \mathrm{Id}(R)$ such that $r + e - f \in I$. If $f = 0$ then $x = r + e$. Also $(x - r)^2$=$x - r$, which shows $r(1 - r) \in I$. But $1 - r$ is unit. Hence $r \in I$. If $e = 0$, then $x = r + f$. Also $(x - r)^2 = x - r$, which shows $r(1 - r) \in I$. But $1 - r$ is a unit. Hence $r \in I$.

The converse of Lemma 2.6 is not true. Take $I = 3\mathbb{Z}$ in $\mathbb{Z}$, $J(\mathbb{Z})$= {0}, Also {0} ⊆ $3\mathbb{Z}$, But $I = 3\mathbb{Z}$ is not feebly r-clean ideal.

Proposition 2.8. Let I be an ideal of a commutative ring. Then I is feebly r-clean ideal of R if and only if the ideal $I[[x]]$ is feebly r-clean ideal of $R[[x]]$
Proof: (⇐) Suppose I[[x]] is feebly clean ideal of $R[[x]]$, as a homomorphic copy of $I[[x]]$, then I is a feebly clean ideal of R.

(⇒) Let I be a feebly r-clean ideal of ring R. Let $f(x) = \sum a_i x^i \in I[[x]]$, then for $a_0 \in I$, there exist a regular $r_0 \in Reg(R)$ and orthogonal idempotents $e_0, f_0 \in Id(R)$ such that $a_0 = r_0 + e_0 - f_0$. Then $f(x) = \sum a_i x^i = e_0 - f_0 + r_0 + a_1 x + a_2 x^2 + \ldots$ where $r_0 + a_1 x + a_2 x^2 + \ldots \in Reg(R[[x]])$ and e_0, $f_0 \in Id(R) \subseteq Id(R[[x]])$ with $e_0 f_0 = f_0 e_0 = 0$. Hence $I[[x]]$ is feebly r-clean ideal of $R[[x]]$.

Theorem 2.9. Let I be an ideal of a ring R containing $J(R)$ and idempotent can be lifted modulo $J(R)$, then I is feebly r-clean ideal of R if and only if $I/J(R)$ is feebly r-clean ideal of $R/J(R)$.
Proof: (⇒) Suppose I is feebly r-clean ideal of R. Let $x \in I$, then there exist a regular $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $x = r + e - f$. For $r \in Reg(R)$, then $r + J(R) \in Reg(R/J(R))$. Since e, f are orthogonal idempotents of R, then $e + J(R) \in Id(R/J(R))$ and $f + J(R) \in Id(R/J(R))$ are orthogonal idempotents of $R/J(R)$. Let $\overline{x} = x + J(R) \in I/J(R)$, then $x + J(R) = r + J(R) + e + J(R) - f + J(R)$, which implies $\overline{x} = \overline{r} + \overline{e} - \overline{f}$. Therefore, $I/J(R)$ is feebly r-clean ideal of $R/J(R)$.

(⇐) Suppose $I/J(R)$ is feebly r-clean ideal of $R/J(R)$. Let $x \in I$, then $\overline{x} = \overline{r} + \overline{e} - \overline{f}$, where $\overline{r} \in Reg(R/J(R))$ and $\overline{e}, \overline{f} \in Id(R/J(R))$ with $\overline{e}\overline{f} = \overline{f}\overline{e} = 0$. Hence, $x - r + e - f \in J(R)$. So $x = r + e - f + j$, where $j \in J(R)$. Since idempotents can be lifted modulo $J(R)$, I is feebly r-clean ideal of R.

Proposition 2.10. Let I_1 and I_2 be two feebly r-clean ideal of a ring R and either $I_1 \subseteq J(R)$ or $I_2 \subseteq J(R)$ then $I_1 + I_2$ is feebly r-clean ideal of R.
Proof: Suppose I_1 and I_2 are feebly r-clean ideal of R. Without loss of generality we assume that $I_2 \subseteq J(R)$. Take $x \in I_1 + I_1$ then $x = x_1 + x_2$ where $x_1 \in I_1$ and $x_2 \in I_2 \subseteq J(R)$. Since I_1 is feebly r-clean ideal of R, we can write $x_1 = r_1 + e_1 - f_1$, where $r_1 \in Reg(R)$ and $e_1, f_1 \in Id(R)$ with $e_1 f_1 = f_1 e_1 = 0$. So $x = r_1 + (e_1 - f_1) + r_2$, thus x is feebly r-clean element of R. Therefore, $I_1 + I_2$ is feebly r-clean ideal of R.

Proposition 2.11. Let R be a commutative ring and $R(M)$ be the idealization of R and R-module M, Then an ideal I of a ring R is a feebly r-clean ideal of R if and only if for any submodule M' of M, $I(M')$ is a feebly r-clean ideal of $R(M)$.
Proof: (⇐) Suppose I be feebly r-clean ideal of R. Consider an ideal $I(M')$ of $R(M)$ for some submodule $I(M')$ of M. Take $(x, m) \in I(M')$, then there exist a regular element $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $x = r + e - f$. Then $(x, m) = (r, m) + (e, 0) - (f, 0)$, where $(r, m) \in Reg(R(M))$ and $(e, 0), (f, 0)$ are orthogonal idempotent of $Id(R(M))$.

(⇒) Suppose $I(M')$ is a feebly r-clean ideal of $R(M)$. Take $x \in I$, then $(x, 0) \in I(M')$. Since $I(M')$ is feebly r-clean ideal, there exist a regular $(r, 0) \in Reg(R(M))$ and orthognal idempotents $(e, 0), (f, 0) \in Id(R(M))$ such that $(x, 0) = (r, 0) + (e, 0) - (f, 0)$. Therefore, $x = r + e - f$ where $r \in Reg(R)$ and e, f are orthogonal idempotents of $Id(R)$

3. Feebly ∗-clean ideal and feebly ∗-r-clean ideal

Definition 3.1. An ideal I of a $*$-ring R is called feebly $*$-clean ideal if for every $x \in I$ such that $x = u + p - q$ where $u \in U(R)$ and p, q are orthogonal projections of R.

Proposition 3.2. Homomorphic image of feebly $*$-clean ideal is feebly $*$-clean ideal.

Theorem 3.3. Let R be a ring and I_1 be an ideal containing the feebly $*$-clean ideal I, then I_1 is a feebly r-clean ideal of R if and only if I_1/I is a feebly $*$-clean ideal of R/I.
Proof: (⇒)Let I_1 is a feebly $*$-clean ideal of R, then clearly I_1/I is feebly $*$-clean ideal of R/I.

(⇐) Let I_1/I be a feebly $*$-clean ideal of R/I and $x \in I_1$, then $\overline{x} = \overline{u} + \overline{p} - \overline{q}$, where $\overline{p}, \overline{q} \in P(R/I)$ and $\overline{u} \in U(R/I)$. Since idempotents can be lifted modulo ideal, so lift $\overline{e}$ to $e \in I_1$ and $\overline{f}$ to $f \in R$. Then $x - e + f$ is a unit in I_1 modulo I. Hence $x + e - f$ is unit in I_1.
Theorem 3.4. Let $\{R_i\}$ be a family of rings and I_i's are ideals of R_i. then the ideal $I = \prod_{i=1}^{m} I_i$ of $R = \prod_{i=1}^{m} R_i$ is feebly $*$-clean ideal if and only if each I_i is feebly $*$-clean ideal of $\{R_i\}$.
Proof: (⇒) This is immediate, since the homomorphic image of a unit (resp., projection) is a unit (resp., projection).

(⇐) Suppose each I_i is feebly $*$-clean ideal of R_i. Let $x = (x_i) \in \prod I_i$. For each i there exist nilpotent $u_i \in U(R_i)$ and two orthogonal idempotents $e_i, f_i \in Id(R_i)$ such that $x_i = u_i + e_i - f_i$. Then $x = u + e - f$ where $u = (u_i) \in U(\prod R_i)$ and $e = (e_i), f = (f_i) \in Id(\prod R_i)$. Hence $\prod I_i$ is feebly $*$- clean ideal.

Theorem 3.5. If I ba an feebly ideal of a ring R. Then $M_n(I)$ is a feebly clean ideal of $M_n(R)$.

Proof: Clearly, the resul holds for n=1. Assume that result holds for $n = k-1$, $(k \geq 2)$. Suppose that $A \in M_k(I)$, write $A = \begin{pmatrix} a & t \\ s & B \end{pmatrix}$, where $r \in I$, $B \in M_{k-1}(I)$. Since I is a feebly clean ideal of R, for $r \in I$ then there exist unit $u \in U(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $r = u + e - f$. Since $B - su^{-1}t \in M_{k-1}(I)$, there exist orthogonal idempotents $E = E^2 \in M_{k-1}(R)$, $F = F^2 \in M_{k-1}(R)$ and unit $V \in GL_{k-1}(R)$ such that $B - su^{-1}t = V + E - F$. Set $E' = \begin{pmatrix} e & 0 \\ 0 & E \end{pmatrix}$, $F' = \begin{pmatrix} f & 0 \\ 0 & F \end{pmatrix}$ and $U = \begin{pmatrix} u & t \\ s & V + su^{-1}t \end{pmatrix}$. Also $E' = E'^2$, $F' = F'^2$ and

$$U \begin{pmatrix} u^{-1} + u^{-1}tV^{-1}su^{-1} & u^{-1}tV^{-1} \\ -V^{-1}su^{-1} & V^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} u^{-1} + u^{-1}tV^{-1}su^{-1} & u^{-1}tV^{-1} \\ -V^{-1}su^{-1} & V^{-1} \end{pmatrix} U$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & I_{k-1} \end{pmatrix} \in M_k(R).$$

Thus, $U \in GL_k(R)$. Clearly, $A = U + E' - F'$, where E', F' are orthogonal idempotents of $M_k(R)$ and U is a unit of $M_k(R)$. Therefore, $M_k(I)$ is feebly clean ideal of $M_k(R)$. By induction, we complete the proof.

Proposition 3.6. Let I be an ideal of a commutative ring. Then I is feebly $*$-clean ideal of R if and only if the ideal $I[[x]]$ is feebly $*$-clean ideal of $R[[x]]$

Proof: ($\Leftarrow$) Suppose I[[x]] is feebly $*$clean ideal of $R[[x]]$, as a homomorphic copy of $I[[x]]$, then I is a feebly $*$-clean ideal of R.

($\Rightarrow$) Suppose I be a feebly $*$-clean ideal of ring R. Let $f(x) = \sum a_i x^i \in I[[x]]$, then for $a_0 \in I$, there exist unit $u_0 \in U(R)$ and orthogonal projections $p_0, q_0 \in P(R)$ such that $a_0 = u_0 + p_0 - q_0$. Then $f(x) = \sum a_i x^i = p_0 - q_0 + u_0 + a_1 x + a_2 x^2 + ..$ where $u_0 + a_1 x + a_2 x^2 + .. \in U(R[[x]])$ and $p_0, q_0 \in P(R) \subseteq P(R[[x]])$ with $p_0 q_0 = p_0 q_0 = 0$. Hence $I[[x]]$ is feebly $*$-clean ideal of $R[[x]]$.

Definition 3.7. An ideal I of a $*$-ring R is called feebly $*$-r-clean ideal if for every $x \in I$ such that $x = r + p - q$ where $r \in Reg(R)$ and p, q are orthogonal projections of R.

Proposition 3.8. Homomorphic image of feebly $*$-r-clean ideal is feebly $*$-r-clean ideal.

Theorem 3.9. Let $\{R_i\}$ be a family of rings and I_i's are ideals of R_i. then the ideal $I = \prod_{i=1}^{m} I_i$ of $R = \prod_{i=1}^{m} R_i$ is feebly $*$-r-clean ideal if and only if each I_i is feebly $*$-r-clean ideal of $\{R_i\}$.

Proof: Similar to the proof of Theorem 3.4.

Proposition 3.10. Let $M =_B M_A$ be a bimodule. If $I = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ a formal triangular matrix ideal is feebly $*$-r-clean then A and B are feebly $*$-r-clean ideal.

Proof: Let $x \in A$, $y \in B$ and $m \in M$. Take $a = \begin{pmatrix} x & 0 \\ m & y \end{pmatrix} \in I$, Then $a = \begin{pmatrix} x & 0 \\ m & y \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix} + \begin{pmatrix} p_1 & 0 \\ p_2 & p_3 \end{pmatrix} - \begin{pmatrix} q_1 & 0 \\ q_2 & q_3 \end{pmatrix}$, where $\begin{pmatrix} r_1 & 0 \\ r_2 & r_3 \end{pmatrix} \in Reg(T)$ and $\begin{pmatrix} p_1 & 0 \\ p_2 & p_3 \end{pmatrix}$, $\begin{pmatrix} q_1 & 0 \\ q_2 & q_3 \end{pmatrix}$ are orthogonal idempotents of I. Clearly, p_1, q_1 are orthogonal projections in A and p_3, q_3 are orthogonal projections in B respectively. Also r_1, r_2 regular element in A and B respectively. Then $x = r_1 + p_1 - q_1$ and $y = r_3 + p_3 - q_3$. Hence A and B are both feebly $*$-r-clean ideals.

Theorem 3.11. Let $M_2(R)$ be a 2 ×2 upper triangular matrix ring over R. Then an ideal $\begin{pmatrix} I_1 & R \\ 0 & I_2 \end{pmatrix}$ of $M_2(R)$ is a feebly $*$-r-clean ideal if and only if I_1 and I_2 are feebly $*$-r-clean ideal of R.

Proof: Suppose I_1 and I_2 are feebly $*$-r-clean ideal of R. Let $A = \begin{pmatrix} i_1 & R \\ 0 & i_2 \end{pmatrix} \in \begin{pmatrix} I_1 & R \\ 0 & I_2 \end{pmatrix}$. Since I_1 is feebly $*$-r-clean ideal of R, then there exist a regular element $r_1 \in Reg(I_1)$ and orthogonal projections $p_1, q_1 \in P(R)$ such that $i_1 = r_1 + p_1 - q_1$. Since I_2 is feebly $*$-r-clean ideal of R, then there exist nilpotent $n_2 \in N(I_2)$ and orthogonal projections $p_2, q_2 \in Id(R)$ such that $i_2 = r_2 + p_2 - q_2$. Then $A = \begin{pmatrix} r_1 & r \\ 0 & r_2 \end{pmatrix} + \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} - \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$, where $\begin{pmatrix} r_1 & r \\ 0 & r_2 \end{pmatrix} \in N(M_2(R))$ and $\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$, $\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \in Id(M_2(R))$ with $\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, $\begin{pmatrix} I_1 & R \\ 0 & I_2 \end{pmatrix}$ is a feebly $*$-r-clean ideal of $M_2(R)$.

Conversely, Suppose not I_1 and I_2 are not feebly $*$-r-clean ideal of R, As I_1 is not a feebly $*$-r-clean ideal of R, then there exists $i_1 \in I_1$ such that $i_1 \neq r_1 + p_1 - q_1$, where $r_1 \in Reg(R)$ and orthogonal projections $p_1, q_1 \in P(R)$, the same argument, As I_2 is not a feebly $*$-r-clean ideal of R, then there exists $i_2 \in I_2$ such that $i_2 \neq r_2 + p_2 - q_2$, where $r_2 \in Reg(R)$ and orthogonal projections $p_2, q_2 \in P(R)$, which shows $\begin{pmatrix} i_1 & 0 \\ 0 & i_2 \end{pmatrix}$ is not a feebly $*$-r-clean element of $M_2(R)$.

3. Conclusion

In this paper, we introduce the feebly r-clean ideal, feebly $*$-r-clean ideal and investigate its properties. The future scop of this study is to investigate its properties in an amalgamated ring

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