Annals of Pure and Applied Mathematics
Vol. 20, No. 2, 2019, 57-61
SSN: 2279-087X (P), 2279-0888(online)
Published on 31 October 2019
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.634v20n2a2

Annals of
Pure and Applied
Mathematics

On Solutions of the Diophantine Equation $11^x + 23^y = z^2$ with Consecutive Positive Integers x, y

Nechemia Burshtein

117 Arlozorov Street, Tel – Aviv 6209814, Israel
Email: anb17@netvision.net.il

Received 19 October 2019; accepted 30 October 2019

Abstract. In this article, we consider the equation $11^x + 23^y = z^2$ in which x, y, z are positive integers, and x, y are also consecutives. We examine all the possibilities when x is even, odd, and when $x > y$, $x < y$. It is established that the equation has a unique solution which is exhibited.

Keywords: Diophantine equations, Sophie Germain primes

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this article, we investigate the equation $11^x + 23^y = z^2$ in which x, y, z are positive integers, and x, y are also consecutives. We establish that the equation has exactly one solution which is demonstrated.

2. The solutions of $11^x + 23^y = z^2$ with consecutives x, y

In Theorem 2.1 we determine all the possible solutions of $11^x + 23^y = z^2$ when x, y are consecutives.

Theorem 2.1. Let x, y, z be positive integers. Suppose

$$11^x + 23^y = z^2 \tag{1}$$

where x, y are consecutive integers. Let n be an integer.

(a) If $x = 2n + 2$, $y = 2n + 3$, $n \geq 0$, then $11^x + 23^y = z^2$ has no solutions.
(b) If $x = 2n + 2$, $y = 2n + 1$, $n \geq 0$, then $11^x + 23^y = z^2$ has a unique solution.
(c) If $x = 2n + 1$, $y = 2n + 2$, $n \geq 0$, then $11^x + 23^y = z^2$ has no solutions.
(d) If $x = 2n + 1$, $y = 2n$, $n \geq 1$, then $11^x + 23^y = z^2$ has no solutions.

Proof: Although similarities exist in all four cases, we shall nevertheless consider the four cases separately so that each case is self-contained.

(a) Suppose

$$11^{2n+2} + 23^{2n+3} = z^2, \qquad n \geq 0, \qquad z^2 \text{ is even.} \tag{2}$$

When $n = 0$, we obtain from (2) that $11^2 + 23^3 = 12288 \neq z^2$, and the equation has no solution since an even square z^2 does not end in the digit 8.

Let $n \geq 1$. We shall assume that for some value n, the equation has a solution and reach a contradiction.

From (2) we obtain

$$23^{2n+3} = z^2 - 11^{2n+2} = z^2 - 11^{2(n+1)} = (z - 11^{n+1})(z + 11^{n+1}).$$

Denote

$$z - 11^{n+1} = 23^G, \qquad z + 11^{n+1} = 23^H, \qquad G < H, \qquad G + H = 2n + 3,$$

where G, H are integers. Then $23^H - 23^G$ results in

$$2\cdot 11^{n+1} = 23^G(23^{H-G} - 1). \tag{3}$$

If $G > 0$, the power 23^G does not divide the left side of (3), and hence $G = 0$. When $G = 0$, then $H = 2n + 3$ implying

$$2\cdot 11^{n+1} = 23^{2n+3} - 1. \tag{4}$$

For all values $n \geq 1$, it follows from (4) that

$$2\cdot 11^{n+1} + 1 < 11\cdot 11^{n+1} + 1 = 11^{n+2} + 1 < 23^{n+2}\cdot 23^{n+1} = 23^{2n+3},$$

and hence $2\cdot 11^{n+1} \neq 23^{2n+3} - 1$.

Our assumption that for some value $n \geq 1$, the equation $11^{2n+2} + 23^{2n+3} = z^2$ has a solution is therefore false, and the equation has no solutions.

The proof of **(a)** is complete.

(b) Suppose

$$11^{2n+2} + 23^{2n+1} = z^2, \qquad n \geq 0. \tag{5}$$

When $n = 0$, we obtain from (5)

Solution 1. $11^2 + 23^1 = 12^2 = z^2$.

Let $n \geq 1$. We shall assume that for some value n, the equation has a solution and reach a contradiction.

From (5) we have

$$23^{2n+1} = z^2 - 11^{2n+2} = z^2 - 11^{2(n+1)} = (z - 11^{n+1})(z + 11^{n+1}).$$

Denote

$$z - 11^{n+1} = 23^K, \qquad z + 11^{n+1} = 23^L, \qquad K < L, \qquad K + L = 2n + 1,$$

where K, L are integers. Then $23^L - 23^K$ yields

$$2\cdot 11^{n+1} = 23^K(23^{L-K} - 1). \tag{6}$$

If $K > 0$, the power 23^K does not divide the left side of (6), and therefore $K = 0$. When $K = 0$, then $L = 2n + 1$, and (6) results in

$$2\cdot 11^{n+1} = 23^{2n+1} - 1. \qquad (7)$$

For all values $n \geq 1$, (7) yields

$$2\cdot 11^{n+1} + 1 < 11\cdot 11^{n+1} + 1 = 11^{n+2} + 1 < 23^{n+2}\cdot 23^{n-1} = 23^{2n+1}$$

Implying that $2\cdot 11^{n+1} \neq 23^{2n+1} - 1$.

Our assumption that for some value $n \geq 1$, the equation $11^{2n+2} + 23^{2n+1} = z^2$ has a solution is therefore false, and the equation has no solutions.

This concludes part **(b)**.

(c) Suppose

$$11^{2n+1} + 23^{2n+2} = z^2, \qquad n \geq 0. \qquad (8)$$

When $n = 0$, we have from (8) that $11^1 + 23^2 = 540 \neq z^2$, and the equation has no solution.

Let $n \geq 1$. We shall assume that for some value n, the equation has a solution and reach a contradiction.

From (8) we obtain

$$11^{2n+1} = z^2 - 23^{2n+2} = z^2 - 23^{2(n+1)} = (z - 23^{n+1})(z + 23^{n+1}).$$

Denote

$$z - 23^{n+1} = 11^A, \qquad z + 23^{n+1} = 11^B, \qquad A < B, \qquad A + B = 2n + 1,$$

where A, B are integers. Then $11^B - 11^A$ implies

$$2\cdot 23^{n+1} = 11^A(11^{B-A} - 1). \qquad (9)$$

If $A > 0$, the power 11^A does not divide the left side of (9). Hence $A = 0$ and accordingly $B = 2n + 1$. Then (9) yields

$$2\cdot 23^{n+1} = 11^{2n+1} - 1. \qquad (10)$$

For all values $n \geq 1$, the power 11^{2n+1} has a last digit equal to 1. Therefore $11^{2n+1} - 1$ ends in the digit 0. Hence $11^{2n+1} - 1$ is a multiple of 5. Since the left side of (10) is not a multiple of 5, it follows that $2\cdot 23^{n+1} \neq 11^{2n+1} - 1$.

Our assumption that for some value $n \geq 1$, the equation $11^{2n+1} + 23^{2n+2} = z^2$ has a solution is therefore false, and the equation has no solutions.

The proof of **(c)** is complete.

(d) Suppose

$$11^{2n+1} + 23^{2n} = z^2, \qquad n \geq 1, \qquad z^2 \text{ is even}. \qquad (11)$$

For all values $n \geq 1$, the power 11^{2n+1} has a last digit which is equal to 1. When

$n = 2a$ $(a \geq 1)$, the power 23^{4a} ends in the digit 1. Therefore, $11^{4a+1} + 23^{4a} = z^2$ ends in the digit 2. Since an even square z^2 does not end in the digit 2, it follows that $n \neq 2a$.

Suppose that $n = 2\beta + 1$ $(\beta \geq 0)$. Then (11) yields

$$11^{4\beta+3} + 23^{4\beta+2} = z^2. \tag{12}$$

We shall assume that for some value β (12) has a solution, and reach a contradiction.

From (12) we have

$$11^{4\beta+3} = z^2 - 23^{4\beta+2} = z^2 - 23^{2(2\beta+1)} = (z - 23^{2\beta+1})(z + 23^{2\beta+1}).$$

Denote

$$z - 23^{2\beta+1} = 11^C, \qquad z + 23^{2\beta+1} = 11^D, \qquad C < D, \qquad C + D = 4\beta + 3,$$

where C, D are integers. Then $11^D - 11^C$ yields

$$2 \cdot 23^{2\beta+1} = 11^C(11^{D-C} - 1). \tag{13}$$

For all values $C > 0$, the right side of (13) is a multiple of 11^C, whereas the left side of (13) is not. Therefore $C = 0$. Then $D = 4\beta + 3$, and (13) yields

$$2 \cdot 23^{2\beta+1} = 11^{4\beta+3} - 1. \tag{14}$$

For all values β, the last digit of $11^{4\beta+3}$ is equal to 1, and hence $11^{4\beta+3} - 1$ has a last digit equal to 0. Therefore $11^{4\beta+3} - 1$ is a multiple of 5. Since the left side of (14) is not a multiple of 5, it follows that $2 \cdot 23^{2\beta+1} \neq 11^{4\beta+3} - 1$.

Our assumption that for some odd value n the equation $11^{2n+1} + 23^{2n} = z^2$ has a solution is therefore false, and the equation has no solutions.

This concludes the proof of **(d)**, and of Theorem 2.1. □

3. The equation $11^x + 23^y = z^2$ and the Sophie Germain primes

First we shall provide the reader with few basic facts on a particular class of primes, namely the Sophie Germain primes.

Sophie Germain (1776 – 1831) was a French lady mathematician, physicist and philosopher. Among other fields, she was also known in Number Theory for her work on Fermat's Last Theorem, and for the Sophie Germain prime numbers.

A Sophie Germain prime is a prime number $\boldsymbol{P}$ such that $\boldsymbol{2P+1}$ is also prime. The first few Sophie Germain primes are $\boldsymbol{P} = 2, 3, 5, 11, 23, 29, \ldots$.

Numerous articles have been written on the Sophie Germain primes, for example [3, 4, 5, 6, 7]. It is conjectured that there are an infinite number of Sophie Germain pairs $\boldsymbol{(P, 2P+1)}$. The conjecture is extremely difficult to prove. From [8] we also cite: As of 29.2.2016, the largest known proven Sophie Germain prime $\boldsymbol{P}$ is

$$\boldsymbol{P} = 2618163402417 \cdot 2^{1290000} - 1. \tag{15}$$

In this article, we have considered the equation $p^x + q^y = z^2$ in (1) in which the pair of primes p, q satisfies $(p, q) = (11, 23) = \boldsymbol{(P, 2P+1)}$. We have established that the equation $11^x + 23^y = z^2$ has exactly one solution with consecutives $x = 2$, $y = 1$, where

$z = 12$. For each prime $\boldsymbol{P}$, there exists an equation $\boldsymbol{P}^x + (\mathbf{2}\boldsymbol{P} + \mathbf{1})^y = z^2$ which has a unique solution with consecutives $x = 2$, $y = 1$ and $z = \boldsymbol{P} + 1$. The resulting equality $\boldsymbol{P}^2 + (\mathbf{2}\boldsymbol{P} + \mathbf{1})^1 = (\boldsymbol{P} + 1)^2$ is an identity valid for each and every Sophie Germain prime $\boldsymbol{P}$.

So far, quite a large but finite number of primes $\boldsymbol{P}$ exist, therefore the same number of the above identities also exists. If we denote by $\boldsymbol{P_L}$ the largest known prime $\boldsymbol{P}$ demonstrated in (15), then the largest known equation $\boldsymbol{P}_{\boldsymbol{L}}^{x} + (\mathbf{2}\boldsymbol{P_L} + \mathbf{1})^y = z^2$ has a unique solution with consecutives $x = 2$, $y = 1$ and $z^2 = (\boldsymbol{P_L} + 1)^2$.

Remark 3.1. When the pair $(p, q) = (\boldsymbol{P}, \mathbf{2}\boldsymbol{P} + \mathbf{1})$ is replaced by the pair $(A, 2A + 1)$ where A is a positive integer, then the above identity with consecutives $x = 2$, $y = 1$ is the identity $A^2 + (2A + 1)^1 = (A + 1)^2$ valid for each and every integer $A \geq 1$. The values A and $(2A + 1)$ range over primes and composites accordingly.

4. Conclusion

In this article, we have shown that $11^x + 23^y = z^2$ has exactly one solution when x, y are consecutives, namely $11^2 + 23^1 = 12^2$ (**Solution 1**).

In this discussion, we have utilized our technique which uses the last digit of powers such as 11^u and 23^v. This technique is rather very elementary, but quite efficient in solving equations of this kind.

REFERENCES

1. N. Burshtein, On solutions to the diophantine equations $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$ with positive integers x, y, z, *Annals of Pure and Applied Mathematics*, 19 (1) (2019) 75–77.
2. N. Burshtein, On the diophantine equation $p^x + q^y = z^2$, *Annals of Pure and Applied Mathematics*,13 (2) (2017) 229 – 233.
3. N. Burshtein, On Sophie Germain primes, *Journal for Algebra and Number Theory Academia*, 6 (1) (2016) 37–41.
4. C. K. Caldwell, An amazing prime heuristic, 2000. //*www.utm.edu./caldweii*//
   H. Dubner, Large Sophie Germain primes, *Math. of Comput.*, 65 (1996) 393 – 396.
5. Karl-Heinz Indlekofer and A. Járai, Largest known Twin primes and Sophie Germain primes, *Math. of Comput.*, 68 (1999) 1317–1324.
6. F. Liu, On the Sophie Germain prime conjecture, *WSEAS Transaction on Mathematics,*10 (2011) 421–430.
7. Primegrid, *www.primegrid.com*/