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Symmetrical Trapezoidal Normal Distribution and its Parameter Estimation

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Abstract. The trapezoidal rule is applied for construction a new distribution based on normal distribution, is called trapezoidal normal distribution under the same conditions. Furthermore, we illustrate the parameter estimation of the trapezoidal normal distribution using standard differential evolution (DE) algorithm for the sample sizes equal to 10 to 100. The results show the K-S statistic of the trapezoidal normal distribution is less than the K-S statistic of the normal distribution of each sample size data.

Keywords: Trapezoidal normal distribution, Differential evolution algorithm.

AMS Mathematics Subject Classification (2019): 62F10

1. Introduction

Normal distribution is one of the most important continuous probability distribution which is widely used in statistical analysis and other fields such as biology, economics, engineering, and so forth. The random variable X is called a normal distribution with the real location parameter μ and the positive scale parameter σ, denoted by $X \sim N\left(\mu,\sigma^2\right)$ and its probability density function is of the form

$$\phi_X\left(x;\mu,\sigma\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},x\in\mathbb{R}.$$

The cumulative distribution function is of the form

$$F_X\left(x;\mu,\sigma\right)=\int_{-\infty}^{x}\phi_X\left(x;\mu,\sigma\right)dx.$$

Obviously, it is too complicated to directly calculate the value of the cumulative distribution function. However, there are many articles that estimate $F_X\left(x;\mu,\sigma\right)$ using different methods see e.g. [1-4]. In addition, an application of normal distribution was used to join exponential distribution and to determine the inefficiency level of the firms due to the effect of exogenous factors [5]. Furthermore, a few researchers studied some properties of generalized trapezoidal distribution and its applications which triangular and uniform distributions are special cases of the trapezoidal family. Van Dorp and Kotz [6] allows for a non-linear behavior at its tails and a linear incline (or decline) in the central stage and another. Van Dorp et al. [7] developed two novel elicitation procedures for the parameters of a special case of the generalized trapezoidal family by restricting ourselves to a uniform (horizontal) central stage in accordance with the central stage of the original trapezoidal distribution.

In this paper, we shall introduce a new distribution based on normal distribution which is called trapezoidal normal distribution by applying trapezoidal rule.

2. Definitions and main theorems

We propose definition, theorem and corollary that relate to the probability density function $g(x;\mathrm{P},\phi)$ in case of $2n+1$ nodes and $2n+2$ nodes.

Definition 1. Let $k \geq 3$ be an integer, $\ell, d > 0$ be real numbers. A set $\mathrm{P} = \{x_1, x_2, \ldots, x_k\}$ is called a (k, x_1, ℓ, d) node if $x_1 < x_2 < \cdots < x_k$ such that $x_2 - x_1 = x_k - x_{k-1} = \ell$ and $x_{i+1} - x_i = d$, $i = 2, 3, \ldots, k-2$.

The node P is denoted by the notation $Node(k, x_1, \ell, d)$.

Remark 1. Let $k \geq 3$ be an integer, $\ell, d > 0$ be real numbers.

1.1 If P be a (k, x_1, ℓ, d) node, $x_2 = x_1 + \ell$, $x_k = x_1 + 2\ell + (k-3)d$ and $x_{i+1} = x_1 + \ell + (i-1)d$, where $i = 2, 3, \ldots, k-2$.

1.2 Let f be a function and I be the identity function on P, we denote

$$\begin{aligned} \Delta f(x_k) &= f(x_k) - f(x_{k-1}) \\ \Delta x_k &= \Delta I(x_k) = x_k - x_{k-1} \\ \overline{f(x_k)} &= \frac{1}{2}\left(f(x_k) + f(x_{k-1})\right) \\ \overline{x_k} &= \overline{I(x_k)} = \frac{1}{2}(x_k + x_{k-1}). \end{aligned}$$

Remark 2. Let $k \geq 3, n \geq 1$ be integers, $\ell, d > 0$ be real numbers and C be a center node.

2.1 $CeNode(2n+2, C, \ell, d) = Node\left(2n+2, C - \left(n - \frac{1}{2}\right)d - \ell, \ell, d\right)$ is called even node.

2.2 $CoNode(2n+1, C, \ell, d) = Node(2n+1, C - (n-1)d - \ell, \ell, d)$ is called odd node.

Theorem 3. Let $\mu \in \mathbb{R}, \sigma > 0, d > 0, n \in \mathbb{N}, \phi$ be a probability density function of $N\left(\mu, \sigma^2\right)$, and $\mathrm{P} = \left\{x_1, x_2, \ldots, x_{2n+1}\right\}$ be $CoNode\left(2n+1, \mu, \ell, d\right)$. If $0 < \sum_{k=3}^{2n} \overline{\phi\left(x_k\right)}\, d < 1$, $\ell = \dfrac{1}{\phi\left(x_2\right)}\left[1 - \sum_{k=3}^{2n} \overline{\phi\left(x_k\right)}\, d\right]$ and

$$g\left(x; \mathrm{P}, \phi\right) = \begin{cases} \phi\left(x_2\right) + \dfrac{\phi\left(x_2\right)}{\ell}\left(x - x_2\right) & ; x_1 < x \le x_2, \\ \phi\left(x_k\right) + \dfrac{\Delta\phi\left(x_k\right)}{d}\left(x - x_k\right) & ; x_{k-1} < x \le x_k, k = 3, 4, \ldots, 2n \\ \phi\left(x_{2n}\right) - \dfrac{\phi\left(x_{2n}\right)}{\ell}\left(x - x_{2n}\right) & ; x_{2n} < x \le x_{2n+1}, \end{cases} \tag{1}$$

when the other of $g\left(x; \mathrm{P}, \phi\right)$ is zero. Then $g\left(x; \mathrm{P}, \phi\right)$ is a probability density function.

Proof: We will show that $g\left(x; \mathrm{P}, \phi\right) \ge 0$ for all $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} g\left(x; \mathrm{P}, \phi\right)\, dx = 1$.
Since $\phi(x)$ is increasing on $\left[x_1, x_{n+1}\right]$ and $\phi(x)$ is decreasing on $\left[x_{n+1}, x_{2n+1}\right]$, $g\left(x; \mathrm{P}, \phi\right) \ge 0$ for all x. Next, we consider $\int_{-\infty}^{\infty} g\left(x; \mathrm{P}, \phi\right)\, dx = \int_{x_1}^{x_{2n+1}} g\left(x; \mathrm{P}, \phi\right)\, dx$ so we have

$$\begin{aligned} \int_{-\infty}^{\infty} g\left(x; \mathrm{P}, \phi\right)\, dx &= \int_{x_1}^{x_2} g\left(x; \mathrm{P}, \phi\right)\, dx + \sum_{k=3}^{2n} \int_{x_{k-1}}^{x_k} g\left(x; \mathrm{P}, \phi\right)\, dx + \int_{x_{2n}}^{x_{2n+1}} g\left(x; \mathrm{P}, \phi\right)\, dx \\ &= \frac{\phi\left(x_2\right)\ell}{2} + \sum_{k=3}^{2n} \overline{\phi\left(x_k\right)}\, d + \frac{\phi\left(x_{2n}\right)\ell}{2} \\ &= \frac{\phi\left(x_2\right)\ell}{2} + \sum_{k=3}^{2n} \overline{\phi\left(x_k\right)}\, d + \frac{\phi\left(x_2\right)\ell}{2} \\ &= \phi\left(x_2\right)\ell + \sum_{k=3}^{2n} \overline{\phi\left(x_k\right)}\, d. \end{aligned}$$

Since $\ell = \dfrac{1}{\phi\left(x_2\right)}\left[1 - \sum_{k=3}^{2n} \overline{\phi\left(x_k\right)}\, d\right]$ is an appropriate value,

$$\int_{x_1}^{x_{2n+1}} g\left(x; \mathrm{P}, \phi\right)\, dx = 1 = \int_{-\infty}^{\infty} g\left(x; \mathrm{P}, \phi\right)\, dx$$

Therefore, $g\left(x; \mathrm{P}, \phi\right)$ is a probability density function. □

Theorem 4. Let $\mu \in \mathbb{R}, \sigma > 0, d > 0, n \in \mathbb{N}, \phi$ be a probability density function of $N(\mu, \sigma^2)$, and $\mathrm{P} = \{x_1, x_2, \ldots, x_{2n+2}\}$ be $CeNode(2n+2, \mu, \ell, d)$. If $0 < \sum_{k=3}^{2n+1} \overline{\phi(x_k)}\, d < 1$,

$\ell = \frac{1}{\phi(x_2)}\left[1 - \sum_{k=3}^{2n+1} \overline{\phi(x_k)}\, d\right]$ and

$$g(x; \mathrm{P}, \phi) = \begin{cases} \phi(x_2) + \dfrac{\phi(x_2)}{\ell}(x - x_2) & ; x_1 < x \le x_2, \\ \phi(x_k) + \dfrac{\Delta\phi(x_k)}{d}(x - x_k) & ; x_{k-1} < x \le x_k, k = 3, 4, \ldots, 2n+1 \\ \phi(x_{2n+1}) - \dfrac{\phi(x_{2n+1})}{\ell}(x - x_{2n+1}) & ; x_{2n+1} < x \le x_{2n+2}, \end{cases} \tag{2}$$

when the other of $g(x; \mathrm{P}, \phi)$ is zero. Then $g(x; \mathrm{P}, \phi)$ is a probability density function.

Proof: The proof of Theorem 4 follows from Theorem 3. □

Definition 2. The random variable X has trapezoidal-normal distribution for parameters (μ, σ) such that $\mu \in \mathbb{R}$ and $\sigma > 0$ corresponding to $Conode(2n+1, \mu, \ell, d)$ or $Cenode(2n+2, \mu, \ell, d)$ denoted by $X \sim TN(\mu, \sigma^2, n, \mathrm{P})$, if there exists a probability density function satisfying equations (1) and (2), respectively.

Definition 3. A probability distribution is said to be symmetric if and only if there exists a value x_0 such that $f(x_0 - \delta) = f(x_0 + \delta)$ for all real number δ, where f is the probability density function if the distribution is continuous or the probability mass function if the distribution is discrete.

Theorem 5. Let $g(x; \mathrm{P}, \phi)$ be the probability density function of trapezoidal-normal distribution for parameters (μ, σ),

$$g(\mu - x; \mathrm{P}, \phi) = g(\mu + x; \mathrm{P}, \phi)$$

for all real number x, $g(x; \mathrm{P}, \phi)$ is called symmetric.

Proof: Let $x \in [x_k, x_{k+1}]$ for all $x \in \mathbb{R}$. Let x lie in a segment that has the first point, $(x_k, \phi(x_k))$ and the end point, $(x_{k+1}, \phi(x_{k+1}))$. We have a straight line equation in the form

$$y_k(x) = \frac{\Delta\phi(x_{k+1})}{\Delta x_{k+1}}(x - x_k) + \phi(x_k) .$$

Since $\phi(x)$ is symmetry,

$$\phi(\mu - x) = \phi(\mu + x) .$$

We will show that $g(\mu + x; \mathrm{P}, \phi) = g(\mu - x; \mathrm{P}, \phi)$. Here, we consider only 5 cases of odd node and similarity to consider of even node:

Case 1. For $\mu + x < x_1$, we get $\mu - x > x_{2n+1}$.

Since $\mu - x > 2\mu - x_1 = x_{2n+1}$, we have $g(\mu + x; \mathrm{P}, \phi) = 0$ and $g(\mu - x; \mathrm{P}, \phi) = 0$.

Thus, $g(\mu + x; \mathrm{P}, \phi) = g(\mu - x; \mathrm{P}, \phi)$.

Case 2. For $\mu + x > x_{2n+1}$, by similar process as in Case 1, we get

$$g(\mu + x; \mathrm{P}, \phi) = g(\mu - (-x); \mathrm{P}, \phi) = g(\mu - x; \mathrm{P}, \phi)$$

Case 3. For $x_1 < \mu + x < x_2$, we get $(\mu + x, g(\mu + x; \mathrm{P}, \phi))$ lie in a segment that has a straight line equation

$$y_1(x) = \frac{\Delta g(x_2; \mathrm{P}, \phi)}{\Delta x_2} + g(x_1; \mathrm{P}, \phi) = \frac{\phi(x_2)}{\ell}(x - x_1)$$

Thus,

$$g(\mu + x; \mathrm{P}, \phi) = y_1(\mu + x) = \frac{\phi(x_2)}{\ell}(\mu + x - x_1) = \frac{\phi(x_2)}{\ell}(\mu + x - x_2) + \phi(x_2)$$

Conversely, we consider for $x_{2n} < \mu - x < x_{2n+1}$, we get $(\mu - x, g(\mu - x; \mathrm{P}, \phi))$ lie in segment that has straight line equation

$$y_{2n}(x) = \frac{\Delta g(x_{2n+1}; \mathrm{P}, \phi)}{\Delta x_{2n+1}}(x + x_{2n}) + g(x_{2n}; \mathrm{P}, \phi) = -\frac{\phi(x_{2n})}{\ell}(x - x_{2n}) + \phi(x_{2n})$$

Thus,

$$\begin{aligned} g(\mu - x; \mathrm{P}, \phi) &= y_{2n}(\mu - x) \\ &= -\frac{\phi(x_{2n})}{\ell}(\mu - x - x_{2n}) + \phi(x_{2n}) \\ &= \frac{\phi(x_2)}{\ell}(\mu + x - x_2) + \phi(x_2) . \end{aligned}$$

Therefore, $g(\mu + x; \mathrm{P}, \phi) = g(\mu - x; \mathrm{P}, \phi)$.

Case 4. For $x_{2n} < \mu + x < x_{2n+1}$, by similar process as in Case 3, we have $(\mu + x, g(\mu + x; \mathrm{P}, \phi))$ lie in a segment that has a straight line equation

$$y_{2n}(x) = \frac{\Delta g(x_{2n+1}; \mathrm{P}, \phi)}{\Delta x_{2n+1}}(x - x_{2n}) + g(x_{2n}; \mathrm{P}, \phi) = -\frac{\phi(x_{2n})}{\ell}(x - x_{2n}) + \phi(x_{2n})$$

thus
$$
\begin{aligned}
g(\mu+x;\mathrm{P},\phi) &= y_{2n}(\mu+x)\\
&= -\frac{\phi(x_{2n})}{\ell}(\mu+x-x_{2n})+\phi(x_{2n})\\
&= \frac{\phi(x_2)}{\ell}(\mu-x+x_2)+\phi(x_2).
\end{aligned}
$$

Conversely, we consider for $x_1<\mu-x<x_2$, we have $(\mu+x, g(\mu+x;\mathrm{P},\phi))$ lie in segment that has straight line equation

$$y_1(x)=\frac{\Delta g(x_2;\mathrm{P},\phi)}{\Delta x_2}(x-x_1)+g(x_1;\mathrm{P},\phi)=\frac{\phi(x_2)}{\ell}(x-x_1),$$

thus

$$g(\mu-x;\mathrm{P},\phi)=y_1(\mu-x)=\frac{\phi(x_2)}{\ell}(\mu-x-x_1)=\frac{\phi(x_2)}{\ell}(\mu-x-x_2)+\phi(x_2).$$

Therefore, $g(\mu-x;\mathrm{P},\phi)=g(\mu+x;\mathrm{P},\phi)$.

Case 5. For $x_k<\mu+x<x_{k+1}$ and $k=2,3,\ldots,2n-1,$ we have $(\mu+x, g(\mu+x;\mathrm{P},\phi))$ lie in segments that have straight line equations

$$y_k(x)=\frac{\Delta g(x_{k+1};\mathrm{P},\phi)}{\Delta x_{k+1}}(x-x_k)+g(x_k;\mathrm{P},\phi)=\frac{\Delta\phi(x_{k+1})}{\Delta x_{k+1}}(x-x_k)+\phi(x_k),$$

thus
$$g(\mu+x;\mathrm{P},\phi) = y_k(\mu+x)=\frac{\Delta\phi(x_{k+1})}{\Delta x_{k+1}}(\mu+x-x_k)+\phi(x_k).$$

Consider
$$
\begin{aligned}
g(\mu-x;\mathrm{P},\phi) &= y_k(\mu-x)\\
&= \frac{\Delta\phi(x_{2n-k+4})}{\Delta x_{2n-k+4}}(\mu-x-x_{2n-k+3})+\phi(x_{2n-k+3})\\
&= \frac{\Delta\phi(x_{k+1})}{\Delta x_{k+1}}(\mu+x-x_k)+\phi(x_k).
\end{aligned}
$$

Therefore, $g(\mu+x;\mathrm{P},\phi)=g(\mu-x;\mathrm{P},\phi)$.

The proof is complete. □

Theorem 6. Let $\mu\in\mathbb{R},\sigma>0,d>0,n\in\mathbb{N},\phi$ be a probability density function of $N(\mu,\sigma^2)$, and $\mathrm{P}=\{x_1,x_2,\ldots,x_{2n+1}\}$ be $CoNode(2n+1,\mu,\ell,d)$. If $0<\sum_{k=3}^{2n}\overline{\phi(x_k)}\,d<1$, $\ell=\frac{1}{\phi(x_2)}\left[1-\sum_{k=3}^{2n}\overline{\phi(x_k)}\,d\right]$, $N(t)=\max\{k:x_k\le t\}$, $G(x;\mathrm{P},\phi)=\int_{-\infty}^{x}g(s;\mathrm{P},\phi)\,ds$, and

$$G(x;\mathrm{P},\phi)=\begin{cases} 0 & ;x\le x_1, \\ \phi(x_2)(x-x_1)+\dfrac{\phi(x_2)}{\ell}\left(\dfrac{x^2-x_1^2}{2}-x_2(x-x_1)\right) & ;x_1<x\le x_2, \\ \displaystyle\sum_{k=3}^{N(x)-1}\overline{\phi(x_k)}d+\phi\left(x_{N(x)}\right)\left(x-x_{N(x)}\right) & \\ +\dfrac{\Delta\phi\left(x_{N(x)}\right)}{d}\left(\dfrac{x^2-x_{N(x)-1}^2}{2}-x_{N(x)}\left(x-x_{N(x)-1}\right)\right) & ;x_2<x\le x_{2n}, \\ \phi(x_{2n})(x-x_{2n})-\dfrac{\phi(x_{2n})}{\ell}\left(\dfrac{x^2-x_{2n}^2}{2}-x_{2n}(x-x_{2n})\right) & ;x_{2n}<x<x_{2n+1,} \\ 1 & ;x\ge x_{2n+1}, \end{cases}$$

then $G(x;\mathrm{P},\phi)$ is called the cumulative distribution function.

Proof: By using Theorem 3, we get obviously

$$\int_{-\infty}^{x_1} g(x;\mathrm{P},\phi)\,dx=0=\int_{x_{2n+1}}^{\infty} g(x;\mathrm{P},\phi)\,dx .$$

We consider in case of $x_1<x<x_2$,

$$\begin{aligned} G(x;(x_1,x_2),\phi) &=\int_{-\infty}^{x} g(s;(x_1,x_2),\phi)\,ds \\ &=\int_{-\infty}^{x}\left[\phi(x_2)+\frac{\phi(x_2)}{\ell}(s-x_2)\right]ds \\ &=\phi(x_2)(x-x_1)+\frac{\phi(x_2)}{2\ell}\left(\frac{x^2-x_1^2}{2}-x_2(x-x_1)\right). \end{aligned}$$

For $x=x_2$, we get

$$G(x;(x_1,x_2),\phi)=\frac{\phi(x_2)}{2}\ell.$$

Next, we consider in case of $x_{k-1}<x<x_k$, $k=3,4,\ldots,2n$ and since $N(t)=\max\{k:x_k\le t\}$, so we get

$$\begin{aligned} G(x;(x_2,x_{2n}),\phi) &=\int_{x_2}^{x} g(s;(x_2,x_{2n}),\phi)\,ds \\ &=\int_{x_2}^{x}\left[\phi(x_k)+\frac{\phi(x_k)}{d}(s-x_k)\right]ds \\ &=\sum_{k=3}^{N(x)-1}\overline{\phi(x_k)}d+\phi\left(x_{N(x)}\right)\left(x-x_{N(x)}\right)+ \end{aligned}$$

$$\frac{\Delta\phi\left(x_{N(x)}\right)}{d}\left(\frac{x^2-x_{N(x)-1}^2}{2}-x_{N(x)}\left(x-x_{N(x)-1}\right)\right).$$

For all $x=x_k$, $k=3,4,\ldots,2n$, we get

$$G\left(x;\left(x_2,x_{2n}\right),\phi\right)=\sum_{k=3}^{2n}\overline{\phi\left(x_k\right)}\,d\ .$$

Next, we consider in case of $x_{2n}<x<x_{2n+1}$,

$$\begin{aligned}G\left(x;\left(x_{2n},x_{2n+1}\right),\phi\right) &= \int_{x_{2n}}^{x} g\left(s;\left(x_{2n},x_{2n+1}\right),\phi\right)\,ds \\ &= \int_{x_2}^{x}\left[\phi\left(x_{2n}\right)-\frac{\phi\left(x_{2n}\right)}{\ell}\left(s-x_{2n}\right)\right]ds \\ &= \phi\left(x_{2n}\right)\left(x-x_{2n}\right)-\frac{\phi\left(x_{2n}\right)}{\ell}\left(\frac{x^2-x_{2n}^2}{2}-x_{2n}\left(x-x_{2n}\right)\right).\end{aligned}$$

For all $x=x_{2n+1}$, we get

$$G\left(x;\left(x_{2n},x_{2n+1}\right),\phi\right)=\frac{\phi\left(x_{2n}\right)}{2}\ell\ .$$

We combine all cases of $x\in\left(x_1,x_{2n+1}\right)$, and since $\ell=\dfrac{1}{\phi\left(x_2\right)}\left[1-\sum_{k=3}^{2n}\overline{\phi\left(x_k\right)}\,d\ \right]$,

consequently,

$$\int_{-\infty}^{\infty} g\left(x;\mathrm{P},\phi\right)\,dx=\frac{\phi\left(x_2\right)}{2}\ell+\sum_{k=3}^{2n}\overline{\phi\left(x_k\right)}\,d+\frac{\phi\left(x_{2n}\right)}{2}\ell=1.$$

The proof is complete. □

Theorem 7. Let $\mu\in\mathbb{R},\sigma>0,d>0,n\in\mathbb{N},\phi$ be a probability density function of $N\left(\mu,\sigma^2\right)$, and $\mathrm{P}=\left\{x_1,x_2,\ldots,x_{2n+2}\right\}$ be $CeNode\left(2n+2,\mu,\ell,d\right)$. If $0<\sum_{k=3}^{2n+1}\overline{\phi\left(x_k\right)}\,d<1$, $\ell=\dfrac{1}{\phi\left(x_2\right)}\left[1-\sum_{k=3}^{2n+1}\overline{\phi\left(x_k\right)}\,d\ \right]$, $N\left(t\right)=\max\left\{k:x_k\le t\right\}$, $G\left(x;\mathrm{P},\phi\right)=\int_{-\infty}^{x} g\left(s;\mathrm{P},\phi\right)\,ds$, and

$$G(x;\mathrm{P},\phi)$$

$$=\begin{cases} 0 & ;x \le x_1, \\ \phi(x_2)(x-x_1)+\dfrac{\phi(x_2)}{\ell}\left(\dfrac{x^2-x_1^2}{2}-x_2(x-x_1)\right) & ;x_1 < x \le x_2, \\ \displaystyle\sum_{k=3}^{N(x)-1}\overline{\phi(x_k)}d+\phi\left(x_{N(x)}\right)\left(x-x_{N(x)}\right) & \\ \qquad +\dfrac{\Delta\phi\left(x_{N(x)}\right)}{d}\left(\dfrac{x^2-x_{N(x)-1}^2}{2}-x_{N(x)}\left(x-x_{N(x)-1}\right)\right) & ;x_2 < x \le x_{2n+1}, \\ \phi(x_{2n+1})(x-x_{2n+1})-\dfrac{\phi(x_{2n+1})}{\ell}\left(\dfrac{x^2-x_{2n+1}^2}{2}-x_{2n+1}(x-x_{2n+1})\right) & ;x_{2n+1} < x < x_{2n+2}, \\ 1 & ;x \ge x_{2n+2}. \end{cases}$$

Then $G(x;\mathrm{P},\phi)$ is called the cumulative distribution function.

Proof: The proof of Theorem 7 follows from Theorem 6. □

Let $X_1 \sim TN(\mu,\sigma^2)$ be a random variable of trapezoidal-normal distribution, we found that $\Pr(|X_1| \le M)=1$, for some $M>0$, i.e., X_1 is a bounded random variable. At the same time, if $X_2 \sim N(\mu,\sigma^2)$ is a random variable of normal distribution, we found that $0<\Pr(|X_2|>M)<1$ for all $M>0$.

We illustrate the probability density function of trapezoidal-normal distribution, $TN(0,1)$ for $n=6$ as shown in Figure 1. We can see that the probability density function of different d has also different kurtosis.

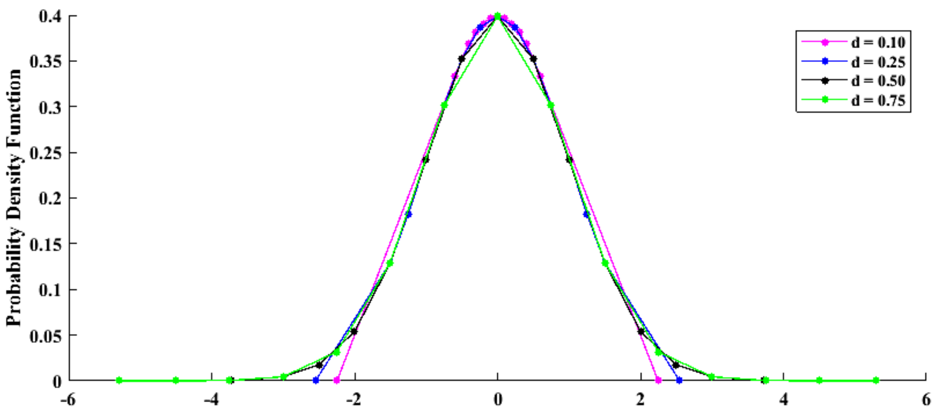
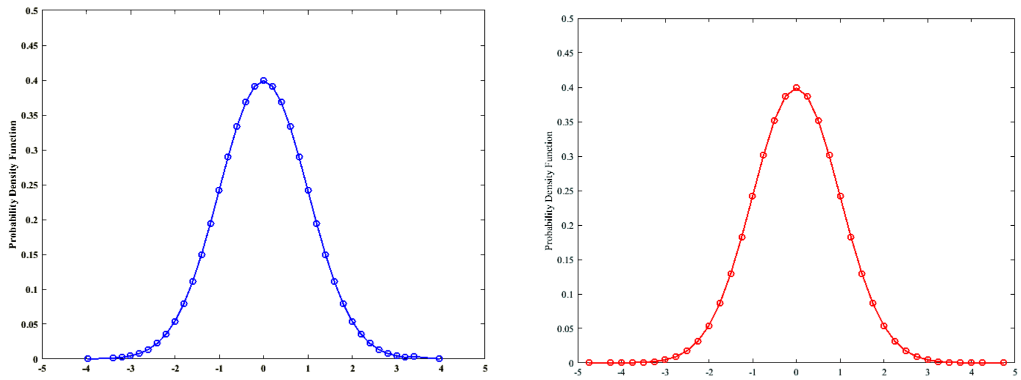


Figure 1: The probability density function of standard trapezoidal normal distribution for $n=6$ and difference of d

Next, we illustrate the probability density function of standard trapezoidal normal distribution, $TN(0,1)$ for $n=17$, $d=0.20$ (a) and $d=0.25$ (b) as shown in Figure 2.



(a) (b)

Figure 2: The probability density function of standard trapezoidal normal distribution

3. Parameters estimation method

In this section, we propose a parameter estimation by using standard differential evolution (DE) algorithm. In part of experiment, we random a sample size of standard normal distribution varying from 10 to 100 data (increase by 10) in each test. After that, we approximate parameters using DE algorithm by setting $F=2,\ CR=0.8$. The results of these experiments are shown in Table 1 and Table 2. The last figure shows the K-S statistic in some cases.

Table 1: The results of parameter estimation of trapezoidal-normal distribution for $F=2,\ CR=0.8$ by using DE algorithm

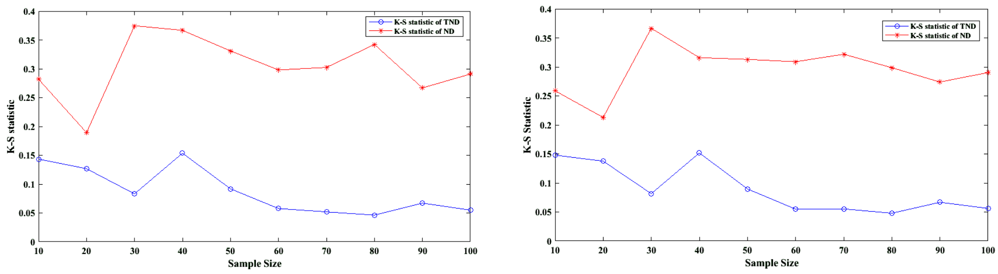
Iterations	μ	σ^2	d	n	K-S Statistic	μ	σ^2	d	n	K-S Statistic
		Sample Size=100						**Sample Size=90**		
150000	0.07930	0.97158	0.01983	93	0.06720	-0.06336	0.95725	0.04793	39	0.06654
100000	0.07930	0.97158	0.05004	31	0.07550	-0.06336	0.95725	0.10366	8	0.05708
50000	0.07930	0.97158	0.01121	19	0.05616	-0.06336	0.95725	0.04055	105	0.06678
10000	0.07930	0.97158	0.00550	24	0.05497	-0.06336	0.95725	0.02598	20	0.06704
		Sample Size=80						**Sample Size=70**		
150000	0.03236	0.99864	0.04220	38	0.05476	0.04277	1.00452	0.01640	60	0.04798
100000	0.03236	0.99864	0.01021	58	0.04944	0.04277	1.00452	0.01995	50	0.04873
50000	0.03236	0.99864	0.00884	95	0.04787	0.04277	1.00452	0.03388	64	0.05539
10000	0.03236	0.99864	0.00867	41	0.04633	0.04277	1.00452	0.02641	58	0.05187
		Sample Size=60						**Sample Size=50**		
150000	-0.16482	0.93273	0.00601	44	0.05744	0.12342	1.02578	0.00737	8	0.08961
100000	-0.16482	0.93273	0.07477	6	0.05603	0.12342	1.02578	0.04383	21	0.10222
50000	-0.16482	0.93273	0.09480	7	0.05493	0.12342	1.02578	0.00958	25	0.08973
10000	-0.16482	0.93273	0.02900	104	0.05770	0.12342	1.02578	0.02541	15	0.09188

		Sample Size=40						**Sample Size=30**		
150000	-0.44127	1.02128	0.04353	70	0.15545	-0.02563	1.09464	0.09477	70	0.08398
100000	-0.44127	1.02128	0.02074	59	0.14520	-0.02563	1.09464	0.07912	48	0.08474
50000	-0.44127	1.02128	0.03457	74	0.15179	-0.02563	1.09464	0.10440	17	0.08161
10000	-0.44127	1.02128	0.03963	63	0.15370	-0.02563	1.09464	0.08341	102	0.08314
		Sample Size=20						**Sample Size=10**		
150000	0.11367	0.75473	0.01080	39	0.12893	0.10983	0.76474	0.01446	14	0.148247
100000	0.11367	0.75473	0.00847	96	0.12899	0.10983	0.76474	0.01798	10	0.149168
50000	0.11367	0.75473	0.02413	105	0.13761	0.10983	0.76474	0.01105	52	0.147956
10000	0.11367	0.75473	0.00508	33	0.12673	0.10983	0.76474	0.00470	14	0.143389

Table 2: The results of parameter estimation normal distribution for $F = 2, CR = 0.8$ by using DE algorithm

Iterations	μ	σ^2	K-S Statistic	μ	σ^2	K-S Statistic
		Sample Size=100			**Sample Size=90**	
150000	0.22599	0.97165	0.27227	0.00481	1.17743	0.39991
100000	0.27899	0.98001	0.27849	0.18436	1.00489	0.30379
50000	0.11309	0.99637	0.29032	0.09503	0.97272	0.27392
10000	0.39801	0.98497	0.29094	-0.00359	0.96379	0.26686
		Sample Size=80			**Sample Size=70**	
150000	0.19754	1.02924	0.31309	0.23089	1.07715	0.34367
100000	0.11769	1.02289	0.30878	0.13398	1.02191	0.30813
50000	0.03704	1.00809	0.29861	0.23396	1.04248	0.32184
10000	0.10523	1.07488	0.34222	0.05478	1.01354	0.30240
		Sample Size=60			**Sample Size=50**	
150000	0.15024	0.94613	0.28936	0.06817	1.08629	0.34906
100000	0.05286	0.93960	0.25572	0.23876	1.02708	0.31154
50000	0.15628	0.97604	0.30893	0.23782	1.02915	0.31292
10000	0.06606	0.99806	0.29799	0.28146	1.05709	0.33109
		Sample Size=40			**Sample Size=30**	
150000	-0.29152	1.04021	0.35826	0.29018	1.15093	0.39023
100000	-0.41545	1.10236	0.35858	0.34995	1.12379	0.37673
50000	-0.39285	1.02440	0.31591	0.02528	1.11353	0.36658
10000	-0.40140	1.11653	0.36672	0.11752	1.12652	0.37464
		Sample Size=20			**Sample Size=10**	
150000	0.24826	0.76134	0.22952	0.07190	0.90608	0.24649
100000	0.14258	0.80298	0.21002	0.11593	0.76654	0.18650
50000	0.12668	0.81678	0.21297	0.19940	0.81705	0.25883
10000	0.06381	0.82125	0.18932	0.13269	0.92577	0.28197

The results are shown in Table 1 and Table 2 show that the K-S statistic of the trapezoidal normal distribution is less than the K-S statistic of normal distribution of each sample size data. The Figure 3 shows some K-S statistic of the trapezoidal normal distribution compared with the normal distribution.



(c) (d)

Figure 3: K-S statistic of normal distribution (ND) and trapezoidal normal distribution (TND) for 10,000 iterations (c) and 50,000 iterations (d).

4. Conclusions and recommendations

This research presents the approach to construct the trapezoidal normal distribution. We obtain that its properties are similar to the normal distribution, but trapezoidal normal distribution has bounded, i.e., $\Pr(|X| < M) = 1$ for some $M > 0$ such that it explains some situations better than the normal distribution.

In the future, we will compare the normal distribution with the trapezoidal-normal distribution through parameter estimator. In order to show that it is credible and useful for data analysis as well as the normal distribution. Moreover, we will study trapezoidal-normal distribution and its application for a real world situation. In addition, we should compare our algorithm with other algorithms such as Particle Swarm Optimization algorithm [8] or fuzzy number to reverse order trapezoidal [9].

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