A Note on Star Critical Ramsey \((C_n, K_6)\) Numbers for Large \(n\)

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Received 2 June 2019; accepted 22 June 2019

Abstract. The study of Ramsey theory was initiated by the paper on a problem of formal logic written by Ramsey. Let \(K_n\) denote the complete graph on \(n\) vertices. For any red/blue colouring of \(K_n\), let \(H_R\) and \(H_B\) denote the red and blue subgraphs of \(K_n\) respectively so that \(K_n = H_R \oplus H_B\). Let \(H, G\) be simple graphs. If there exists a red copy of \(H\) in \(H_R\) or a blue copy of \(G\) in \(H_B\), we say that \(K_n \rightarrow (H, G)\). One branch of Ramsey theory, deals with the exact determination of Ramsey number, \(r(H, G)\), defined as the smallest positive integer \(n\) such that \(K_n \rightarrow (H, G)\). For small size graphs \(H\) and \(G\), Ramsey number \(r(H, G)\) has been studied extensively in the last five decades. In the special case \(H = G = K_n\), the exact determination of \(r(K_n, K_n)\), swiftly expedientiously from the apparent \(r(K_3, K_3) = 6\), to the unmanageable \(r(K_5, K_5)\). Currently, the best known lower and upper bounds for \(r(K_5, K_5)\) are 43 and 48 ([7,8]).

A closely related recent development in this area of study is the determination of Star critical Ramsey number \(r^*(H, G)\) defined as the largest integer \(k\) such that \(K_{r(G,H)-1} \cup K_{1,k} \rightarrow (H, G)\). In this work, we find \(r^*(C_n, K_6)\) when \(n \geq 10\).

Keywords: Graph theory, Ramsey theory, Ramsey critical graphs

AMS Mathematics Subject Classification (2010): 05C55, 05C38, 05D10

1. Introduction

After the introduction of Ramsey Theory, many authors have investigated how to extend the initial result of Erdős and Szekeres, namely \(r(K_3, K_3) = 6\). However, even after a century, very little has been done despite the use of ‘state of the art’ computers. This sentiment has been eloquently expressed by the great mathematician Paul Erdős as exemplified by the following quotation (see 1990 Scientific American article by Ronald Graham and Joel Spencer).

‘Suppose aliens invade the earth and threaten to obliterate it in a year’s time unless human beings can find the Ramsey number for red five and blue five \(r(K_5, K_5)\). We could marshal the world’s best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red
six and blue six $r(K_6, K_6)$, however, we would have no choice but to launch a pre-emptive attack’.

Paul Erdős

(1990 Scientific American article by Ronald Graham & Joel Spencer)

One other branch of mathematics that emerged subsequently is the determination of $r(C_n, K_m)$. In fact it has been conjectured by Bondy and Erdős in 1978 that $r(C_n, K_m) = (n - 1)(m - 1) + 1$ for all $n \geq m \geq 3$, with the exception of $n = m = 3$ ([1]). However, according to the current survey papers on Ramsey theory, the best-known cycle-complete Ramsey numbers related to $r(C_n, K_m)$ are currently known for $n \geq 4m + 2$, $m \geq 3$ (see [8]).

The fact that, Star critical Ramsey numbers [3], give a more detailed insight into understanding the behaviour of corresponding Ramsey numbers has triggered an interest on finding $r^*(C_n, K_m)$. Working along this line, in this paper, we concentrate on the special case of $r^*(C_n, K_6)$ for large values of $n$ ($n \geq 15$).

2. Terminology

The total number of edges of a complete graph on $n$ vertices will be denoted by $|E(K_n)|$ so that $|E(K_n)| = n(n - 1)/2$. The independence number, which is defined as the size of the largest independent set of a graph will be denoted by $\alpha(G)$. Given a vertex $v \in V(G)$ of a graph $G$, the neighbourhood of $v$ in $G$ that is defined as the set of vertices adjacent to $v$ in $G$ will be denoted by $\Gamma(v)$. The degree of a vertex $v$ in $G$, denoted by $d(v)$ is defined as the cardinality of $|\Gamma(v)|$, where $\Gamma(v)$ represents the set of vertices adjacent to $v$. The minimum degree of a graph $G = (V, E)$ denoted by $\delta(G)$, is defined as the minimum of the degrees of the vertices of $G$. Similarly, the maximum degree of a graph $G = (V, E)$ denoted by $\Delta(G)$, is defined as the maximum of the degrees of the vertices of $G$. Furthermore, we say that a graph is a $r$ regular graph (or simply regular), if degree of each vertex of graph $G$ is equal to $r$. (i.e., $\delta(G) = \Delta(G) = r$). Given a graph $G$ and a non-empty subset $S$ of $V(G)$, the induced subgraph of $S$ in $G$ denoted by $G[S]$ is defined as the subgraph obtained by deleting all the vertices of the complement of $S$ from $G$. For a graph $G$ and two disjoint subgraphs $H$ and $H'$ of $G$, we denote the set of edges between $H$ and $H'$ by $E(H, H')$.

3. Methodology

We use three lemmas to calculate $r^*(C_n, K_6)$ when $n \geq 15$. The Lemmas we use are already proven results and relevant references are indicated in parentheses.

Lemma 1. ([5], Lemma 2) A $C_n$ - free graph $G$ of order $N$ with independent number less than or equal to $m$ has minimal degree greater than or equal to $N - r(C_n, K_m)$.

Lemma 2. ([4], Lemma 8) A $C_n$ - free graph (where $n \geq 15$) of order $5(n - 1)$ with no independent set of size 6 contains a $5K_{n-1}$.

Lemma 3. ([2], Lemma 5)

Suppose $G$ contains the cycle $(u_1, u_2, \ldots u_{n-1}, u_1)$ of length $n - 1$ but no cycle of length $n$. Let $Y = V(G) \setminus \{u_1, u_2, \ldots u_{n-1}\}$. Then,
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(a) No vertex \(x \in Y\) is adjacent to two consecutive vertices on the cycle.
(b) If \(x \in Y\) is adjacent to \(u_i\) and \(u_j\) then \(u_{i+1}u_{j+1} \notin E(G)\).
(c) If \(x \in Y\) is adjacent to \(u_i\) and \(u_j\) then no vertex \(x \in Y\) is adjacent to both \(u_{i+1}\) and \(u_{j+2}\).
(d) Suppose \(\alpha(G) \leq m - 1\) where \(m \leq (n + 2)/2\) and \(\{x_1,x_2,\ldots,x_{m-1}\} \subseteq Y\) is an \((m - 1)\) - element independent set. Then, no member of this set is adjacent to \(m - 2\) or more vertices on the cycle (We have taken the liberty of making a slight correction to the inequality \(m \leq (n + 2)/2\) of the original [2], Lemma 5(d)).

4. Main result
Here we provide a theorem with proof to find \(r^*(C_n,K_6)\).

**Theorem 1.** If \(n \geq 15\) then \(r^*(C_n,K_6) = 4n - 2\).

**Proof:** By Lemma 1 and Lemma 2 (see [6,9]), \(r(C_n,K_6) = 5(n - 1) + 1 = 5n - 4\). To find a lower bound for \(r^*(C_n,K_6)\), colour the graph \(K_{5n-4}\setminus K_{1,n-2}\) such that the red graph consists of a \(5K_{n-1} \cup K_{1,1}\) as illustrated in Figure 3, where a thick dotted line represents blue edges bundle (containing \((n - 1)^2\) or \((n - 1)\) number of edges) except and a thin solid line represents a single red edge. Hence, \(K_{5n-4}\setminus K_{1,n-2} \in (C_n,K_m)\). Therefore, \(r^*(C_n,K_6) \geq 4n - 2\). To show that \(r^*(C_n,K_6) \leq 4n - 2\), assume that there exists a red\(C_n^\circ\) free red/blue coloring of a graph \(G = K_{5n-4}\setminus K_{1,n-3}\) that contains no blue \(K_6\). Let \(H\) be the graph obtained by deleting the vertex of degree \(4n - 2\) (say \(v\)) from \(G\) (i.e., \(H = G\setminus v\)). Since \(G\) is a graph on \(5n - 4\) vertices, \(H\) is a graph on \(5(n - 1)\) vertices containing no red \(C_n\) or a blue \(K_6\). Therefore, by Lemma 4, \(H\) contains a red \(5K_{n-1}\). Let \(V_1, V_2, \ldots, V_5\) denote the vertex sets of these five \(K_{n-1}\) components. In order to avoid a red \(C_n\), \(v\) can be adjacent in red to at most one red neighbor in each of the 5 sets \(V_1, V_2, \ldots, V_5\). We exercise the prerogative of assuming that \(V_1, V_2, \ldots, V_5\) represent in the descending order with respect to the number of vertices connected to \(v\) (disregarding the color). For each \(i \in \{1,2,\ldots,m-1\}\), let \(U_i\) represents the number of vertices of \(V_i\) not connected to \(v\).

![Figure 1: A red \(C_n\) free colouring of \(K_{5n-4}\setminus K_{1,n-2}\) with no blue \(K_6\)](image-url)
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Claim 2. $|U_5| \leq n - 3$ and $|U_4| \leq n - 7$.

Proof of Claim 2. $|U_1| + \cdots + |U_5| \leq n - 3$ together with $0 \leq |U_1| \leq |U_2| \leq |U_3| \leq |U_4| \leq |U_5| \leq n - 3$ gives $|U_5| \leq n - 3$.

To prove $|U_4| \leq n - 7$, on the contrary assume that $|U_4| \geq n - 6$. Then,

$$n - 3 = |U_1| + |U_2| + |U_3| + |U_4| + |U_5| \geq |U_4 + U_5| \geq 2|U_4| \geq 2(n - 6).$$

That is $n \leq 9$, a contradiction as $n \geq 10$. Hence, the claim.

Next continue with the proof of Theorem 1. By Claim 2, as $|U_5| \leq n - 3$, we get $V_3$ has at least 2 vertices adjacent to $v$. In order to avoid a red $C_n$, at least one of these two vertices have to be adjacent to $v$ in blue. That is, there exists $v_5 \in V_5$ such that $(v, v_5)$ is blue. Next, $V_4$ has at least 6 vertices connected to $v$. Out of these 6 vertices, at most two vertices can be connected to $v$ in red. Therefore, there exists a vertex $v_4 \in V_4$ adjacent to $\{v_5, v\}$, in blue. That is $\{v_4, v_5, v\}$, induces a blue $K_4$. Next, we get that there is a vertex $v_3 \in V_3$ such that $\{v_3, \ldots, v_5, v\}$ induces a blue $K_4$.

Arguing in the same manner, we conclude that there exists $v_i \in V_1 (1 \leq i \leq 5)$ such that $v_1$ is adjacent in blue to all the vertices of $\{v_2, \ldots, v_5, v\}$, where $\{v_2, \ldots, v_5, v\}$, induces a blue $K_5$. Therefore, $\{v_1, v_2, \ldots, v_5, v\}$, will induce a blue $K_5$, a contradiction.

5. Discussion

First of all, the authors would like to thank the reviewers for their useful comments.

From Theorem 1 obtained for $m = 6$, we envisage the following generalisation.

Result. A $C_n$-free graph, where $n \geq (m - 3)(m - 1)$ with $m \geq 7$, of order $(n - 1)(m - 1)$ with no independent set of order $m$ contains a copy of $(m - 1)K_{n-1}$. Moreover, for $n \geq (m - 3)(m - 1)$ with $m \geq 7$, $r^c(C_m, K_n) = (m - 2)(n - 1) + 2$.

REFERENCES


