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Some Properties of Semi-Prime Ideals in Lattices

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Abstract. Recently Yehuda Rav has given the concept of Semi-prime ideals in a general lattice by generalizing the notion of 0-distributive lattices. In this paper we have included several characterizations of Semi-prime ideals. Here we give a simpler proof of a prime Separation theorem in a general lattice by using semi-prime ideals. We also studied different properties of minimal prime ideals containing a semi prime ideal in proving some interesting results. By defining a p-algebra L relative to a principal semi prime ideal J, we have proved that when L is 1-distributive, then L is a relative S-algebra if and only if every prime ideal containing J contains a unique minimal prime ideal containing J, which is also equivalent to the condition that for any $x, y \in L$, $x \wedge y \in J$ implies $x^{+} \vee y^{+} = 1$. Finally, we have proved that every relative S-algebra is a relative D- algebra if L is 1-distributive and modular with respect to J.

Keywords. Semi-prime ideal, 0-distributive lattice, Annihilator ideal, Maximal filter, Minimal prime ideal.

AMS Mathematics Subject Classifications (2010): 06A12, 06A99, 06B10

1. Introduction

In generalizing the notion of pseudo complemented lattice, J. C. Varlet [7] introduced the notion of 0-distributive lattices. [1] has given several characterizations of these lattices. On the other hand, [4] have studied them in meet semi lattices. A lattice L with 0 is called a *0-distributive* lattice if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. Of course every distributive lattice with 0 is 0-distributive. 0-distributive lattice L can be characterized by the fact that the set of all elements disjoint to $a \in L$ forms an ideal. So every pseudo complemented lattice is 0-distributive. Similarly, a lattice L with 1 is called a *1-distributive* lattice if for all $a,b,c \in L$, $a \vee b = 1 = a \vee c$ imply $a \vee (b \wedge c) = 1$.

Y. Rav [5] has generalized this concept and gave the definition of semi prime ideals in a lattice. For a non-empty subset I of L, I is called a *down set* if for $a \in I$ and $x \leq a$ imply $x \in I$. Moreover I is an *ideal* if $a \vee b \in I$ for all $a,b \in I$. Similarly, F is called a *filter* of L if for $a,b \in F$, $a \wedge b \in F$ and for $a \in F$ and $x \geq a$ imply $x \in F$. F is called a *maximal filter* if for any filter $M \supseteq F$ implies either $M = F$ or $M = L$. A proper ideal (down set) I is called a *prime ideal (down set)* if for $a,b \in L$, $a \wedge b \in I$ imply either $a \in I$ *or* $b \in I$. A prime ideal P is called a *minimal prime ideal* if it does not contain any other prime ideal. Similarly, a proper filter Q is called a *prime filter* if $a \vee b \in Q$ $(a,b \in L)$ implies either $a \in Q$ or $b \in Q$. It is very easy to check that F is a filter of L if and only if $L - F$ is a prime down set. Moreover, F is a prime filter if and only if $L - F$ is a prime ideal.

An ideal I of a lattice L is called a *semi prime ideal* if for all $x, y, z \in L$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$. Thus, for a lattice L with 0, L is called *0-distributive* if and only if $(0]$ is a semi prime ideal. In a distributive lattice L, every ideal is a semi prime ideal. Moreover, every prime ideal is semi prime. In a pentagonal lattice $\{0, a, b, c, 1; a < b\}$, $(0]$ is semi prime but not prime. Here $(b]$ and $(c]$ are prime, but $(a]$ is not even semi prime. Again in
$M_3 = \{0, a, b, c, 1; a \wedge b = b \wedge c = a \wedge c = 0; a \vee b = a \vee c = b \vee c = 1\}$
$(0], (a], (b], (c]$ are not semi prime.

Following lemmas are due to [2].

Lemma 1. *Every filter disjoint from an ideal I is contained in a maximal filter disjoint from I.*

Lemma 2. *Let I be an ideal of a lattice L. A filter M disjoint from I is a maximal filter disjoint from I if and only if for all $a \notin M$, there exists $b \in M$ such that $a \wedge b \in I$*

Let L be a lattice with 0. For $A \subseteq L$, We define
$A^{\perp} = \{x \in L : x \wedge a = 0 \text{ for all } a \in A\}$. $A^{\perp}$ is always a down set of L,but not necessarily an ideal.

Following result is an improvement of [2, Theorem 6].

Theorem 3. *Let L be a 0-distributive lattice. Then for $A \subseteq L$, $A^{\perp}$ is a semi-prime ideal.*

Proof: We have already mentioned that $A^{\perp}$ is a down set of L. Let $x, y \in A^{\perp}$. Then $x \wedge a = 0 = y \wedge a$ for all $a \in L$. Hence $a \wedge (x \vee y) = 0$ for all $a \in A$. This implies $x \vee y \in A^{\perp}$ and so $A^{\perp}$ is an ideal.

Now let $x \wedge y \in A^{\perp}$ and $x \wedge z \in A^{\perp}$. Then $x \wedge y \wedge a = 0 = x \wedge z \wedge a$ for all $a \in A$. This implies $x \wedge a \wedge (y \vee z) = 0$ for all $a \in L$ as L is 0-distributive. Hence $x \wedge (y \vee z) \in A^{\perp}$ and so $A^{\perp}$ is a semi prime ideal. ●

Let $A \subseteq L$ and J be an ideal of L. We define
$A^{\perp_J} = \{x \in L : x \wedge a \in J \text{ for all } a \in A\}$. This is clearly a down set containing J. In presence of distributivity, this is an ideal. $A^{\perp_J}$ is called an annihilator of A relative to J.

Following Theorem due to [2] gives some nice characterizations of semi prime ideals.

Theorem 4. *Let L be a lattice and J be an ideal of L. The following conditions are equivalent.*
(i) J is semi prime.
(ii) $\{a\}^{\perp_J} = \{x \in L : x \wedge a \in J\}$ is a semi prime ideal containing J.
(iii) $A^{\perp_J} = \{x \in L : x \wedge a \in J \text{ for all } a \in A\}$ is a semi prime ideal containing J.
(iv) Every maximal filter disjoint from J is prime. ●

Following prime Separation Theorem due to [5] was proved by using Glevinko congruence. But we have a simpler proof.

Theorem 5. Let J *be an ideal of a lattice L. Then the following conditions are equivalent:*
(i) J is semi prime

(ii) For any proper filter F disjoint to J there is a prime filter Q containing F *such that $Q \cap J = \phi$.*

Proof. (i)=>(ii). Since $F \cap J = \phi$, so by Lemma 1, there exists a maximal filter $Q \supseteq F$ such that $Q \cap J = \phi$. Then by Theorem 4, Q is prime.
(ii)=>(i). Let F be a maximal filter disjoint to J. Then by (ii) there exists a prime filter $Q \supseteq F$ such that $Q \cap J = \phi$. Since F is maximal, so $Q = F$. This implies F is prime and so by theorem 4, J must be semi prime. ●

Now we give another characterization of semi-prime ideals with the help of Prime Separation Theorem using annihilator ideals. This is also an improvement of the result
[4, Theorem 8].

Theorem 6. *Let J be an ideal in a lattice L. J is semi- prime if and only if for all filter F disjoint to $A^{\perp_J}$ $(A \subseteq L)$, there is a prime filter containing F disjoint to $A^{\perp_J}$.*

Proof. Suppose J is semi prime and F is a filter with $F \cap A^{\perp J} = \phi$. Then by Theorem 4, $A^{\perp J}$ is a semi prime ideal. Now by Lemma 1, we can find a maximal filter Q containing F and disjoint to $A^{\perp_J}$. Then by Theorem 4 (iv), Q is prime.
Conversely, let $x \wedge y \in J, x \wedge z \in J$. If $x \wedge (y \vee z) \notin J$, then $y \vee z \notin \{x\}^{\perp J}$. Thus $[y \vee z) \cap \{x\}^{\perp_J} = \varphi$. So there exists a prime filter Q containing $[y \vee z)$ and disjoint from $\{x\}^{\perp_J}$. As $y, z \in \{x\}^{\perp_J}$, *so* $y, z \notin Q$. Thus $y \vee z \notin Q$, as Q is prime. This implies, $[y \vee z)$ □ Q a contradiction. Hence $x \wedge (y \vee z) \in J$, and so J is semi-prime. ●

Let J be any ideal of a lattice L and P be a prime ideal containing J. We define $J(P) = \{x \in L : x \wedge y \in J \text{ for some } y \in L - P\}$. Since P is a prime ideal, so $L - P$ is a prime filter. Clearly $J(P)$ is a down set containing J and $J(P) \subseteq P$.

Lemma 7. *If P is a prime ideal of a lattice L containing any semi prime ideal J, then $J(P)$ is a semi prime ideal.*

Proof. Let $a, b \in J(P)$. Then $a \wedge v \in J$ and $b \wedge s \in J$ for some $v, s \in L - P$. Thus $a \wedge v \wedge s \in J$ and $b \wedge v \wedge s \in J$. Since J is semiprime, so

$v \wedge s \wedge (a \vee b) \in J$ and $v \wedge s \in L - P$ as it is a filter. Hence $a \vee b \in J(P)$ and so $J(P)$ is an ideal as it is a down set.
Now suppose $x \wedge y, x \wedge z \in J(P)$.
Then $x \wedge y \wedge v, x \wedge z \wedge s \in J$ for some $v, s \in L - P$. Then by the semi primeness of J, $[(x \wedge y) \vee (x \wedge z)] \wedge v \wedge s \in J$ where $v \wedge s \in L - P$.
This implies $(x \wedge y) \vee (x \wedge z) \in J(P)$, we have $J(P)$ is semi prime. ●

Lemma 8. *Let J be a semi prime ideal of a lattice L and P be a prime ideal containing J. If Q is a minimal prime ideal containing $J(P)$ with $Q \subseteq P$, then for any $y \in Q - P$, there exists $z \notin Q$ such that $y \wedge z \in J(P)$.*

Proof. If this is not true, then suppose for all $z \notin Q$, $y \wedge z \notin J(P)$.
Set $D = (L - Q) \vee [y)$. We claim that $J(P) \cap D = \phi$. If not, let $t \in J(P) \cap D$.
Then $t \in J(P)$ and $t \geq a \wedge y$ for some $a \in L - Q$.
Now $a \wedge y \leq t$ implies $a \wedge y \in J(P)$, which is a contradiction to the assumption .
Thus, $J(P) \cap D = \phi$.
Then by Lemma 1, there exists a maximal filter $R \supseteq D$ such that, $R \cap J(P) = \phi$.
Since $J(P)$ is semiprime , so by Theorem 4, R is a prime filter. Therefore $L - R$ is a mminimal prime ideal containing $J(P)$. Moreover $L - R \subseteq Q$ and $L - R \neq Q$ as $y \in Q$ but $y \notin L - R$. This contradicts the minimality of Q.
Therefore there must exist $z \notin Q$ such that $y \wedge z \in J(P)$. ●

Lemma 9. *Let P be a prime ideal containing a semi prime ideal J. Then each minimal prime ideal containing $J(P)$ is contained in P.*

Proof. Let Q be a minimal prime ideal containing $J(P)$. If $Q \square P$, then choose $y \in Q - P$. Then by lemma 8, $y \wedge z \in 0(P)$ for some $z \notin Q$. Then $y \wedge z \wedge x \in J$ for some $x \notin P$. As P is prime, $y \wedge x \notin P$. This implies $z \in J(P) \subseteq Q$, which is a contradiction. Hence $Q \subseteq P$.●

Proposition 10. *If in a lattice L, P is a prime ideal containing a semi prime ideal J, then the ideal $J(P)$ is the intersection of all the minimal prime ideals containing J but contained in P.*

Proof. Let Q be a prime ideal containing J such that $Q \subseteq P$. Suppose $x \in J(P)$
.
Then $x \wedge y \in J$ for some $y \in L - P$. Since $y \notin P$, so $y \notin Q$.
Then $x \wedge y \in J \subseteq Q$ implies $x \in Q$.

Thus $J(P) \subseteq Q$. Hence $J(P)$ is contained in the intersection of all minimal prime ideals containing J but contained in P. Thus $J(P) \subseteq \cap\{Q,$ the prime ideals containing J but contained in $P\} \subseteq \cap \{Q,$ the minimal prime ideals containing J but contained in $P\} = X$ (say).

Now, $J(P) \subseteq X$. If $J(P) \neq X$, then there exists $x \in X$ such that $x \notin J(P)$. Then $[x) \cap J(P) = \phi$. So by Zorn's lemma as in lemma 1 there exists a maximal filter $F \supseteq [x)$ and disjoint to $J(P)$. Hence by Theorem 4, F is a prime filter as $J(P)$ is semiprime. Therefore $L - F$ is a minimal prime ideal containing $J(P)$. But $x \notin L - F$ implies $x \notin X$ gives a contradiction. Hence $J(P) = X = \cap\{Q,$ the minimal prime ideals containing J but contained in $P\}$.●

An algebra $L = \langle L; \wedge, \vee, *, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is called a p-*algebra* if

(i) $\langle L; \wedge, \vee, *, 0, 1\rangle$ is a bounded lattice, and

(ii) for all $a \in L$, there exists an a^* such that $x \leq a^*$ if and only if $x \wedge a = 0$.

The element a^* is called the *pseudo complement* of a.

Let J be an ideal of a lattice L with 1. For an element $a \in L$, a^+ is called the *pseudo complement of a relative to* J if $a \wedge a^+ \in J$ and for any $b \in L$, $a \wedge b \in J$ implies $b \leq a^+$. L is called a *pseudo complemented lattice relative to* J if its every element has a pseudo complement relative to J.

Theorem 11. *For an ideal J of a lattice L with 1, if L is pseudo complemented relative to J, then J must be a principal semi prime ideal.*

Proof. Let L be pseudo complemented relative to J. Now for all $a \in L$, $1 \wedge a = a$. So the relative pseudo complement of 1 must be the largest element of J. Hence J must be principal. Now suppose $a,b,c \in L$ with $a \wedge b, a \wedge c \in J$. Then $b,c \leq a^+$, and so $b \vee c \leq a^+$. Thus $a \wedge (b \vee c) \in J$, and hence J is semi prime. ●

An algebra $L = \langle L; \wedge, \vee, +, J, 1\rangle$ is called a p-*algebra relative to* J if

(i) $\langle L; \wedge, \vee, J, 1\rangle$ is a lattice with 1 and a principal semi prime ideal J, and

(ii) for all $a \in L$, there exists a pseudo complement a^+ relative to J.

Suppose $J = (t]$. An element $a \in L$ is called a *dense element relative to* J if $a^+ = t$.

We denote the set of all dense elements relative to J by $D_J(L)$. It is easy to check that $D_J(L)$ is a filter of L.

Lemma 12. *Let $L = (L; \wedge, \vee, +, J, 1)$ be a p-algebra relative to J and P be a prime ideal of the lattice L containing J. Then the following conditions are equivalent.*

(i) *P is a minimal prime ideal containing J.*

(ii) *$x \in P$ implies $x^+ \notin P$.*

(iii) *$x \in P$ implies $x^{++} \in P$.*

(iv) *$P \cap D_J(L) = \phi$.*

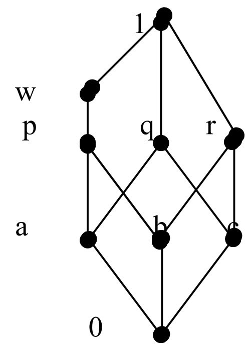
Proof. (i) implies (ii). Let P be minimal and let (ii) fail, that is, $a^+ \in P$ for some $a \in P$. Let $D = (L-P) \vee [a)$. We claim that $J \cap D = \phi$. Indeed, if $j \in J \cap D$, then $j \geq q \wedge a$ for some $q \in L - P$, which implies that $q \wedge a \in J$, and so $q \leq a^+$. Thus $q \in P$ gives a contradiction. Then $a^+ \notin D$, for otherwise $a \wedge a^+ \in J \cap D$. Hence $D \cap (a]^+ = D \cap \{a\}^{\perp_J} = \phi$. Then by Theorem 5, there exists a prime filter $F \supseteq D$ and disjoint to $(a]^+$. Hence $Q = L - F$ is a prime ideal disjoint to D. Then $Q \subseteq P$, since $Q \cap (L-P) = \phi$ and $Q \neq P$, as $a \notin Q$, cotradicting the minimality of P.

(ii) implies (iii). Indeed $x^+ \wedge x^{++} \in J \subseteq P$ for any $x \in L$; thus if $x \in P$, then by (ii), $x^+ \notin P$, implying that $x^{++} \in P$.

(iii) implies (iv). If $a \in P \cap D_J(L)$ for some $a \in L$, then $a^{++} = 1 \notin P$, a contradiction to (iii). Thus $P \cap D_J(L) = \phi$.

(iv) implies (i). If P is not minimal prime ideal containing J, then $Q \subset P$ for some prime ideal Q of L containing J. let $x \in P - Q$. Then $x \wedge x^+ \in J \subseteq Q$ and $x \notin Q$; therefore $x^+ \in Q \subset P$, which implies that $x \vee x^+ \in P$. But $x \vee x^+ \in D_J(L)$; thus we obtain $x \vee x^+ \in P \cap D_J(L)$, contradictong (iv). ●

A relative p-algebra $L = \langle L; \wedge, \vee, +, J, 1 \rangle$ is called a *relative S-algebra* if $a^+ \vee a^{++} = 1$. L is said to be a relative D-algebra if for all $a, b \in L$, $(a \wedge b)^+ = a^+ \vee b^+$. Of course every relative D-algebra is a relative S-algebra, but the following example due to [3] shows that the converse need not be true.



L

Here, L is an S-algebra, but $(q \wedge r)^* = c^* = w \neq p = b \vee a = q^* \vee r^*$ shows that it is not a D-algebra.

Two prime ideals P and Q are called *co-maximal* if $P \vee Q = L$.
Following result on 1-distributive lattices is due to [6].

Theorem 13. *Let L be a lattice with 1. Then the following conditions are equivalent.*
(i) L *is 1-distributive.*
(ii) *Every maximal ideal is a prime* ideal.
(iii) *Each $a \neq 1$ of L is contained in a prime ideal.* ●

Theorem 14. *In a relative p-algebra $L = (L; \wedge, \vee, +, J, 1)$ where L is 1-distributive the following conditions are equivalent*
(i) L is a relative S-algebra.
(ii) Any two distinct minimal prime ideals containing J are co-maximal.
(iii) Every prime ideal containing J contains a unique minimal prime ideal containing J.
(iv) For each prime ideal P containing J, $J(P)$ is a prime ideal.
(v) For any $x, y \in L$, $x \wedge y \in J$, implies $x^+ \vee y^+ = 1$.

Proof. (i) implies (ii). Suppose L is a relative S-algebra. Let P and Q be two distinct minimal prime ideals containing J. Choose $x \in P - Q$. Then by Lemma 12, $x^+ \notin P$ but $x^{++} \in P$. Now $x \wedge x^+ \in J \subseteq Q$ implies $x^+ \in Q$, as Q is prime. Therefore, $1 = x^{++} \vee x^+ \in P \vee Q$. Hence $P \vee Q = L$. That is P, Q are co-maximal.
(ii) implies (iii) is trivial.

(iii) implies (iv). By Theorem 11, J is a semi ptrime ideal. So by Proposition 10, (iv) holds.
(iv) implies (v). Suppose (iv) holds and yet (v) does not. Then there exists $x, y \in L$ with $x \wedge y \in J$ but $x^{+} \vee y^{+} \neq 1$. Since L is 1-distributive, so by Theorem 13(iii), there is prime ideal P containing $x^{+} \vee y^{+}$. If $x \in J(P)$, then $x \wedge r \in J$ for some $r \in L - P$. This implies $r \leq x^{+} \in P$ gives a contradiction. Hence $x \notin J(P)$. Similarly $y \notin J(P)$. But by (iv), $J(P)$ is prime, and so $x \wedge y \in J \subseteq J(P)$ is contradictory. Thus (iv) imples (v).
(v) implies (i). Since $x \wedge x^{+} \in J$, so by (v) $x^{++} \vee x^{+} = 1$, and L is an S-algebra relative to J.●

A lattice L with 0 is called 0-modular if for all $x, y, z \in L$ with $z \leq x$ and $x \wedge y = 0$ imply $x \wedge (y \vee z) = z$. Now we generalize the concept. Let J be an ideal of a lattice L. We define L to be modular with respect to J if for all $x, y, z \in L$ with $z \leq x$ and $x \wedge y \in J$ imply $x \wedge (y \vee z) = z$.
We conclude the paper with the following result.

Theorem 15. *Let* $\langle L : \wedge, \vee, +, J, 1 \rangle$ *be a relative P-algebra such that* L *is both modular with respect* J *and 1-distributive. If* L *is a relative S-algebra, then it is a relative D- algebra.*

Proof. Suppose L is an S-algebra and $a, b \in L$. Now $a^{+} \vee a^{++} = 1 = b^{+} \vee b^{++}$. Thus $(a^{+} \vee b^{+}) \vee b^{++} = 1 = (a^{+} \vee b^{+}) \vee a^{++}$. Since L is 1-distributive, so $a^{+} \vee b^{+} \vee (a^{++} \wedge b^{++}) = 1$. Now $a \wedge b \wedge a^{+} \in J$ and $a \wedge b \wedge b^{+} \in J$ imply $a^{+},\ b^{+} \leq (a \wedge b)^{+}$, and so $a^{+} \vee b^{+} \leq (a \wedge b)^{+}$. Also, $(a \wedge b)^{+} \wedge (a^{++} \wedge b^{++}) = (a \wedge b)^{+} \wedge (a \wedge b)^{++} \in J$. Thus by J-modularity of L, $(a \wedge b)^{+} = (a \wedge b)^{+} \wedge 1 = (a \wedge b)^{+} \wedge [(a^{++} \wedge b^{++}) \vee (a^{+} \vee b^{+})] = a^{+} \vee b^{+}$, and so L is a relative D- algebra. ●

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