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Semi-Prime Ideals of Gamma Rings

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Abstract. In this paper, we have developed some characterizations of semi-prime ideals of gamma rings. At last we have proved that an ideal Q of a Γ-ring M is semi-prime if and only if B(Q)=Q.

Keywords: Ideals, semi-ideals, gamma rings

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1. Introduction

The concepts of a Γ-ring was first introduced by Nobusawa [6] in 1964. His concept is more general than a ring. Now a day, his Γ-ring is called a Γ-ring in the sense of Nobusawa.

Barnes [3] gave a definition of a Γ-ring which is more general. He introduced the notation of Γ-homomorphisms, Prime and Primary ideals, m-systems and the radical of an ideal for Γ-rings.

The general radical theory for rings had been introduced by Kurosh [4] and Amitsur [1,2]. They studied the generalizations of a general radical. McCoy [5] studied prime and semi-prime ideals and prime radicals of classical rings. In this paper, our results are the generalizations of McCoy [5], which is not in Barnes [3].

2. Preliminaries

2.1 Gamma ring

Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x\alpha y$) such that

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$
$x(\alpha + \beta)z = x\alpha z + x\beta z$
$x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$, where x, y, z$\in$M and $\alpha, \beta\in\Gamma$. Then M is called a Γ-ring in the sense of Barnes [3].

2.2 Ideal of Γ-rings

A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c\in M, \alpha\in\Gamma, a\in A\}(A\Gamma M)$ is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two-sided ideal of M.

If A and B are both left (respectively right or two-sided) ideals of M, then $A + B = \{a + b \mid a\in A, b\in B\}$ is clearly a left (respectively right or two-sided) ideal, called the sum of A and B. We can say every finite sum of left (respectively right or two-sided) ideal of a Γ-ring is also a left (respectively right or two-sided) ideal.

It is clear that the intersection of any number of left (respectively right or two-sided) ideal of M is also a left (respectively right or two-sided) ideal of M.

If A is a left ideal of M, B is a right ideal of M and S is any non-empty subset of M, then the set, $A\Gamma S = \{\sum_{i=1}^{n} a_i\gamma s_i \mid a_i\in A, \gamma\in\Gamma, s_i\in S$, n is a positive integer$\}$ is a left ideal of M and $S\Gamma B$ is a right ideal of M. $A\Gamma B$ is a two-sided ideal of M.

If a$\in$M, then the **principal ideal generated by a** denoted by $\langle \mathbf{a}\rangle$ is the intersection of all ideals containing **a** and is the set of all finite sum of elements of the form $na + x\alpha a + a\beta y + u\gamma a\mu v$, where n is an integer, x, y, u, v are elements of M and $\alpha, \beta, \gamma, \mu$ are elements of Γ. This is the smallest ideal generated by a. Let a$\in$M. The smallest **left (right) ideal generated by a** is called the **principal left (right) ideal $\langle \mathbf{a}|$ ($|\, \mathbf{a}\rangle$).**

2.3 Nilpotent element. Let M be a Γ-ring. An element x of M is called nilpotent if for some $\gamma\in\Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^n x = (x\gamma x\gamma ... \gamma x\gamma)x = 0$.

2.4 Nilpotent ideal. An ideal A of a Γ-ring M is called nilpotent if $(A\Gamma)^n A = (A\Gamma A\Gamma.. ....\Gamma A\Gamma)A = 0$, where n is the least positive integer.

2.5 Radical of a Γ-ring. Let M be a Γ-ring with minimum condition. The two sided ideal which is the sum of all nilpotent left ideals of M is called the radical of M and is denoted by rad M.

2.6 Quotient Γ-ring. Let M be a Γ-ring. Let A be an ideal of M. Then the set $\{m + A \mid m\in M\}$ is called the quotient Γ-ring of M by A. It is denoted by M/A, where $(m_1 + A)\gamma(m_2 + A) = m_1\gamma m_2 + A$ and $(m_1 + A) + (m_2 + A) = (m_1 + m_2) + A$ for all $m_1, m_2\in M$ and $\gamma\in\Gamma$.

2.7 Γ-homomorphism. Let M and N be two Γ-rings. Let φ be a map from M to N. Then φ is a Γ- homomorphism if and only if φ(x + y) = φ(x) + φ(y) and φ(xγy) = φ(x)γφ(y) for all x, y∈M and all γ∈Γ. If φ is a Γ homomorphism of M into N, then kernel of φ, denoted by ker φ defined as $\{x\in M \mid \varphi(x)=0\}$ is an ideal of M.

3. Semi- Prime Ideals of Gamma Rings

Definition 3.1. An ideal P in a Γ-ring M is said to be a prime ideal if and only if it has the following property:

If A and B are ideals in M such that $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

Theorem 2.3. If P is an ideal in a Γ-ring M all of the following conditions are equivalent:

(i) P is a prime ideal of M.

(ii) If $a, b \in M$ such that $a\Gamma M\Gamma b \subseteq P$, then $a \in P$ or $b \in P$.

(iii) If <a> and <b> are principal ideals in M such that $<a> \Gamma <b> \subseteq P$, then $a \in P$ or $b \in P$.

(iv) If U and V are right ideals in M such that $U \Gamma V \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

(v) If U and V are left ideals in M such that $U \Gamma V \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

The proof is given in Barnes [3].

If P is an ideal in M, let us denote by C(P), the complement of P in M, that is, C(P) is the set of elements of M which are not elements of P. Now each of the equivalent condition of the theorem 3.2 can be used to characterize a prime ideal in terms of some property of C(P). In this connection, we shall find condition (ii) of theorem 3.2 to be of special interest.

Definition 3.3. A set N of elements of a Γ-ring M is said to be an n-system if and only if it has the following property:

If $a,b \in N$, there exists $x \in M$ such that $a\alpha x\beta b \in N$, $\alpha, \beta \in \Gamma$.

For our purpose, the significance of this concept stems from the fact that the equivalence of theorem 3.2 (i) and (ii) asserts that an ideal P in a Γ-ring M is a prime ideal in M if and only if C(P) is an n-system.

It is trivial that M itself is a prime ideal in M. Clearly C(M) is empty, so in order for the proceeding statement to be true without exception. We explicitly agree that the empty set is to be considered as an n-system.

A set of elements of a Γ-ring which is closed under multiplication is often called a multiplicative system. It is obvious that any multiplicative system L is also an n-system. For if $a,b \in L$, then $a\alpha x\beta b \in L, \alpha, \beta \in \Gamma$ for $x = a$ or $x = b$. Hence the concept of n-system is a generalization of that of multiplicative system.

We now define another concept whose significance will be indicated in the theorem to follow.

Definition 4.3. The prime radical B(A) of the ideal A in a Γ-ring M is the set consisting of those elements m of M with the property that every n-system in M which contains m meets A (that is, has not-empty intersection with A).

It is not obvious that B(A) is an ideal in M, but the next theorem will show that this is the case. However, let us observe that A and B(A) are contained in precisely the same prime ideal which contains B(A) necessarily contains A. Suppose that P is a prime ideal in M such that $A \subset P$ and let $m \in B(A)$. If $m \notin P$, C(P) would be an n-system containing m and therefore we would have $C(P) \bigcap A$ is non-empty. However, since $A \subset P$, $C(P) \bigcap A$ is empty and this contradiction shows that $m \in P$. Hence $B(A) \subseteq P$ as we wished to show. If $m \in M$, then the set $\left\{ (m\alpha)^i m \middle| i = 1, 2, 3 .... \right\}$ is a multiplicative system and hence also an n-system.

Theorem 3.5. If $m \in B(A)$, then there exists a positive integer n such that $(m\alpha)^n m \in A$.

The proof is given in W. E. Barnes [3].

Theorem 3.6. If A is an ideal in the Γ-ring M, then B(A) coincides with the intersection of all the prime ideals in M which contain A.

The proof is given in W. E. Barnes [3].

Suppose that M is commutative and that $m \in M$. Let N be any n-system in M which contains m. Then there exists $x \in M$ such that $m\alpha x\alpha m = m\alpha m\alpha x \in N$, $\alpha \in \Gamma$. Again applying the definition of n-system, there exists $y \in M$ such that $(m\alpha m\alpha x)\alpha y\alpha m = (m\alpha)^2 m\alpha x\alpha y \in N$. Continuing in this way, it is clear that for each positive integer n there exists $t \in M$ such that $(m\alpha)^n m\alpha t \in N$. Now if A is an ideal in M such that $(m\alpha)^n m \in A$, then $(m\alpha)^n m\alpha t \in A$ and $N \bigcap A$ is non-empty. This shows that, if $(m\alpha)^n m \in A$, then every n-system containing m meets A and hence that $m \in B(A)$. the following result is then a consequence of this observation and Theorem 3.5.

Theorem 3.7. If A is an ideal in the commutative Γ-ring M, then $B(A) = \left\{ m \middle| (m\alpha)^n m \in A, \alpha \in \Gamma \; and \; for \; some \; positive \; \text{int} eger \; n \right\}$.

The proof is given in Barnes [3].

Definition 3.8. An ideal Q in a Γ-ring M is said to be a semi-prime ideal if and only if it has the following property:
If A is an ideal in M such that $A \Gamma A \subseteq Q$, then $A \subseteq Q$.

Several simple facts are almost immediate consequences of this definition. It is clear that a prime ideal is semi-prime. Moreover, the intersection of any set of semi-prime ideals is a semi-prime ideal.

Although Definition 3.8 refers to the square of an ideal A, it follows easily by induction that if Q is a semi-prime ideal and $(A\Gamma)^n A \subseteq Q$ for an arbitrary positive integer n, then $A \subseteq Q$.

The following important result is fairly easy to prove but for the sake of completeness, we write out a proof.

Theorem 3.9. An ideal Q in a Γ-ring M is a semi-prime ideal in M if and only if M/Q contains no nonzero nilpotent ideals.

Proof. Let f be the natural Γ-homomorphism of M on to M/Q, with kernel Q. Suppose Q is a semi-prime ideal in M and U is a nilpotent ideal in M/Q, say $(U\Gamma)^n U = 0$. then $f^{-1}\left((U\Gamma)^n U\right)$ = Q and it follows that $\left(f^{-1}(U)\Gamma\right)^n f^{-1}(U) \subseteq f^{-1}\left((U\Gamma)^n U\right) = Q$ and hence U = 0.

Conversely suppose that M/Q contains no non-zero nilpotent ideals and that A is an ideal in M such that $A\Gamma A \subseteq Q$. Then $f(A)\Gamma f(A) = f(A\Gamma A) = 0$. Hence $f(A) = 0$ and $A \subseteq Q$.

Although it is possible to prove results about semi-prime ideals that are analogous to all of those established for prime ideals proceeding section, we shall present only a few that are essential for later applications. First, let us state the following theorem whose proof we omit since it can be established by very easy modifications of the proof of theorem 3.2.

Theorem 3.10. If Q is an ideal in a Γ-ring M, all of the following conditions are equivalent.

(i) Q is a semi-prime ideal

(ii) If $a \in M$ such that $a\Gamma M\Gamma a \subseteq Q$, then $a \in Q$.

(iii) If <a> is a principal ideal in M such that $<a>\Gamma<a> \subseteq Q$, then $a \in Q$.

(iv) If U is a right ideal in M such that $U\Gamma U \subseteq Q$, then $U \subseteq Q$.

(v) If U is a lift ideal in M such that $U\Gamma U \subseteq Q$, then $U \subseteq Q$.

Let us next make the following definition which is analogous to the definition of an n-system.

Definition 3.11. A set T of elements of a Γ-ring is said to be a t-system if and only if it has the following property:

If $a \in T$, there exist $x \in M$ such that $a\alpha x\alpha a \in T, \alpha \in \Gamma$.

It is clear that an n-system is also a t-system. Also, the equivalence of conditions (i) and (ii) of theorem 3.10 assures us that an ideal Q in M is a semi-prime ideal if and only if C(Q) is a t-system.

The following lemma will play a central role in the proof of the next theorem.

Lemma 3.12. If T is a t-system in the Γ-ring M and there exists an n-system N in M such that $a \in N$ and $N \subseteq T$.

Proof. Let $N = \{a_1, a_2, a_3, ...\}$, where the elements of this sequence are defined inductively as follows. First we define $a_1 = a$. Since now $a_1 \in N$, $a_1\Gamma M\Gamma a_1 \cap T$ is non-empty and we choose a_2 as some element of $a_1\Gamma M\Gamma a_1 \cap T$. In general, if a_i has been defined, with $a_i \in T$, we choose a_{i+1} as an element of $a_i\Gamma M\Gamma a_i \cap T$. Thus a set N is defined such that $a \in N$ and $N \subseteq T$. To complete the proof, we only need to show that N is an n-system. Suppose that $a_i, a_j \in N$ and for convenience, let us assume that $i \leq j$. Then $a_{j+1} \in a_j\Gamma M\Gamma a_j \subseteq a_i\Gamma M\Gamma a_j$ and $a_{j+1} \in N$. A similar argument takes care of the case in which i>j, so we conclude that N in indeed an n-system. Thus the proof is completed.

We can now easily prove the following theorem.

Theorem 3.13. An ideal Q in a Γ-ring M is a semi-prime in M if and only if B(Q) = Q.
Proof. The "if" part of this theorem is an immediate consequence. of Theorem 3.6 and the fact that any intersection of prime ideals is a semi-prime ideal. To prove the "only if" part, suppose that Q is a semi-prime ideal in M. certainly $Q \subseteq$ B(Q), so let us assume that Q$\subset$B(Q) and seek a contradiction. Suppose that $a \in B(Q)$ with $a \notin Q$. Hence C(Q) is a t-system and a $\in$C(Q). By the Lemma 3.12, there exists an n-system N such that $a \in N \subseteq$C(Q), Now $a \in$B(Q) and by definition of B(Q), every n-system which contains a meets Q. But Q$\cap$C(Q). is empty and therefore N$\cap$Q is empty. This giver the desired contradiction and completes the proof of the theorem.

In view of theorem 3.6 and the fact that an intersection of prime (or semi-prime) ideals is a semi-prime ideal, we have the following immediate corollary to the preceding theorem.

Corollary 3.14. An ideal Q in a Γ-ring M is a semi-prime ideal if and only if Q is an intersection of prime ideals in M.

If A in an ideal in a Γ-ring M, the intersection of all the semi-prime ideals which contain A is the unique smallest semi-prime ideal which contains A. We may also state the following consequence of theorems 3.6 and 3.13.

Corollary 3.15. If A is an ideal in the Γ-ring M, then B(A) is the smallest semi-prime ideal in M which contains A.

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