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Annulets in a Distributive Nearlattice

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Abstract. Here the authors show that in a distributive near lattice S with 0, set of all ideals of the form (x]*, denoted by $A_0(S)$ is a join semi lattice with lower bound property. It is a sub semi lattice of lattice of ideals if and only if S is normal. We show that $A_0(S)$ is relatively complimented if and only if S is sectionally quasi-complemented. Moreover, $A_0(S)$ is Boolean when S is quasi-complemented.

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1. Introduction

In a distributive lattice L with 0, set of all ideals of the form $(x]^*$ can be made into a lattice $A_0(L)$, which is by [1] called the lattice of annulates of L. In this paper we have studied annulates of nearlattices and generalized several results of [1].

By a *near lattice*, we mean a meet semilattice with the property that any two elements possessing a common upper bound have a supremum. By [2], this property is known as the "**upper** bound property". For detailed literature on nearlattices see [2], [3] and [7].

A nearlattice S is called a *distributive nearlattice* if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, provided $y \vee z$ exists. A nonempty subset I of a nearlattice S is called an *ideal* if

i) For $x, y \in I$, $x \vee y \in I$, provided $x \vee y$ exists, and

ii) For $x \in I, t \leq x$ ($t \in S$) implies $t \in I$.

A non empty subset of S is called a filter if i) for all $x, y \in F, \quad x \wedge y \in F$,
and ii) $t \in S, t \geq x \; and \; x \in F$ imply $t \in F$.

For a distributive nearlattice S with 0, $I(S)$ denotes the set of all ideals, which is a distributive lattice.

A distributive nearlattices S with 0 is called normal if every prime ideal of S contains a unique minimal prime ideal. A distributive near lattice S with 0 is called generalized Stone if for each $x \in S$, $(x]^* \vee (x]^{**} = S$.

In this paper by a "dual nearlattice" we will mean a join semilattice with the lower bound property. That is, its notion is dual to a nearlattice.

2. Annulets

For a distributive nearlattice S with 0, $I(S)$ the lattice of ideals of S is pseudo complemented. An ideal J of S is called an annihilator ideal if $J = J^{**}$. The pseudo complement of an ideal J is the annihilator ideal $J^* = \{x \in S : x \wedge j = 0$ for all $j \in J\}$. It is well known by [4, Theorem 4, p-58] that the set of annihilator ideals $A(S)$ is a Boolean algebra, where the supremum of J and K in $A(S)$ is given by $J \underline{\vee} K = (J^* \cap K^*)^*$. Thus for two annulets $(x]^*$ and $(y]^*$. $(x]^* \underline{\vee} (y]^* = ((x]^{**} \cap (y]^{**})^* = ((x \wedge y]^{**})^* = (x \wedge y]^*$. Hence, the set of all annulets $A_0(S)$ of S is a join sub semilattice of $A(S)$. Of course, $A_0(S)$ is not necessarily a meet semilattice. But for any $x, y \in S$ if $x \vee y$ exists then $(x]^* \cap (y]^* = (x \vee y]^*$.

Proposition 2.1. *Let S be a distributive nearlattice with 0. Then $A_0(S)$ is a dual nearlattice and it is a dual subnearlattice of $A(S)$. Moreover, $A_0(S)$ has the same largest element $S=(0]^*$ as $A(S)$.*

Proof: We have already shown that $A_0(S)$ is a join subsemilattice of $A(S)$. Now suppose $(x]^* \supseteq (t]^*$ and $(y]^* \supseteq (t]^*$ for some $x, y, t \in S$. Then $(x]^* \cap (y]^*$ $=((x]^* \cap (y]^*) \veebar (t]^* = ((x]^* \veebar (t]^*) \cap ((y]^* \veebar (t]^*) = (x \wedge t]^* \cap (y \wedge t]^*$ $=((x \wedge t) \vee (y \wedge t)])^*$ as $(x \wedge t) \vee (y \wedge t)$ exists by the upper bound property of S. This shows that $A_0(S)$ has the lower bound property. Hence $A_0(S)$ is a dual nearlattice and so a dual subnearlattice of $A(S)$. ■

Proposition 2.2. *Let S be a distributive nearlattice with 0. $A_0(S)$ has a smallest element (then of course, it is a lattice) if and only if S possesses an element d such that $(d]^*=(0]$.*

Proof. If there is an element $d \in S$ with $(d]^*=(0]$, then clearly $(0]$ is the smallest element in $A_0(S)$.

Conversely, if $A_0(S)$ has a smallest element $(d]^*$, then for any $x \in S$, $(x]^*=(x]^* \veebar (d]^* = (x \wedge d]^*$. Thus $x \wedge d = 0$ implies $(x]^*=(0]^*=S$, and hence $(d]^*=(0]$. ■

Following result gives a characterization of a normal nearlattice which is a generalization of [1, Proposition 2.2].

Theorem 2.3. *A distributive nearlattice S with 0 is normal if and only if $A_0(S)$ is a join subsemilattice of $I(S)$.*

Proof. By Proposition 2.1, $A_0(S)$ is a join subsemilattice of $A(S)$, and for any $x, y \in S, (x]^* \veebar (y]^* = (x \wedge y]^*$. Now by [6, Theorem 1.9], S is normal if and only if $(x]^* \vee (y]^* = (x \wedge y]^*$ for all $x, y \in S$. Hence $(x]^* \veebar (y]^* = (x]^* \vee (y]^*$ for all $x, y \in S$. This proves the theorem. ■

A distributive nearlattice S with 0 is called disjunctive if for $0 \le a < b$

$(a, b \in S)$ there is an element $x \in S$ such that $a \wedge x = 0$ where $0 < x \le b$. It is easy to check that S is disjunctive if and only if $(a]^* = (b]^*$ implies $a = b$ for any $a, b \in S$. Then we have the following result.

Theorem 2.4. *A disjunctive normal nearlattice S is dual isomorphic to $A_0(S)$. Hence S has a largest element (in that case S is a lattice) if and only if there exists $d \in S$ such that $(d]^* = (0]$.*

Proof. If S is normal, then by theorem 2.3, $A_0(S)$ is a join subsemilattice of $I(S)$, and for any $x, y \in S$, $(x \wedge y]^* = (x]^* \vee (y]^*$. Also for any near lattice S, $(x]^* \cap (y]^*$
$= (x \vee y]^*$ if $x \vee y$ exists in S. Hence the map $x \to (x]^*$ is a dual homomorphism from S onto $A_0(S)$. If S is disjunctive then obviously this map is one-one and so is a dual isomorphism. Second part is trivial. ■

By [5] a distributive near lattice S with 0 is called quasi-complemented if for each $x \in S$, there is an $x^{/}$ such that $x \wedge x^{/} = 0$ and $(x]^* \cap (x^{/}]^* = (0]$. The following result generalizes [1, Proposition 2.4].

Theorem 2.5. *A distributive nearlattice S with 0 is quasi-complemented if and only if $A_0(S)$ is a Boolean subalgebra of $A(S)$.*

Proof. Suppose S is quasi-complemented. Then by [5, Th. 2.1] S has an element d such that $(d]^* = (0]$. Then by Proposition 2.2, $A_0(S)$ has a smallest element and so it is a sublattice of $A(S)$. Moreover for each $x \in S$ there exists $x^{/} \in S$ such that $x \wedge x^{/} = 0$ and $(x]^* \cap (x^{/}]^* = (0]$. Then $(x]^* \veebar (x^{/}]^* = (x \wedge x^{/}]^* = (0]^* = S$. Therefore $A_0(S)$ is a Boolean subalgebra of $A(S)$.

Conversely, if $A_0(S)$ is a Boolean subalgebra of $A(S)$, then for any $x \in S$ there exists $y \in S$ such that $(x]^* \cap (y]^* = (0]$ and $(x]^* \veebar (y]^* = S$.
But $(x]^* \veebar (y]^* = (x \wedge y]^*$, and so $x \wedge y = 0$. Therefore S is a quasi-complemented.. ■

Now we generalize [1, Proposition 2.5]. To prove this we need the following lemma. The proof of lemma is trivial.

Lemma 2.6. *Let* $I = [0, x]$, $0 < x$ *be an interval in a distributive nearlattice* S *with* 0. *For* $a \in I$, $(a]^{+} = \{y \in I : y \wedge a = 0\}$ *is the annihilator of* $(a]$ *with respect to* I. *Then*

(i) *If* $a, b \in I$ *and* $(a]^{+} \subseteq (b]^{+}$ *then* $(a]^{*} \subseteq (b]^{*}$

(ii) *If* $w \in S$, $(w]^{*} \cap I = (w \wedge x]^{+}$ ■

Theorem 2.7. *For a distributive nearlattice* S *with* 0, $A_0(S)$ *is relatively complemented if and only if* S *is sectionally quasi -complemented.*

Proof. Suppose $A_0(S)$ is relatively complemented. Consider the interval $I = [0, x]$ and let $a \in I$, then $(x]^{*} \subseteq (a]^{*} \subseteq (0]^{*} = S$. Since the interval $[(x]^{*}, S]$ in $A_0(S)$ is complemented, there exists $w \in S$ such that $(a]^{*} \cap (w]^{*} = (x]^{*}$ and $(a]^{*} \underline{\vee} (w]^{*} = S$. Then $(a]^{*} \underline{\vee} (w]^{*} = (a \wedge w]^{*}$ gives $a \wedge w = 0$ Then $a \wedge w \wedge x = 0$ and $w \wedge x \in I$. Moreover, intersecting $(a]^{*} \cap (w]^{*} = (x]^{*}$ with $(x]$ and using above lemma, we have $(a]^{+} \cap (w \wedge x]^{+} = (0]$. This shows that I is quasi -complemented.

Conversely, suppose S is sectionally quasi –complemented. Since $A_0(S)$ is distributive, it suffices to prove that the interval $[(a]^{*}, S]$ is complemented for each $a \in S$. Let $(b]^{*} \in [(a]^{*}, S]$. Then $(a]^{*} \subseteq (b]^{*} \subseteq S$, so $(b]^{*} = (a]^{*} \underline{\vee} (b]^{*} = (a \wedge b]^{*}$. Now consider the interval $I = [0, a]$ in S. Then $a \wedge b \in I$. Since I is quasi-complemented, there exists $w \in I$ such that $w \wedge a \wedge b = 0$ and $(w]^{+} \cap (a \wedge b]^{+} = (0] = (a]^{+}$. This implies $(w \vee (a \wedge b)]^{+} = (a]^{+}$, as $w \vee (a \wedge b)$ exists in S. Then by lemma 2.6(i), $(a]^{*} = (w \vee (a \wedge b)]^{*} = (w]^{*} \cap (a \wedge b]^{*} = (w]^{*} \cap (b]^{*}$. Also from $w \wedge a \wedge b = 0$ we have $w \wedge b = 0$, hence $(w]^{*} \underline{\vee} (b]^{*} = S$. Therefore $A_0(S)$ is relatively complemented. ■

Since by [5, Theorem 2.3], a nearlattice S is generalized Stone if and only if it is normal and sectionally quasi-complemented, combining Theorems 2.7 and 2.3 we have the following result:

Theorem 2.8. *A nearlattice* S *with* 0 *is a generalized Stone nearlattice if and only if* $A_0(S)$ *is a relatively complemented dual subnearlattice of* $I(S)$. ■

REFERENCES

1. W. H. Cornish, Annulets and α-ideals in a distributive lattice, *J. Austral. Math. Soc.,* 15(1) (1973) 70-77.
2. W. H. Cornish and R.C. Hickman, Weakly distributive semilattices, *Acta. Math. Acad. Soci. Hung.,* 32 (1978) 5-16.
3. W. H. Cornish and A. S. A. Noor, Standard elements in a nearlattice, *Bull. Austral. Math. Soc.*, 26(2) (1982), 185-213.
4. G. Gratzer, *Lattice theory: First concepts and distributive lattices*, Freeman, SanFrancisco, 1971.
5. A. S. A. Noor and A. K. M. S. Islam, Generalized Stone nearlattces, *Jahangirnagar University Journal of Science*, 33 (1) (2010) 105-111.
6. A. S. A. Noor and M. A. Latif, Normal nearlattices, *The Rajshahi University Studies*, 9 (1982) 69-74.
7. A. S. A. Noor and M. B. Rahaman, Separation properties in nearlattices, *The Rajshahi University Studies*, 22 (1994), Part-B.