Size Multipartite Ramsey Numbers for Small Paths vs. $K_{2,n}$

C. J. Jayawardene

Department of Mathematics, University of Colombo, Colombo 3, Sri Lanka
Email: c_jayawardene@yahoo.com

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Abstract. Let $G$ and $H$ be finite graphs without loops and multiple edges. We use the notation $K_{js} \rightarrow (G,H)$ to mean that if the edges of the complete graph $K_{js}$ are coloured by the two colours red and blue, then either the red subgraph of $K_{js}$ contains a copy of $G$, or the blue subgraph of $K_{js}$ contains a blue copy of $H$. The size Ramsey multipartite number $m_j(P_3,K_{2,n})$ is defined as the smallest natural number $s$ such that $K_{js} \rightarrow (P_3,K_{2,n})$. In this paper, we obtain the exact values of the size Ramsey numbers $m_j(P_3,K_{2,n})$ and $m_j(P_4,K_{2,n})$ for $j \geq 3$.

Keywords: Ramsey theory, Multipartite Ramsey numbers

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1. Introduction

Let $G$ and $H$ be two finite graphs without loops and multiple edges. Let the complete graph on $n$ vertices and the complete balance multipartie graph having $j$ multipartite sets of size $s$ be denoted by $K_n$ and $K_{js}$ respectively. The write $K_{js} \rightarrow (G,H)$ to mean that if the edges of the complete graph $K_{js}$ are coloured by the two colours red and blue, then there is a copy of $G$ in red, or a copy of $H$ in blue. In particular, the smallest positive integer $n$ such that $K_n \rightarrow (G,H)$ is defined as the Ramsey number $r(G,H)$. Moreover, diagonal classical Ramsey number $r(n,n)$ is defined as $r(K_n,K_n)$. The exact determination of these diagonal classical Ramsey numbers have been studied for a few decades (see Radziszowski, 2017 for a survey) but sadly not much progress has been done even in the case of $r(5,5)$. In the last decade, using this idea of the original classical Ramsey numbers and of the size Ramsey numbers, the notion of size multipartite Ramsey numbers were introduced by Burger and Vuuren (i.e., Burger et al., 2004) by exploring the two colourings of multipartite graph $K_{js}$ instead of the complete graph. More formally, size Ramsey multipartite number $m_j(G,H)$ is defined as the smallest natural number $s$ such that $K_{js} \rightarrow (G,H)$. Size multipartite Ramsey numbers have not been studied in detail up to now. Some of the known results are by Syafrizal, Baskaro and Uttunggadewa (i.e., Syfrizal et al., 2005).

Notation

Given a simple graph $G = (V,E)$, the order and the size of the graph are defined as $|V(G)|$ and $|E(G)|$ respectively. Given a vertex $v \in V(G)$ we define the neighborhood of $v$ as the
set of vertices adjacent to \( v \) in \( G \) and is denoted by \( N(v) \). We also define \( |N(v)| \) as the degree of such a vertex. A path on \( n \) vertices in a graph \( G = (V,E) \) denoted by \( P_n(V,E) \), is the subgraph consisting of the vertex set \( \{a_1,a_2,...,a_n\} \) and the edge set \( \{(a_1,a_2), (a_2,a_3), \ldots, (a_{n-1},a_n)\} \) respectively.

2. Size Ramsey numbers related to paths of size three versus a certain class of complete bipartite graphs

**Lemma 1.** Let \( j \geq 3 \) and \( n \geq 1 \). Then,

\[
\left\lfloor \frac{n+1}{j-1} \right\rfloor \leq m_j(P_3, K_{2,n}) \leq \left\lfloor \frac{n+2}{j-1} \right\rfloor.
\]

**Proof:** \( m_j(P_3, K_{2,n}) = 1 \) when \( n \leq j - 2 \). Therefore, if \( \left\lceil \frac{n+1}{j-1} \right\rceil = 1 \) the result follows trivially.

So, for the rest of the proof, we assume that \( \left\lceil \frac{n+1}{j-1} \right\rceil \geq 2 \).

First to find an upper bound, consider any red \( P_3 \)-free red/blue colouring of \( K_{j\times s} \), where \( s = \left\lfloor \frac{n+2}{j-1} \right\rfloor \). Let \( G = H_R \oplus H_B \) where \( H_R \) and \( H_B \) are the red and blue subgraph of \( G \) induced by the red and blue colouring, respectively. Since \( H_R \) has no \( P_3 \), \( H_R \) will contain two vertices, \( v_1 \) and \( v_2 \) belonging to the same partition \( A \), such that each of these vertices will have at most red degree one. This will force a \( K_{2,m} \) in \( H_B \), where

\[
m = (j - 1)s - 2 = (j - 1)\left\lfloor \frac{n+2}{j-1} \right\rfloor - 2 \geq n
\]

with the highest degree vertices of \( K_{2,m} \) chosen to be \( v_1 \) and \( v_2 \). Therefore,

\[
m_j(P_3, K_{2,n}) \leq \left\lfloor \frac{n+2}{j-1} \right\rfloor
\]

Next to find a lower bound, consider the colouring given by \( K_{j\times s} = H_R \oplus H_B \), where \( s = \left\lfloor \frac{n+1}{j-1} \right\rfloor - 1 \), such that \( H_R \) is a matching as illustrated in the following graphs corresponding to the two cases \( s \) even and \( s \) is odd. If \( s \) is odd and \( j \) is odd as indicated in the second figure, one vertex will be an isolated vertex in red. If \( s \) is odd and \( j \) is odd as indicated in the second figure, the red graph will consist of a perfect matching and the edge \((x,y)\) will be coloured red.

**Figure 1:** (a) If \( s \) is even
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![Figure 1:](image)

**Figure 1**: (b) If $s$ is even, $(x,y)$ is a red edge. Otherwise $x$ is an isolated vertex in $H_R$.

As seen above, in both these cases the graph has no red $P_3$. Also note that if $s \geq 2$, then, $s(j - 2) \leq s(j - 1) - 2$ if $s \geq 2$. Therefore, if $K_{2, m}$ is contained in the graph then

$$m \leq (j - 1)s - 1 = \left(\frac{n + 2}{j - 1} - 1\right)(j - 1) < \left(\frac{n + 2}{j - 1}\right)(j - 1) - 1 = n$$

if $s \geq 2$ and

$$m \leq s(j - 2) = (j - 2) \leq n$$

if $s = 1$

Therefore, in both cases the graph contains no blue $K_{2,n}$.

Hence, $m_j(P_3, K_{2,n}) \geq \left\lceil \frac{n+1}{j-1} \right\rceil$. Hence the result.

**Theorem 2.** $m_j(P_3, K_{2,n}) = 1$ if $n \leq j - 2$.

Also, if $n > j - 2$,

$m_j(P_3, K_{2,n})$

**Proof:** As seen in lemma 1, $m_j(P_3, K_{2,n}) = 1$ if $n \leq j - 2$.

Hence, assume that $s = \left\lceil \frac{n+2}{j-1} \right\rceil \geq 2$. When, $n + 1 \neq 0 \mod(j - 1)$, we know that $\left\lceil \frac{n+1}{j-1} \right\rceil = \left\lfloor \frac{n+2}{j-1} \right\rfloor$. Therefore, when $n + 1 \neq 0 \mod(j - 1)$ the theorem directly follows from the lemma 1.

Hence, we may assume that $n + 1 = 0 \mod(j - 1)$ and $s \geq 2$. Thus, we are left with only the following three cases.

**Case 1:** If $s = \left\lceil \frac{n+1}{j-1} \right\rceil$ is even.

Consider the colouring given by $K_{jm} = H_R \oplus H_B$, where $s = \left\lfloor \frac{n+2}{j-1} \right\rfloor - 1$, such that $H_R$ is a perfect matching as shown in the following diagram.
Then, the graph has no red \( P_3 \). Moreover, if \( K_{2,n} \) is contained in the graph then as \( s(j - 2) \leq s(j - 1) - 2 \) we get
\[
m \leq (j - 1)s - 2 = \left( \left\lfloor \frac{n + 2}{j - 1} \right\rfloor - 1 \right) (j - 1) - 2 < \frac{n + 2}{j - 1} (j - 1) - 2 = n
\]
Therefore, the graph contains no blue \( K_{2,n} \). Hence, \( m_j(P_3, K_{2,n}) \geq \left\lfloor \frac{n + 2}{j - 1} \right\rfloor \).
Therefore, by the lemma 1, in this case we get \( m_j(P_3, K_{2,n}) = \left\lfloor \frac{n + 2}{j - 1} \right\rfloor \).

**Case 2:** If \( j \) is even.
Consider the colouring given by \( K_{n_s} = H_R \oplus H_B \), where \( s = \left\lfloor \frac{n + 2}{j - 1} \right\rfloor - 1 \), such that \( H_R \) is a matching as illustrated in the following diagram.

Then, this graph has no red \( P_3 \). Moreover, if the corresponding blue graph has no \( K_{2,n} \). Therefore, in this case, we get \( m_j(P_3, K_{2,n}) \geq \left\lfloor \frac{n + 2}{j - 1} \right\rfloor \).
That is \( m_j(P_3, K_{2,n}) = \left\lfloor \frac{n + 2}{j - 1} \right\rfloor \).

**Case 3:** If \( \left\lfloor \frac{n + 1}{j - 1} \right\rfloor \) and \( j \) are odd.
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Consider any red $P_3$-free red/blue colouring of $K_{j,s}$, where $s = \left\lceil \frac{n+1}{j-1} \right\rceil$. Let $G = H_R \oplus H_B$ where $H_R$ and $H_B$ are the red and blue subgraph of $G$ induced by the red and blue colouring, respectively. Since $H_R$ has no $P_3$ and $j \times s$ is odd, $H_R$ will contain one isolated vertex $v_1$. Let $v_2$ be another vertex of the same partition $v_1$ belongs to. This will force a $K_{2,m}$ in $H_B$, where $m = (j - 1)s - 1 = \left\lceil \frac{n+1}{j-1} \right\rceil - 1 \geq n$

with the highest degree vertices of $K_{2,m}$ chosen to be $v_1$ and $v_2$. Therefore,

$$m_j(P_3, K_{2,n}) \leq \left\lceil \frac{n+1}{j-1} \right\rceil.$$

Hence in this case, $m_j(P_3, K_{2,n}) = \left\lceil \frac{n+1}{j-1} \right\rceil$ as required.

3. Size Ramsey numbers related to paths of size four versus certain class of complete bipartite graphs

The following definitions follow from a paper by author et al 2016.

**Definition 3. (Bad colourings)** A (red and blue) colouring of $K_{j,s}$ is called a bad colouring if the red connected components of $H_R$ consists of three cycles and at most two disjoint edges.

**Definition 4. (Colouring of $K_{j,s}$ generated by a $s \times t$ matrix)** Let $A = (a_{i,j})_{s \times t}$ represent an matrix consisting of distinct elements in each column. Then $G = G(A)$ the multipartite graph with $j$ partite sets generated by $A$, is defined by $V(G) = \{v_{k,i} | 1 \leq i \leq s, 1 \leq k \leq j\}$ and $E(G) = \{(v_{k,i}, v_{k',i'}) | a_{k,i} = a_{k',i'}\}$, where the $j$ partite sets are respectively given by $V_k = \{v_{k,i} | i = 1, \ldots, s\}$ for $k = 1, \ldots, j$.

The red and blue colouring of $K_{j,s}$ given by $K_{j,s} = H_R \oplus H_B$ such that $H_R = G$ is said to be the two colouring generated by $A$.

**Theorem 5.** Let $j \geq 6$.

a) If $n < 3j - 7$ then $m_j(P_3, K_{2,n}) \in \{1, 2, 3\}$.

b) If $n \geq 3j - 7$ then,

$$m_j(P_3, K_{2,n}) = \left\lceil \frac{n+1}{j-1} \right\rceil$$

**Proof:** (a) The above theorem is a direct consequence of the following two propositions.

**Proposition 6.**

a) $m_j(P_4, K_{2,n}) \leq \left\lceil \frac{n+4}{j-1} \right\rceil$.

(b) If $\left\lceil \frac{n+3}{j-1} \right\rceil \not\equiv 0 \mod 3$ then $m_j(P_4, K_{2,n}) \leq \left\lceil \frac{n+3}{j-1} \right\rceil$.

**Proof:** Before we start off with the proofs noting that, clearly (a) is true if $\left\lceil \frac{n+4}{j-1} \right\rceil \leq 1$ and (b) is true if $\left\lceil \frac{n+3}{j-1} \right\rceil \leq 1$ as $m_j(P_4, K_{2,n}) = 1$ if $n \leq j - 3$. 

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(a) To find an upper bound, consider any arbitrary red \( P_s \)-free red/blue colouring of \( K_{ps} = H_R \oplus H_B \), where \( s = \left\lceil \frac{n+4}{j-1} \right\rceil > 1 \).

Claim 1a: All connected components of \( H_R \) consists of at most 3 vertices.

Assume the claim is false. Then in order to avoid a \( P_4 \), there will exist a vertex \( v \) that will be adjacent to at least three mono-valent vertices \( v_2, v_3 \) and \( v_4 \) in \( H_R \). These vertices belong to distinct partite sets; as otherwise any two of these vertices belonging to the same partite set will be forced to be in some blue \( K_{2,n} \) (this is true because

\[
(j-1)s - 1 = (j-1) \left\lceil \frac{n+4}{j-1} \right\rceil - 1 \geq n,
\]

a contradiction. Therefore, assume that the three three mono-valent vertices \( v_2, v_3 \) and \( v_4 \) in \( H_R \) belong to distinct partite sets \( A_1, A_2 \) and \( A_3 \) respectively. Consider the vertex \( v_2 \) and let \( u \) be any other vertex in the same partite set \( v_2 \) belongs to. Repeating the previous argument we will get that \( u \) will be adjacent to at least three mono-valent vertices \( u_2, u_3 \) and \( u_4 \) belong to distinct partite sets \( B_2, B_3 \) and \( B_3 \). Proceeding in this manner, we will arrive at a contradiction as the number of monovalent vertices of \( H_R \) is finite. Hence the claim.

Therefore, by the previous claim, given any pair of vertices \( v_1 \) and \( v_2 \) belonging to the same partitions \( A \), we get that \( v_1 \) and \( v_2 \) are adjacent to at most four vertices in \( H_R \{A\} \). This forces a \( K_{2,n} \) where

\[
m \geq (j-1)s - 4 = (j-1) \left\lceil \frac{n+4}{j-1} \right\rceil - 4 \geq n.
\]

Thus, every colouring of \( K_{ps} \) contains a red \( P_4 \) or a blue \( K_{2,n} \). Hence

\[
m(\{P_4, K_{2,n}\}) \leq \left\lceil \frac{n+4}{j-1} \right\rceil.
\]

(b) In this case to find an upper bound, consider any arbitrary red \( P_s \)-free red/blue colouring of \( K_{ps} = H_R \oplus H_B \), where \( s = \left\lceil \frac{n+3}{j-1} \right\rceil > 1 \).

Claim 1b: All connected components of \( H_R \) consists of at most 3 vertices.

Assume the claim is false. Then in order to avoid a \( P_s \), there will exist a vertex \( v \) that will be adjacent to at least three mono-valent vertices \( v_2, v_3 \) and \( v_4 \) in \( H_R \). These vertices belong to distinct partite sets; as otherwise any two of these vertices belonging to the same partite set will be in some blue \( K_{2,n} \) (as \( (j-1)s - 1 = (j-1) \left\lceil \frac{n+3}{j-1} \right\rceil - 1 \geq n \)), a contradiction. Therefore, assume that the three three mono-valent vertices \( v_2, v_3 \) and \( v_4 \) in \( H_R \) belong to distinct partite sets \( A_1, A_2 \) and \( A_3 \) respectively. Consider the vertex \( v_2 \) and let \( u \) be any other vertex in the same partite set \( v_2 \) belongs to. Repeating the previous argument we will get \( u \) that will be adjacent to at least three mono-valent vertices \( u_2, u_3 \) and \( u_4 \) belong to distinct partite sets \( B_2, B_3 \) and \( B_3 \). Proceeding in this manner we will arrive at a contradiction as the number of monovalent vertices of \( H_R \) is finite. Hence the claim.

By the above claim all the connected components of \( H_R \) must be of size at most three.

However, as \( sj = 0 \ mod \ 3 \) we get that one of the components must be of size one or two. Therefore, there exists a vertex \( v \) of degree at most 1. Let \( v_1 \) be any other vertex belonging to the same partite set \( v \) belongs to. Then by the claim \( V \) can be adjacent in red to at most two other vertices. We get that \( v \) and \( v_1 \) will be adjacent to at most 3 vertices. But as, \( K_{2,n} \)
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(as $(j - 1)s - 3 = (j - 1)\left[\frac{n+3}{j-1}\right] - 3 \geq n$, we get that $v$ and $v_1$ will be the largest blue degree vertices of a blue $K_{1,n}$. Therefore, if $\left[\frac{n+3}{j-1}\right] j \neq 0 \mod 3$ then $m_j(P_4, K_{2,n}) \leq \left[\frac{n+3}{j-1}\right]$. □

Note that from the above proposition (a) part of the proposition follows as directly.

**Proposition 7.** Suppose $n \geq 3j - 7$, (i.e. $s \geq (j - 1)\left[\frac{n+3}{j-1}\right] - 1 \geq 2$). Then we get $m_j(P_4, K_{2,n}) \geq \left[\frac{n+3}{j-1}\right]$. Moreover, if $j \neq 0 \mod 3$ and $s \neq 2 \mod 3$, then we get $m_j(P_4, K_{2,n}) \geq \left[\frac{n+3}{j-1}\right]$.

**Proof:** The above theorem is a direct consequence of the results obtained in the following 4 cases.

**Case 1:** If $j = 0 \mod 3$.

Let $V_1, V_2, V_3$ where $j = 3k$ represent the $j$ partite sets of $K_{j \times s'}$. Consider the colouring $K_{j \times s'} = H_R \oplus H_B$, where $s' = \left[\frac{n+4}{j-1}\right] - 1$, such that $B_R$ is partitioned in to $sk$ disjoint 3 cycles such that $V_1, V_2, V_3$ consists of $s$ disjoint triangles, $V_2, V_3, V_4$ consists of $s$ disjoint triangles and likewise continuing in this manner $V_{3k-1}, V_{3k-2}, V_{3k}$ consists of $s$ disjoint triangles.

Then, the graph has no red $P_4$. Moreover, if $K_{2,m}$ is a contained in $H_B$ then

$m \leq (j - 1)s - 4 = \left(\left[\frac{n+4}{j-1}\right] - 1\right) (j - 1) - 4 \leq \left(\frac{n+4}{j-1}\right) (j - 1) - 4 = n$

Therefore, the graph contains no blue $K_{2,n}$. Hence, $m_j(P_4, K_{2,n}) \geq \left[\frac{n+4}{j-1}\right]$.

**Case 2:** If $s = 2 \mod 3$.

**Case 2.1:** If $s = 2 \mod 3$ and $n + 3 = 0 \mod (j - 1)$.

Then $\left[\frac{n+3}{j-1}\right] + 1 = \left[\frac{n+4}{j-1}\right]$. Consider the colouring generated on $K_{j \times s'}$, where $s' = \left[\frac{n+4}{j-1}\right] + 1 = 3q$ where $s' = 3q$, by the matrix $(A)_{3q}$ given below. Note that in this colouring, all the vertices of $B_R$ are partitioned in to 3 cycles,

$$
\begin{pmatrix}
  a_1 & b_1 & 3 & 3 & 3 & 3 & \ldots & j-2 & j-2 & j-2 & j-2 \\
  b_1 & 2 & 2 & 2 & 5 & \ldots & j-3 & j-3 & a_1 & b_1 \\
  1 & 1 & 1 & 4 & 4 & \ldots & j-4 & a_1 & b_1 & \ldots \\
  a_2 & b_2 & j+1 & j+1 & j+1 & \ldots & 2j-4 & 2j-4 & 2j-4 & 2j-4 \\
  b_2 & j & j & j & j+3 & \ldots & 2j-5 & 2j-5 & 2j-5 & 2j-5 & a_2 \\
  j-1 & j-1 & j-1 & j+2 & j+2 & \ldots & 2j-6 & a_2 & b_2 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_q & b_q & p-(j-5) & \ldots & p & p & p \\
  b_q & p-(j-4) & \ldots & p & p & p & a_q & b_q \\
  p-(j-3) & \ldots & \ldots & p & p & p & a_q & b_q \\
\end{pmatrix}
$$
where \( p = q(j-2)\) and \( a_i's \) and \( b_i's \) distinct and consists of arbitrary assigned \( 2q \) elements of \( \{ p + 1,p + 2,...,jq \} \). If \( K_{2,m} \) is the largest blue \( K_{2,n} \). Then as \( s' \geq 3 \)

\[
m \leq (j-1)s' - 4 = \left( \frac{n+4}{j} - 1 \right) (j-1) - 4 < n
\]

Therefore, the graph contains no blue \( K_{2,n} \). Hence \( m_j(P_q, K_{2,n}) \geq \left\lceil \frac{n+4}{j-1} \right\rceil \).

**Subcase 2.2:** \( s \equiv 2 \mod 3, j = 1 \mod 3 \) and \( n + 3 \neq 0 \mod j - 1 \).

As defined in subcase 2.1, let \( A = (a_{ij})_{3qj} \), where \( s = 3q+2, r = \left\lfloor \frac{j}{3} \right\rfloor \) and \( p = q(j-2)\). Consider the colouring generated on \( K_{jx^i} \), where \( s = \left\lfloor \frac{n+3}{j-1} \right\rfloor - 1 \) by the matrix \( B = (b_{ij})_{3qj} \) given below where \( p' = \max\{a_{ij}\} + 2 \).

Note that in this colouring, all the vertices of \( B_R \) are partitioned in to 3 cycles except when \( v_{l-1} \) and \( v_{4,i} \) incident to the edge \((v_{l-1}, v_{4,i})\) corresponding to the \( p'-1 \) valued double entry of the matrix.

\[
B = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \ldots & a_{1,j-2} & a_{1,j-1} & a_{1,j} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \ldots & a_{2,j-2} & a_{2,j-1} & a_{2,j} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{s-2,1} & a_{s-2,2} & a_{s-2,3} & a_{s-2,4} & \ldots & a_{s-2,j-2} & a_{s-2,j-1} & a_{s-2,j} \\
p' - 1 & p' & \ldots & p' & \ldots & p' + r - 1 & p' + r - 1 & p' + r - 1 \\
p' + r & p' + r & \ldots & p' + r & p' - 1 & p' + 2r - 1 & p' + 2r - 1 & p' + 2r - 1 \\
\end{pmatrix}
\]

If \( m \) is the largest value such that \( K_{2,m} \) is in \( R_B \). Then as \( j \geq 6 \),

\[
m \leq (j-1)s - 3 = \left( \frac{n+3}{j-1} - 1 \right) (j-1) - 3 < n.
\]

Therefore, we get that the graph contains no blue \( K_{2,m} \). But as \( n + 3 \neq 0 \mod (j - 1) \) we have \( \left\lceil \frac{n+3}{j-1} \right\rceil = \left\lceil \frac{n+4}{j-1} \right\rceil \). Therefore, \( m_j(P_q, K_{2,n}) \geq \left\lceil \frac{n+4}{j-1} \right\rceil \).

**Subcase 2.3:** \( s \equiv 2 \mod 3, j = 2 \mod 3 \) and \( n + 3 \neq 0 \mod (j - 1) \).

As defined in subcase 2.1, let \( A = (a_{ij})_{3qj} \), where \( s = 3q+2, r = \left\lfloor \frac{j}{3} \right\rfloor \) and \( p = q(j-2) \). Consider the colouring generated on \( K_{jx^i} \), where \( s = \left\lfloor \frac{n+3}{j-1} \right\rfloor - 1 \) by the matrix \( D = (d_{ij})_{3qj} \) given below with \( p' = \max\{a_{ij}\} + 2 \).

Note that in this colouring, all the vertices of \( B_R \) are partitioned in to 3 cycles except when \( v_{l-1} \) and \( v_{4,i} \) incident to the edge \((v_{l-1}, v_{4,i})\) corresponding to the \( p'-1 \) valued double entry (i.e. \( p'-1 \) appears exactly twice in the matrix) of the matrix or when \( v_{2,i} \) and \( v_{5,i} \) incident to the edge \((v_{2,i}, v_{5,i})\) corresponding to the \( p' + 2r \) valued double entry of the matrix.
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If $m$ is the largest value of $K_{2,m}$ in $R_B$. Then,

\[
m \leq (j-1)s - 3 = \left(\left\lceil \frac{n+3}{j-1} \right\rceil - 1 \right)(j-1) - 3 < n.
\]

Therefore, we get that the graph contains no blue $K_{2,m}$. Hence, $m_j(P_4,K_{2,n}) \geq \left\lceil \frac{n+3}{j-1} \right\rceil$.

**Case 3:** If $s = 1 \mod 3$.

**Subcase 3.1:** If $s = 1 \mod 3$ and $j = 1 \mod 3$.

As defined in subcase 2.1, let $A = (a_{m\ell})_{3qq}$, where $s = 3q + 1$, $r = \left\lfloor \frac{j}{3} \right\rfloor$ and $p = q(j-2)$.

Consider the colouring generated on $K_{j,s}$, where $s = \left\lceil \frac{n+3}{j-1} \right\rceil - 1$ by the matrix $B = (b_{m\ell})_{(3q+1) \times j}$ given below where $p' = \max\{a_{m\ell}\} + 2$. Note that in this colouring, all the vertices of $B_R$ are partitioned in to 3 cycles except when $v_{1s}$ and $v_{4s}$ incident to the edge $(v_{1s},v_{4s})$ corresponding to the $p'-1$ valued double entry (i.e. $p'-1$ appears exactly twice in the matrix) of the matrix or when $v_{2s}$ and $v_{5s}$ incident to the edge $(v_{2s},v_{5s})$ corresponding to the $p'$ valued double entry of the matrix.

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \ldots & a_{1,j-2} & a_{1,j-1} & a_{1,j} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{s-1,1} & a_{s-1,2} & a_{s-1,3} & a_{s-1,4} & \ldots & a_{s-1,j-2} & a_{s-1,j-1} & a_{s-1,j} \\
p' - 1 & p' & p' + 1 & p' + r & \ldots & p' + r - 1 & p' + r - 1 & p' + r - 1 \\
\end{pmatrix}
\]

If $m$ is the largest value of $K_{2,m}$ in $R_B$. Then,

\[
m \leq (j-1)s - 3 = \left(\left\lceil \frac{n+3}{j-1} \right\rceil - 1 \right)(j-1) - 3 < n.
\]

Therefore, we get that the graph contains no blue $K_{2,m}$. Hence, $m_j(P_4,K_{2,n}) \geq \left\lceil \frac{n+3}{j-1} \right\rceil$.

**Subcase 3.2:** $s = 1 \mod 3$ and $j = 2 \mod 3$.

As defined in case 1, let $A = (a_{i\ell})_{3qq}$, where $s = 3q + 1$, $r = \left\lfloor \frac{j}{3} \right\rfloor$ and $p = q(j-2)$. Consider the colouring generated on $K_{j,s}$, where $s = \left\lceil \frac{n+3}{j-1} \right\rceil - 1$ by the matrix $D = (d_{i\ell})_{(3q+2)q}$ given
below with \( p' = \max\{a_{ij}\} + 2 \). Note that in this colouring, all the vertices of \( B_R \) are partitioned in to 3 cycles except when \( v_{1i}, v_{4i} \) incident to the edge \((v_{1i}, v_{4i})\) corresponding to the \( p'-1 \) valued double entry (i.e. \( p'-1 \) value appears exactly twice) of the matrix.

\[
\begin{pmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,j-2} & a_{1,j-1} & a_{1,j} \\
  a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,j-2} & a_{2,j-1} & a_{2,j} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  a_{s-1,1} & a_{s-1,2} & a_{s-1,3} & a_{s-1,4} & \cdots & a_{s-1,j-2} & a_{s-1,j-1} & a_{s-1,j} \\
  p' - 1 & p' & p' & p' & \cdots & p' + r - 1 & p' + r - 1 & p' + r - 1
\end{pmatrix}
\]

If \( m \) is the largest value of \( K_{2,m} \) in \( R_R \). Then as \( j \geq 6 \),

\[
m \leq (j - 1)s - 3 = \left(\left\lceil \frac{n+3}{j-1} \right\rceil - 1\right)(j - 1) - 3 < n.
\]

Therefore, we get that the graph contains no blue \( K_{2,m} \). Hence, \( m_j(P_4, K_{2,n}) \geq \left\lceil \frac{n+3}{j-1} \right\rceil \).

**Case 4:** If \( s \equiv 0 \mod 3 \).

Consider the colouring generated on \( K_{js}, 0 \), where \( s = \left\lceil \frac{n+3}{j-1} \right\rceil - 1 = 3q \) where \( s' = 3q \), by the matrix \((A)_{3q} \) given below.

Note that in this colouring, all the vertices of \( B_R \) are partitioned in to 3 cycles.

\[
\begin{pmatrix}
  a_1 & b_1 & 3 & 3 & 3 & \cdots & j-2 & j-2 & j-2 \\
  b_1 & 2 & 2 & 2 & 5 & \cdots & j-3 & j-3 & a_1 \\
  1 & 1 & 1 & 4 & 4 & \cdots & j-4 & a_1 & b_1 \\
  a_2 & b_2 & j+1 & j+1 & j+1 & \cdots & 2j-4 & 2j-4 & 2j-4 \\
  b_2 & j & j & j & j+3 & \cdots & 2j-5 & 2j-5 & a_2 \\
  j-1 & j-1 & j-1 & j+2 & j+2 & \cdots & 2j-6 & a_2 & b_2 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
  a_q & b_q & p-(j-5) & \cdots & p & p & p \\
  b_q & p-(j-4) & \cdots & p-1 & p-1 & a_q & a_q \\
  p-(j-3) & \cdots & p-2 & a_q & b_q & & & & &
\end{pmatrix}
\]

where \( p = q(j-2) \) and \( a_i \)'s and \( b_i \)'s distinct and consists of arbitrary assigned \( 2q \) elements of \( \{p+1,p+2,\ldots,jq\} \). If \( K_{2,m} \) is the largest blue \( K_{2,m} \). Then as \( j \geq 6 \),

\[
m \leq (j - 1)s - 4 = \left(\left\lceil \frac{n+3}{j-1} \right\rceil - 1\right)(j - 1) - 4 < n - 1.
\]

Therefore, the graph contains no blue \( K_{2,m} \). Hence, \( m_j(P_4, K_{2,n}) \geq \left\lceil \frac{n+3}{j-1} \right\rceil \).}

**REFERENCES**

Size Multipartite Ramsey Numbers for Small Paths vs. $K_{2,N}$