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On the Non-Linear Diophantine Equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$

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Abstract. In this paper, we discussed all the solutions of non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$, where x, y and z are non-negative integers and proved that these non-linear Diophantine equations have no non-negative integer solution.

Keywords: Diophantine Equations, Exponential Equations, Catalan's Conjecture

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The Diophantine equation plays a major role in number theory. There is no general method to determine that a given Diophantine equation has how many solutions. In [5], Catalan conjectured that the Diophantine equation $a^x - b^y = 1$, where a, b, x and y are non-negative integers under condition min {a, b, x, y}>1 has a unique solution (a, b, x, y) = (3, 2, 2, 3). In [10], Sroysang proved that the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers has a unique solution (x, y, z) = (1, 0, 2). In [1], Acu proved that the Diophantine equation $2^x + 5^y = z^2$, where x, y and z are non-negative integers has only two solutions (x, y, z) = (3, 0, 3) and (2, 1, 3). In [8], B. Sroysang proved that the Diophantine equation $8^x + 19^y = z^2$, where x, y and z are non-negative integers has a unique solution (x, y, z) = (1, 0, 3). In [11], Sroysang proved that the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers, has only three solutions (x, y, z) = (0, 1, 2), (3, 0, 3) and (4, 2, 5). In [9], Sroysang proved that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution. In [3], Burshtein proved that the Diophantine equation $2^a + 7^b = c^2$ when a and b both are odd integers, has no solution. In [2], Burshtein discussed an open problem of Chotchaisthit, on the Diophantine equation $2^x + p^y = z^2$, where p is particular prime and y = 1. In [4], Burshtein also discussed on the Diophantine equation $2^x + p^y = z^2$ for odd prime p and x, y and z are positive integers. In [6], S. Kumar et.al. proved that the Diophantine equation $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ have no non-negative integer solution.

In most of these papers, the authors used theory of congruence and/or Catalan's conjecture [7] to find or to show the non-existence of the solutions of the Diophantine equations of form $p^x + q^y = z^2$.

In this paper, we discussed about the solutions of non-linear Diophantine equations

$$31^x + 41^y = z^2 \tag{1}$$

And $$61^x + 71^y = z^2 \qquad (2)$$

where x, y, and z are non-negative integers and proved that these Diophantine equations has no non-negative integers solution. We have used the Catalan's conjecture to solve these non-linear Diophantine equations.

2. Preliminaries

Proposition 2.1. The Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers under condition min {a, b, x, y} >1 has unique solution (a, b, x, y) = (3, 2, 2, 3).
Proof. See in [1].

Lemma 2.1. The Diophantine equation $1+ 41^y = z^2$, where y and z are non-negative integers, has no non-negative integer solution.
Proof: Here we consider three cases.
Case I. If y = 0. Then $z^2 = 2$, which is impossible.
Case II. If y = 1. Then $z^2 = 42$, which is also impossible.
Case III. If y > 1. Then $z^2 = 1+41^y > 42$.
This gives z > 6. Here min {y, z} > 1, so by Proposition, in this case also no solution.

Lemma 2.2. The Diophantine equation $1 + 71^y = z^2$, where y and z are non-negative integers, has no non-negative integer solution.
Proof: Here we consider three cases.
Case I. If y = 0. Then $z^2 = 2$, which is impossible.
Case II. If y = 1. Then $z^2 = 72$, which is also impossible.
Case III. If y > 1. Then $z^2 = 1+71^y > 72$.
This gives z > 8. Here min {y, z} > 1, so by Proposition, in this case also no solution.

Lemma 2.3. The Diophantine equation $31^x + 1 = z^2$, where x and z are non-negative integers, has no non-negative integer solution.
Proof: Here we consider three cases.
Case I. If x = 0. Then $z^2 = 2$, which is impossible.
Case II. If x = 1. Then $z^2 = 32$, which is also impossible.
Case III. If x >1. Then $z^2 = 31^x + 1 > 32$.
This gives z > 5. Here min {x, z} > 1, by Proposition, in this case also no solution.

Lemma 2.4. The Diophantine equation $61^x + 1 = z^2$, where x and z are non-negative integers, has no non-negative integer solution.
Proof: Here we consider three cases.
Case I. If x = 0. Then $z^2 = 2$, which is impossible.
Case II. If x = 1. Then $z^2 = 62$, which is also impossible.
Case III. If x >1. Then $z^2 = 61^x + 1 > 62$.
This gives z > 7. Here min {x, z} > 1, by Proposition 2.1, in this case also no solution.

3. Main results

Theorem 3.1. The non-linear Diophantine equation $31^x + 41^y = z^2$, where x, y, and z are non-negative integers, have no solution.
Proof: Here we consider three cases.

Case I. If $x = 0$, then by Lemma 2.1, there is no non-negative integer solution.
Case II. If $x \geq 1$and $y = 0$, then by Lemma 2.3, there is no non-negative integer solution.
Case II. If $x \geq 1$and $y \geq 1$, then 31^x and 41^y both are odd. Thus z^2 is even, so z is even. Let $z = 2n$ then $z^2 \equiv 4n^2 \pmod 5$ (3)
where n is non negative integer.
Now $31 \equiv 1 \pmod 5$ and $41 \equiv 1 \pmod 5$, therefore $31^x \equiv 1 \pmod 5$ and $41^y \equiv 1 \pmod 5$. Thus $z^2 = 31^x + 41^y \equiv 2 \pmod 5$. (4)
From Eqn. (3) and (4), we obtain

$$4n^2 \equiv 2 \pmod 5 \text{ implies } n^2 \equiv 3 \pmod 5.$$

This is impossible.
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Corollary 3.1.1. The non-linear Diophantine equation $31^x + 41^y = k^{2t}$, where x, y, and z are non-negative integers, k and t are positive integers, has no solution.
Proof: Let $k^t = z$, then this Diophantine equation becomes $31^x + 41^y = z^2$, which has no solution by Theorem 3.1.

Corollary 3.1.2. The non-linear Diophantine equation $31^x + 41^y = k^{2t+2}$, where x, y, and z are non-negative integers, k and t are positive integers, has no solution.
Proof: Let $k^{t+1} = z$, then this Diophantine equation becomes $31^x + 41^y = z^2$, which has no solution by Theorem 3.1.

Theorem 3.2. The non-linear Diophantine equation $61^x + 71^y = z^2$, where x, y, and z are non-negative integers, have no solution.
Proof: Here we consider three cases.
Case I. If $x = 0$, then by Lemma 2.1, there is no non-negative integer solution.
Case II. If $x \geq 1$and $y = 0$, then by Lemma 2.3, there is no non-negative integer solution.
Case II. If $x \geq 1$and $y \geq 1$, then 61^x and 71^y both are odd. Thus z^2 is even, so z is even. Let $z = 2n$, then $z^2 \equiv 4n^2 \pmod 5$. ... (5)
Where n is non negative integer.
Now $61 \equiv 1 \pmod 5$ and $71 \equiv 1 \pmod 5$, therefore $61^x \equiv 1 \pmod 5$ and $71^y \equiv 1 \pmod 5$. Thus $z^2 = 61^x + 71^y \equiv 2 \pmod 5$. … (6)
From Eqn. (3) & (4), we obtain

$$4n^2 \equiv 2 \pmod 5 \text{ implies } n^2 \equiv 3 \pmod 5.$$

This is impossible.

Corollary 3.2.1. The non-linear Diophantine equation $61^x + 71^y = k^{2t}$, where x, y, and z are non-negative integers, k and t are positive integers, has no solution.
Proof: Let $k^t = z$, then this Diophantine equation becomes $61^x + 71^y = z^2$, which has no solution by Theorem 3.2.

Corollary 3.2.2. The non-linear Diophantine equation $61^x + 71^y = k^{2t+2}$, where x, y, and z are non-negative integers, k and t are positive integers, has no solution.
Proof: Let $k^{t+1} = z$, then this Diophantine equation becomes $61^x + 71^y = z^2$, which has no solution by Theorem 3.2.

4. Conclusion

In this paper we discussed the solution of non linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$ and find that these Diophantine equation have no solution for any non negative integers x, y and z.

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