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Epimorphisms and Ideals of Distributive Nearlattices

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Abstract. Preservation of Images and the inverse images of special types of ideals of a distributive nearlattice under an epimorphism with a condition on its kernel is established.

Keywords: Distributive nearlattice, annihilator ideal, ∝-ideal, 0-ideal

AMS Mathematics Subject Classification (2010): 06B99

1. Introduction

As a generalization of the concept of distributive lattices on one hand and the pseudo complemented lattices on the other, 0-distributive lattices are introduced by Varlet [5]. 0-distributive semi lattices arise as a natural generalization of 0-distributive lattices.0-ideals, annihilator ideals and ∝-ideals are special ideals introduced and studied in 0-distributive lattices by many researchers (see [1,2,3,4]. Some properties of 0- ideals in 0-distributive nearlattice and annulets in a distributive nearattice are studied by [6,7] respectively. Analogously we have these special ideals in distributive nearlattice with 0. Several properties of semi prime ideals in nearlattices and their characterizations are studied by [8]. It is well known that homomorphism and their kernels play an important role in abstract algebra. In this paper our aim is to discuss about preservation of the images and inverse images of these special ideals of a distributive nearlattice under an epimorphism with a condition that its kernel contains the smallest element only.

2. Preliminaries

Following are some basic concepts needed in sequel. By a nearlattice we mean a meet semilattice together with the property that any two elements possessing a common upper bound have the supremum. This is called as upper bound property. A nearlattice S is called distributive if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, provided $(y \vee z)$ exists. A nearlattice S with 0 is called 0-distributive if for all $x, y, z \in S$, with $(x \wedge y) = 0 = (x \wedge z)$ and $y \vee z$ exists imply $x \wedge (y \vee z) = 0$. Of course, every distributive nearlattice S with 0 is 0-distributive. A subset I of a nearlattice S is called a down set if $x \in I$ and for $t \in S$ with $t \leq x$ imply $t \in I$. An ideal I in a nearlattice S is a non-empty subset of S such that it is down set and whenever $a \vee b$ exists for $a, b \in I$ then $a \vee b \in I$. An ideal $I$of a nearlattice S is called is called semi prime ideal if for all $x, y, z \in I, x \wedge y \in I$ and $x \wedge y \in I$ imply $x \wedge (y \vee z) \in I$ provided $y \vee z$ exists. A non-empty subset F of S is called a filter if $t \geq a$, $a \in F$ imply $t \in F$ and whenever $a, b \in F$ then $a \wedge b \in F$. Let S be a nearlattice with 0. An element a^* is called the pseudo-complement of a if $a \wedge a^* = 0$ and if $a \wedge x = 0$ for some $x \in S$, then $x \leq a^*$. For any non-empty subset A of S, the set $A^* = \{x \in S / x \wedge a = 0,$ for each $a \in A\}$ is called annihilator of A. An ideal I in S is called dense in Sif $I^* = \{0\}$. An element $x \in S$ is said to be dense in S if $(x]^* = \{x\}^* = \{0\}$. An ideal I of S is called an α-ideal if $(x]^{**} \subseteq I$ for each $x \in I$.

Throughout this paper S and S' will denote distributive nearlattices with 0 and $0'$ respectively. By a homomorphism (i.e. a nearlattice homomorphism)we mean a mapping $f: S \to S'$ satisfying: (i) $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in S$ (ii) $f(0) = 0'$ and (iii) $f(a \vee b) = f(a) \vee f(b)$ whenever $a \vee b$ exists. The kernel of f is the set $\{x \in S / f(x) = 0'\}$ and we denote it by $Ker\, f$.

Lemma 2.1. If $f: S \to S'$ is an epimorphism, then
(i) For any filter F (ideal I) of S, $f(F)\big(f(I)\big)$ is a filter (Ideal) of S'.
(ii) For any filter F' (ideal I') of S', $f^{-I}(F')(f^{-I}(I'))$ is a filter (ideal) of S.
(iii) $Ker\, f$is an ideal in S.

Proof. (i) Let F be a filter of S. To prove $f(F) = \{f(x): x \in F\}$ is a filter of S'. Let $p, q \in f(F)$. Then $p = f(x)$, $q = f(y)$ for some $x, y \in F$ and $p \wedge q = f(x) \wedge f(y) = f(x \wedge y)$, where$x \wedge y \in F$ as F is a filter of S. Therefore $p \wedge q \in f(F)$. Let $x' \in S'$, $a' \in f(F)$ such that $x' \geq a'$. Since $a' \in f(F)$ we have $a' = f(b)$, for some $b \in F$. As$x' \in S'$ and f is a surjection, we have $x' = f(a)$ for some $a \in S$. Thus $x' \geq a'$ implies $f(a) \geq f(b)$. Therefore $f(a \vee b) = f(a) \vee f(b) = f(a)$. As $b \in F$, we have $a \vee b \in F$. Hence $f(a) \in f(F)$ i.e.$x' \in f(F)$ this proves $f(F)$ is a filter.
(ii) Let F'be a filter of S'. To prove $f^{-I}(F')$ is a filter of S. Let $a, b \in f^{-I}(F')$. Then $f(a), f(b) \in F'$ and F' being a filter we have $f(a) \wedge f(b) \in F'$ i.e.$f(a \wedge b) \in F'$. Hence $a \wedge b \in f^{-I}(F')$. Let $x \in S$ and $t \in f^{-I}(F')$ such that $x \geq t$. Then$f(x) \geq f(t)$. As $t \in f^{-I}(F')$, we have $f(t) \in F'$. As F' is a filter we get $f(x) \in F'$ i.e. $x \in f^{-I}(F')$. Thus $f^{-I}(F')$ is an up- set. Therefore $f^{-I}(F')$ is a filter.
(iii) Let $x, y \in Kerf$. Then $f(x) = f(y) = 0'$. Now $f(x \vee y) = f(x) \vee f(y) = 0' \vee 0' = 0'$. Therefore $x \vee y \in Kerf$. Let $a \in Kerf$ and $x \leq a (x \in S)$. Then $f(a) = 0'$ and $a \vee x = a$. Therefore $f(a \vee x) = f(a)$ i.e. $f(a) \vee f(x) = f(a)$ which implies $f(x) \leq f(a) = 0'$. Therefore $f(x) = 0'$proving$x \in Ker\, f$. Hence $Kerf$ is a down set. This proves that $Kerf$ is an ideal.

3. Epimorphisms and 0-ideals

We begin with the following definitions

Definition 3.1. For any filter F of a nearlattice S with 0, define $0(F) = \{x \in S /\, x \wedge f = 0$, for some $f \in F\}$

Definition 3.2. An ideal I in a nearlattice S is called a 0-ideal if $I = 0(F)$ for some filter F of S.

Theorem 3.3. Let $f: S \to S'$ be an epimorphism. $Ker\ f = \{0\}$, then we have
(i) $f\big(0(F)\big) = 0\big(f(F)\big)$ for any filter F of S.
(ii) $0(F) = 0(G)$ if and only if $0\big(f(F)\big) = 0\big(f(G)\big)$ for any filters F and G of S.
(iii) $f^{-I}\big(0(F')\big) = 0(f^{-I}(F')$ for any filter F' of S'.
Proof: (i) Let F be a filter of S. $x \in f\big(0(F)\big) \Rightarrow x = f(t)$ for some $t \in 0(F)$. Therefore $t \wedge k = 0$ for some $k \in F \Rightarrow f(t \wedge k) = f(0) = 0' \Rightarrow f(t) \wedge f(k) = 0'$ where $f(k) \in f(F)$. Therefore $f(t) \in 0\big(f(F)\big)$ i.e. $x \in 0\big(f(F)\big)$. Thus $f\big(0(F)\big) \subseteq 0\big(f(F)\big)$. Now let $f(x) = x' \in 0\big(f(F)\big)$. Then $x' \wedge y' = 0'$ for some $y' \in f(F)$. $y' \in f(F) \Rightarrow y' = f(y)$ where $y \in F$. Therefore $x' \wedge y' = 0'$ becomes $f(x) \wedge f(y) = 0'$ i.e. $f(x \wedge y) = 0'$ which gives $x \wedge y \in Kerf = \{0\}$ and consequently $x \wedge y = 0$ where $y \in F$ leading to $x \in 0(F)$. Therefore $x' = f(x) \in f\big(0(F)\big)$. Thus $0\big(f(F)\big) \subseteq f\big(0(F)\big)$. Combining both the inclusions we get $0\big(f(F)\big) = f\big(0(F)\big)$.
(ii) First suppose $0(F) = 0(G)$ where F and G are filters of S. Then $f\big(0(F)\big) = f\big(0(G)\big)$. by property (i) we get $0\big(f(F)\big) = 0\big(f(G)\big)$. Conversly suppose $0\big(f(F)\big) = 0\big(f(G)\big)$ for the filters F and G of S. To prove $0(F) = 0(G)$. Let $x \in 0(F) \Rightarrow x \wedge y = 0$ some $y \in F \Rightarrow f(x) \wedge f(y) = 0'$ where $f(y) \in f(F) \Rightarrow f(x) \in 0\big(f(F)\big)$ by hypothesis we get $f(x) \in 0\big(f(G)\big) \Rightarrow f(x) \wedge t' = 0'$ for some $t' \in f(G)$. Let $t' = f(w)$, $w \in G$. Therefore $f(x) \wedge f(w) = 0' \Rightarrow f(x \wedge w) = 0' \Rightarrow x \wedge w \in Ker\ f = \{0\} \Rightarrow x \wedge w = 0$ where $w \in G \Rightarrow x \in 0(G)$. Thus $0(F) \subseteq 0(G)$. On the same lines we can prove $0(G) \subseteq 0(F)$. Thus $0(G) = 0(F)$.
(iii) To prove $f^{-1}\big(0(F')\big) = 0(f^{-1}(F')$ for any filter F' of S'. Let $a \in f^{-1}\big(0(F')\big)$. Then $f(a) \in 0(F')$ gives $f(a) \wedge f(b) = 0'$ for some $f(b) \in F'$. Thus $f(a \wedge b) = 0'$ which implies $a \wedge b \in Ker\ f = \{0\}$. Therefore $a \wedge b = 0$. As $f(b) \in F'$, $b \in f^{-I}(F')$. Thus $a \wedge b = 0$ where $b \in f^{-I}(F')$yields $a \in 0\big(f^{-I}(F')\big)$. Thus $f^{-1}\big(0(F')\big) \subseteq 0(f^{-1}(F')$. Proceeding in the reverse manner we have $0(f^{-1}(F') \subseteq f^{-1}\big(0(F')\big)$. Thus $f^{-1}\big(0(F')\big) = 0(f^{-1}(F')$.

Theorem 3.4. Let $f: S \to S'$ be an epimorphism.If $Ker\ f = \{0\}$, then (i) If K is a 0- deal of S then $f(K)$ is a 0-ideal of S'.
(ii) If $K'$is a 0-ideal of S', then $f^{-1}(K')$ is 0-ideal of S.
Proof. (i) Let K be a 0- deal of S, then $K = 0(F)$ for some filter F in S. Hence by Theorem 3.3(i), $f(K) = f\big(0(F)\big) = 0(f(F)$. As $f(F)$ is a filter in S' (see Lemma 2.1), $f(K)$is a 0- ideal of S.
(ii) Let K' be a 0- ideal of S'. Then $K' = 0(F')$, for some filter F'in S'. Hence by theorem 3.3
(iii) $f^{-1}(K') = f^{-1}\big(0(F')\big) = 0(f^{-1}(F')$. As $f^{-I}(F')$ is a filter in S (see Lemma 2.1), $f^{-I}(K')$ is a 0-ideal of S.

4. Epimorphisms and annihilator ideals

We begin with the following definitions.

Definition 4.1. For any non-empty subset A of S, define $A^* = \{x \in S / x \wedge a = 0,$ for each $a \in A\}$. A^* is called Annihilator of A.

Remarks: (i) If $A = \{a\}$, then $\{a\}^* = (a]^*$.

(ii) A directed above nearlattice with 0 is 0-distributive if and only if A^* is an ideal in S, for any non- empty subset A of S.

Definition 4.2. An ideal I in S is called an annihilator ideal if $I = A^*$ for some nonempty subset A of S or equivalently, $I = I^{**}$

Theorem 4.3. Let $f : S \to S'$ be an epimorphism. If $Ker\ f = \{0\}$, then we have (i) $f(A^*) = (f(A))^*$, for any nonempty subset A of S. (ii) $f^{-1}(B^*) = (f^{-1}(B))^*$ for any nonempty subset B of S'. (iii) $A^* = B^*$ if and only if $(f(A))^* = (f(B))^*$ for any nonempty subsets A and B of S.

Proof. (i) Let A be any nonempty subset of S. Let $f(x) \in f(A^*)$. Then $x \in A^* \Rightarrow x \wedge a = 0$ for each $a \in A \Rightarrow f(x \wedge a) = 0'$ for each $a \in A \Rightarrow f(x) \wedge f(a) = 0'$ for each $f(a) \in f(A) \Rightarrow f(x) \in \left(f(A)\right)^*$. Hence $f(A^*) \subseteq \left(f(A)\right)^*$. Conversely suppose $x' = f(x) \in \left(f(A)\right)^*$ where $x \in S$. Then $f(x) \wedge f(a) = 0'$ for each $f(a) \in f(A)$. But then $f(x \wedge a) = 0'$ implies $x \wedge a \in Ker\ f = \{0\}$ for each $a \in A$. Therefore $x \wedge a = 0$ for each $a \in A$. Thus $x \in A^*$ which gives $x' = f(x) \in f(A^*)$. This shows that $\left(f(A)\right)^* \subseteq f(A^*)$. Combining both the inclusions we get $f(A^*) = (f(A))^*$
(ii) Let B be any nonempty subset of S'. Let $x \in f^{-I}(B^*)$. Then $f(x) \in B^* \Rightarrow f(x) \wedge f(b) = 0'$ for each $f(b) \in B \Rightarrow f(x \wedge b) = 0'$ for each $b \in f^{-I}(B) \Rightarrow x \wedge b \in ker\ f = \{0\}$ for each $b \in f^{-I}(B) \Rightarrow x \wedge b = 0$ for each $b \in f^{-I}(B) \Rightarrow x \in (f^{-I}(B))^*$. Hence $f^{-I}(B^*) \subseteq (f^{-I}(B))^*$. Conversely suppose $x \in (f^{-I}(B))^*$ then $x \wedge b = 0$ for each $b \in f^{-I}(B) \Rightarrow f(x \wedge b) = 0'$ for each $b \in f^{-I}(B) \Rightarrow f(x) \wedge f(b) = 0'$ for each $f(b) \in B \Rightarrow f(x) \in B^* \Rightarrow x \in f^{-I}(B^*)$. Hence $(f^{-I}(B))^* \subseteq f^{-I}(B^*)$. Combining both the inclusions we get $f^{-1}(B^*) = (f^{-1}(B))^*$.
(iii) Let A and B be any two subsets of S. Then $A^* = B^* \Rightarrow f(A^*) = f(B^*) \Rightarrow (f(A))^* = (f(B))^*$ (by (i)). Let $(f(A))^* = (f(B))^*$ Now $x \in A^* \Rightarrow x \wedge a = 0$ for each $a \in A \Rightarrow f(x \wedge a) = 0'$ for each $a \in A \Rightarrow f(x) \wedge f(a) = 0'$ for each $f(a) \in f(A) \Rightarrow f(x) \in \left(f(A)\right)^* \Rightarrow f(x) \in \left(f(B)\right)^* \Rightarrow f(x) \wedge f(b) = 0'$ for each $f(b) \in f(B) \Rightarrow f(x \wedge b) = 0'$ for each $b \in B \Rightarrow x \wedge b \in Ker\ f = \{0\}$ for each $b \in B \Rightarrow x \wedge b = 0$ for each $b \in B \Rightarrow x \in B^*$. This shows that $A^* \subseteq B^*$. Similarly we can show that $B^* \subseteq A^*$. From both the inclusions we get $A^* = B^*$.

In the following theorem we prove that the images and inverse images of annihilator ideals in a distributive nearlattice with 0 under an epimorphism with $\ker f = \{0\}$ are annihilator ideals.

Theorem 4.4. Let $f : S \to S'$ be an epimorphism. If $Ker\ f = \{0\}$, then
(i) For any annihilator ideal I of S, $f(I)$ is an annihilator ideal of S'.
(ii) For any annihilator ideal I' of S', is an annihilator ideal of S', $f^{-I}(I')$ is an annihilator ideal of S.

Proof. (i) Let I be any annihilator ideal of S, then $f(I)$ is an ideal of S' (see Lemma 2.1(i)). Further $I = I^{**} \Rightarrow f(I) = f(I^{**}) \Rightarrow f(I) = \{f(I)\}^{**}$ (By theorem 4.3(i)). This shows that $f(I)$ is an annihilator ideal of S'.

(ii) Let I' be any annihilator ideal of S', then $f^{-I}(I')$ is an ideal in S (See lemma 2.1(ii)). Further $I' = I'^{**} \Rightarrow f^{-I}(I') = f^{-I}(I'^{**}) \Rightarrow f^{-I}(I') = \{f^{-1}(I')\}^{**}$ (By theorem 4.3(i)). This shows that $f^{-I}(I')$ is an annihilator ideal of S.

5. Epimorphisms and ∝-ideals

We begin with the following definitions.

Definition 5.1. An ideal I in S is an ∝-ideal if $\{a\}^{**} \subseteq I$ for each $a \in I$.

Remark 5.2. Every annihilator ideal in S is an∝-ideal.

Now we prove that the images and inverse images of ∝-ideals in a distributive nearlattice with 0 under an epimorphism with ker $f = \{0\}$ are again ∝-ideals.

Theorem 5.3. Let$f: S \to S'$ be an epimorphism. If $Ker\ f = \{0\}$, then we have

(i) If I is an ∝-ideal in$S$, $f(I)$ is an ∝-ideal in S'.

(ii) If I' is an ∝-ideal in S', then $f^{-I}(I')$ is an ∝-ideal in S.

Proof. (i) Let I be an ∝-ideal in S, then $f(I)$ is an ideal in S' (see lemma 2.1(i)). Let $x \in f(I)$. Then $x = f(a)$ for some $a \in I$. As I is an ∝-ideal in S, $\{a\}^{**} \subseteq I$. Hence $f(\{a\}^{**}) \subseteq f(I) \Rightarrow (\{f(a)\})^{**} \subseteq f(I)$ (by theorem 4.3(i)) $\Rightarrow \{x\}^{**} \subseteq I$. Hence $f(I)$ is an ∝-ideal inS'.

(ii) Let I' be an ideal in S' then $f^{-I}(I')$ is an ideal in S (See lemma 2.1(i)). Let $x, y \in S$ such that $\{x\}^{*} = \{y\}^{*}$ and $x \in f^{-I}(I')$ but then $\{x\}^{*} = \{y\}^{*} \Rightarrow \{f(x)\}^{*} = \{f(y)\}^{*}$ (By Theorem 4.3 (iii)). As $f(x) \in I'$ and I' is an ∝-ideal in S', we get $f(y) \in I'$ which means $y \in f^{-I}(I')$. Hence$f^{-I}(I')$ is an ∝-ideal in S.(By [5], Proposition 2.5 (i) and (ii))

Theorem 5.4. Let $f: S \to S'$ be an epimorphism. Then for an ∝-ideal I' in $S', f^{-I}(I')$ is an ∝-ideal in S provided $f^{-I}(\{x'\}^{*})$ is an ∝-ideal in S for any x' in S'.

Proof: Let I' be an ∝-ideal in S'. $f^{-I}(I')$is an ideal of S (See Lemma 2.1(ii)). Let $x, y \in S$ such that $\{x\}^{*} = \{y\}^{*}$ and $x \in f^{-I}(I')$. Let $f(t) \in \{f(x)\}^{*}$ for some $t \in S$. Hence $f(x) \in \{f(t)\}^{*} \Rightarrow x \in f^{-1}(\{f(t)^{*}\})$. By assumption $(f^{-1}\{f(t)^{*}\})$ is an ∝-ideal in S. Thus $\{x\}^{*} = \{y\}^{*}$ and $x \in f^{-1}(\{f(t)^{*}\})$ imply $y \in f^{-1}(\{f(t)^{*}\})$ (By [5], Proposition 2.5 (i) and (ii)). Thus $f(t) \wedge f(y) = 0' \Rightarrow f(t) \in \{f(y)\}^{*}$.This shows that $\{f(x)\}^{*} \subseteq \{f(y)\}^{*}$. Similarly we can show that $\{f(y)\}^{*} \subseteq \{f(x)\}^{*}$. Hence $\{f(x)\}^{*} = \{f(y)\}^{*}$. As $f(x) \in I'$ and $I'$is an ∝-ideal in S',$f(y) \in I'$ (By [5], Proposition 2.5 (i) and (ii)). Thus $y \in f^{-I}(I')$. And the result follows (By [5], Proposition 2.5 (i) and (ii))

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