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On the Duhamel's Solutions to the Null Equations of Incompressible Fluids

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Abstract. The incompressible Navier-Stokes equation with null initial conditions or simply null Navier-Stokes equation was developed from the incompressible Navier-Stokes equations by subtracting the incompressible Navier-Stokes equations evaluated at the initial time, 0, from itself at some future time, t. A solution of the null Navier-Stokes is obtained via Laplace transform valid for a finite time interval to obtain Duhamel's solution. Additionally, by setting the kinematic viscosity to zero the solution becomes a solution for the incompressible null Euler Equations for all t in $[0, \infty)$. Theorem 1 and Lemma 1 shows the methodology to prove Duhamel's solution satisfies both the divergence equation for incompressible fluids and the incompressible null Navier-Stokes momentum equations. Theorem 2 shows how to obtain the incompressible Euler solution by taking the limit of the kinematic viscosity to zero on the Duhamel's Navier Stokes solution vector function. This article demonstrates a clear path of how the solutions for the null Navier-Stokes and null Euler equations are obtained.

Keywords: Duhamel's solutions, Incompressible Navier-Stokes equations, Incompressible Euler equations

AMS Mathematics Subject Classification (2010): 76A02, 35Q30, 76D05, 76N10

1. Introduction

This article was inspired by Duhamel's formulas in equations 19 [3] and 1.5 [4] in Professor Terrence Tao impressive articles. This article develops the concepts and methods of solving the incompressible null Navier-Stokes and the incompressible null Euler equations via the Laplace transform and the Newtonian potential found in [3]. The Eulerian velocity, $u_k(t, X_k(t))$, is usually prescribed in the academic literature [1] as $u_k(t, x_k)$were the spatial Eulerian coordinate is a normal size letters, and spatial time coordinates, x_k, not an explicit function of time (i.e. no time arguments). In this article, the approach is different, we treat both Eulerian (Caps Letters) and Lagrangian (normal size letters) spatial coordinates for the same spatial location and time to represent the same fluid parcel, and since there is no difference between them, therefore both Eulerian and Lagrangian velocities are identical for the same location and time as found in the sampling process of reference [6] but if the times are different they may represent different fluid parcels which may happen to pass through the same location. Therefore, the Eulerian velocities field with Eulerian spatial coordinates are denoted as$u_k(t, X_k(t))$. In this article the Eulerian spatial coordinate, $X_k(t)$, is replaced by the Lagrangian spatial coordinate, i.e.

$u_k(t, X_k(t))|_{X_k(t)=x_k(t,x_{ok})} = u_k(t, x_k(t, x_{ok}))$ where the Lagrangian spatial coordinate argument, $x_k(t, x_{ok})$ is a function of time and has a parameter argument initial location, x_{ok},which is not a function of time and it represents where the fluid parcel crossed some stream line at initial coordinate x_{ok}. Therefore, if Lagrangian and Eulerian spatial coordinate locations are identical,$x_k(t, x_{ok}) = X_k(t)$at a given time t (See reference [6] Section 3), then they maybe not at any other time for that fluid parcel, i.e. typically $x_k(t, x_{ok}) \neq x_k(\tau, x_{ok})$, unless of course,$t = \tau$, or a periodic process is occurring. Partial time differentiation of $x_k(t, x_{ok}) = X_k(t)$ yields

$$u_k^{Lagrangian}(t, x_k(t, x_{ok})) = \frac{\partial\, x_k(t, x_{ok})}{\partial t} = \frac{\partial X_k(t)}{\partial t} = \frac{dX_k(t)}{dt} = u_k^{Eulerian}(t, X_k(t))$$

Both velocity representations are numerically equal. Therefore, the spatial representation used in this article is Lagrangian, while the flow field representation is Eulerian[1]. Following reference [6] each time particular time can be treated as a sampling point along all the streamlines with all the path-lines, which may be composed, of extremely large number of different fluid parcels path-lines crossing points in the streamline as shown by the sampling process in [6], although the notation used in [6] was a vector field notation (upper arrows) and the notation used here is component index notation. In this article we treat every point as a possible intersection of streamlines and path-lines in the flow field, not just a sampling of a single streamline as in [6].

The Navier-Stokes and Euler equations are nonlinear equations, which no exact solution has been found so far for the most general cases. For every time t $\geq$ 0, the divergence of the incompressible fluid flow is given by (Eq. 1)[2].

$$\sum_k \frac{\partial u_k(t, x_k(t, x_{ok}))}{\partial X_k} = 0 \tag{1}$$

The incompressible Navier-Stokes momentum equations are given by (Eq. 2).

$$\frac{\partial u_k(t, x_k(t, x_{ok}))}{\partial t} + \sum_j u_j(t, x_k(t, x_{ok}))\frac{\partial u_k(t, x_k(t, x_{ok}))}{\partial X_j} - \nu\Delta u_k(t, x_k(t, x_{ok})) = -(\frac{\partial \phi(x_k(t, x_{ok}))}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p(x_k(t, x_{ok}))}{\partial X_k}) \tag{2}$$

The field or material derivative is given as

$$\frac{du_k(t, x_k(t, x_{ok}))}{dt} = \frac{\partial u_k(t, x_k(t, x_{ok}))}{\partial t} + \sum_j u_j(t, x_k(t, x_{ok}))\frac{\partial u_k(t, x_k(t, x_{ok}))}{\partial X_j}.$$

where $\frac{d}{dt}$ is the material or field derivative [6], ϕ is the external force potential, p is the pressure, ρ_o is the constant density, and ν is the constant kinematic viscosity (in meter squared per second). Moving the Laplacian term to the left side of the equation to obtain (Eq. 3).

$$\left(\frac{d}{dt} - \nu\Delta\right)u_k(t, x_k(t, x_{ok})) = -\left(\frac{\partial \phi(x_k(t, x_{ok}))}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p(x_k(t, x_{ok})}{\partial X_k}\right) = -\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right) \tag{3}$$

Notice that when t = 0, in (Eq. 2) this equation becomes,

1Note: At this point Eulerian and Lagrangian field velocities are shown to be identical, therefore we use Eulerian field velocities nomenclature to follow the historical nomenclature. This statement is a repetition of a statement in Section 3.3 of reference [6].

2Note: The fluid Eulerian velocity is denoted by normal size letter *u*, except for the fluid velocity with null initial conditions, which is denoted by capital letter *U*. In this article, all fluid velocities are Eulerian, but they have Lagrangian spatial coordinates. The Einstein notation convention is not used in this article. Only variables with indices in the explicit sigma symbol are being sum, the indices always are equal to 1, 2 and 3 but not shown.

$$\frac{\partial u_{oi}}{\partial t} + \sum\nolimits_j u_{oj}\frac{\partial u_{oi}}{\partial X_j} - \nu\Delta u_{oi} = -(\frac{\partial \phi_o}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i})$$

With initial conditions along the streamlines of the velocity field are given by (Eq. 4).

$$u_k(0, x_{ok}) = u_{ok} \tag{4}$$

The Euler momentum equation is obtained by setting the kinematic viscosity to zero in (Eqs. 2 or 3) to obtain (Eq. 5).

$$\frac{du_k\big(t, x_k(t, x_{ok})\big)}{dt} = \frac{\partial u_k}{\partial t} + \sum\nolimits_j u_j \frac{\partial u_k}{\partial X_j} = -\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right) \tag{5}$$

The above equations are valid throughout the fluid, but we will concentrate on the movement or flow of a parcel of fluid moving through or crossing a streamline and the center of the fluid parcel volume, *V(t)*, is $\boldsymbol{x_k(t, x_{ok})}$.

In Section 2, the null Navier-Stokes equations was developed from the incompressible Navier-Stokes equations by subtracting the incompressible Navier-Stokes equations evaluated at the initial time, 0, from itself at some future time, t.

In Section 3 consists of finding the solutions to the null Navier-Stokes equations via Laplace transform. Although we will check the Duhamel's solution satisfy the null Navier-Stokes equations with the understanding the time dependent nonlinear terms are nulled out. Section 3 contains two theorems and a lemma, which proves the Duhamel's solutions do indeed, solves both Navier-Stokes equations and Euler equations with the understanding the time dependent nonlinear terms are nulled out for incompressible fluids, although, at the expense of practical applications. This article shows a clear path of how the solution is obtained, but the Duhamel's solution does not contribute to a solution which includes the nonlinear part of a field derivative, since it actually zeros or nulls out the nonlinear time dependent terms (see Appendix A). The Duhamel's solution, which solves the null Navier Stoke equations, was found via Laplace transforms to be a convolution integral as shown by (Eq. 6)

$$U_k\big(t, x_k(t, x_{ok})\big) = u_k\big(t, x_k(t, x_{ok})\big) - u_{ok} = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} d\tau \tag{6}$$

where, Δ is the Laplacian operator with respect to the spatial Eulerian coordinates,X_k,(i.e. $\Delta \equiv \Delta_{\vec{X}}$)of a fluid parcel. Both symbols Δ and $\Delta_{\vec{X}}$are used interchangeably. The Lagrangian coordinate $x_k(t, x_{ok})$in the argument of the Eulerian fluid velocity,$u_k\big(t, x_k(t, x_{ok})\big)$,is the center coordinates of the fluid parcel at time t with spherical control volume, $V(t)$,where as $x_k(\tau, x_{ok})$ in $W_k\big(\tau, x_k(\tau, x_{ok})\big)$ is the Lagrangian spatial coordinate center of other fluid parcel at other previous time τin the spherical control volume, $V(\tau)$. Both may have different spatial locations and different times; therefore, they **may** represent different parcels of fluids even though they may have started from the same location, x_{ok}. This description of x_{ok} can be visualized as a movement of an inserted tiny drop at $x_{ok}$of colored fluid with the same density as the rest of the fluid, at first the tiny drop is concentrated in a very small space and then it moves with the currents as time passes. The kernel,$K(t, \tau) = (e^{(t-\tau)\nu\Delta} - 1)$, is the kernel operator which is also a transfer function of time, and the Laplacian operator which "operates" on the input vector function $\frac{\partial W_k(\tau, x_k(\tau, x_{ok}))}{\partial \tau}$. The kernel operator acts as a momentum averaging and diffusion effect. The vector function $W_k\big(\tau, x_k(\tau, x_{ok})\big)$,is given by (Eq. 7) a Newton potential of the external forces and pressure gradient divided by constant density, ρ_o, within the spherical control volume$V(\tau)$, centered at coordinate,$x_k(\tau, x_{ok})$.

$$W_k\big(\tau, x_k(\tau, x_{ok})\big) = \iiint_{V(\tau)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{ok})-Y_n)^2}} \frac{dY}{\nu} \;+ \psi_k\big(\tau, x_k(\tau, x_{ok})\big) \quad (7)$$

The vector function $W_k\big(\tau, x_k(\tau, x_{ok})\big)$is not unique since adding any arbitrary vector function, $\psi_k\big(\tau, x_k(\tau, x_{ok})\big)$, with the property that it's partial time derivative satisfies the Laplace equation, will also be a solution since $\Delta \frac{\partial}{\partial t}\psi_k = 0$. Note that $dY = dY_1 dY_2 dY_3$ in (Eqs. 7& 9) are the sides of the differential volume (cube). The non-unique Duhamel solution to the incompressible null Euler equations for all times, $t \geq 0$, is given by (Eq. 8).

$$U_k^{Euler}\big(t, x_k(t, x_{ok})\big) = u_k^{Euler}\big(t, x_k(t, x_{ok})\big) - u_{ok} = -\int_0^t \left[(t-\tau)\Delta_{\vec{X}} \frac{\partial W_k^{Euler}(\tau, x_k(\tau, x_{ok}))}{\partial \tau}\right] d\tau \quad (8)$$

The vector function$W_k^{Euler}(\tau, X_k)$ is given by (Eq. 9) below.

$$W_k^{Euler}\big(\tau, x_k(t, x_{ok})\big) = \iiint_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{ok})-Y_n)^2}} dY + \psi_k^{Euler}\big(\tau, x_k(\tau, x_{ok})\big) \quad (9)$$

The vector function $W_k^{Euler}\big(\tau, x_k(\tau, x_{ok})\big)$ is not unique since adding any arbitrary vector function, $\psi_k^{Euler}\big(\tau, x_k(\tau, x_{ok})\big)$, with the property that its partial time derivative satisfies the Laplace equation, will also be a solution since $\Delta_{\vec{X}} \frac{\partial}{\partial \tau}\psi_k^{Euler} = 0$.

Sections 4 and 5 will discuss whether the kernel operator can zero out the vector function $\frac{\partial W_k(\tau, x_k(\tau, x_{ok}))}{\partial \tau}$, and meaning of the kernel operating on vector function $\frac{\partial W_k(\tau, x_k(\tau, x_{ok}))}{\partial \tau}$, respectively.

In Appendix A, the Duhamel's solution is shown to have the property to zero out the nonlinear time dependent part of the material or field derivative operator.

$$U_k\big(t, x_k(t, x_{ok})\big) = u_k\big(t, x_k(t, x_{ok})\big) - u_{ok} = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1) \frac{\partial W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} d\tau$$

That is $U_k\big(t, x_k(t, x_{ok})\big)$ satisfies the following equation, which shows fluid velocity vector$U_j\big(t, x_k(t, x_{ok})\big)$ is in the null space of matrix$\frac{\partial U_k(t, x_k(t, x_{ok}))}{\partial X_j}$ along a streamline.

$$\sum_j U_j\big(t, x_k(t, x_{ok})\big) \frac{\partial U_k\big(t, x_k(t, x_{ok})\big)}{\partial X_j} = 0$$

See Section 2, Section 3, and Appendix A for a direct proof and more details on the proof. The Duhamel's solutions do not meet the criteria of Clay millennium prize [11] due to its finite duration of the solution's validity in time and not treating the nonlinear terms adequately (i.e. not zero them out), whereas the Clay prize requires a full Navier-Stokes solution with finite energy for all time [11]. Although, we hope the Duhamel's solution might lead to the full nonlinear solution to the Navier-Stokes incompressible equations or at least a better understanding of what is required to solve these equations.

There have been many claims to solve the incompressible Navier-Stokes equation, since they are too many space won't permit their inclusion in this article[3]. Others have claimed to obtain the

[3]Note: I am not claiming a general solution, but a specialized solution achieved when the Navier-Stokes equations have zero initial conditions for the flow field, the pressure gradient, and the external force. They

first integral of the incompressible Navier-Stokes equation (for example [2,6]). Scholle et al. [2] wrote the first integral of the Navier-Stokes in terms of a tensor potential method. The tensor potential method was used to obtain the first integral of Navier-Stokes equations is based on the use of classical Maxwell electromagnetics techniques. The highlight of this method is to change the Navier-Stoke PDE to a linear diffusion non-homogeneous equation. Unfortunately, in page 6, equation 27 (1st below),the tensor potential depends on the fluid velocities, which are supposed to be given as curl of a vector potential,$u_i = \varepsilon_{ijn}\partial_j \Psi_n$, but this nonlinearity of the equation is ignored in page 7 second paragraph as the equation is treated as non-homogeneous linear diffusive equation in equation 28 (2ndbelow). (Notice: Below U is the external force potential used in [2], and Einstein summation convention is active for only the following two equations since Scholle et al. used it in [2]).

$$\rho u_i u_j + (p+U)\delta_{ij} = -\partial_k \partial_k \tilde{a}_{ij} \rightarrow^{corrected} \rho(\varepsilon_{ijn}\partial_j \Psi_n)(\varepsilon_{jmn}\partial_m \Psi_n) + (p+U)\delta_{ij} = -\partial_k \partial_k \tilde{a}_{ij}(\rho, \varepsilon_{ijn}\Psi_n \varepsilon_{jmn}\Psi_n, p, U)$$

$$\rho \partial_t \Psi_n - \eta \partial_k \partial_k \Psi_n = -\varepsilon_{nkl}\partial_k \partial_m \tilde{a}_{ml} \rightarrow^{corrected} \rho \partial_t \Psi_n - \eta \partial_k \partial_k \Psi_n = -\varepsilon_{nkl}\partial_k \partial_m \tilde{a}_{ml}(\rho, \varepsilon_{ijn}\partial_j \Psi_n \varepsilon_{jmn}\partial_m \Psi_n, p, U)$$

These equations are a nonlinear diffusion partial differential equation and there is no reduction of the (quadratic)nonlinearity as claimed in page 7 due to $\varepsilon_{ijn}\partial_j \Psi_n \varepsilon_{jmn}\partial_m \Psi_n$ input argument of $\tilde{a}_{ml}$. This problem can be resolved by the assumption of zero initial conditions, since the neglecting of these nonlinear terms in the null Navier-Stokes is allowed as shown in Appendix A. Yet, an additional statement needs to be made in reference to variational principles involving the Navier-Stokes equations found in Section C Self-Adjoint form of [2] in the first paragraph which needs to be clarified:

> "In terms of the latter, it was Millikan who showed the non-existence of a Lagrangian, in terms of the velocity u_i, the pressure p and their first order derivatives, that would enable the NS equations to be written as Euler-Lagrange equations."

This statement gives the impression of authority but obviously, Millikan paper is of a steady state nature and is an incorrect predictor because the incompressible Navier-Stokes equations has already been written in Euler-Lagrangian form [5], and the Bernoulli Principle for incompressible viscous fluids (i.e. first Navier-Stokes momentum equation integral) was obtain in an insight in [5], and mathematical proven in [6]. Other than the above comments, the article [2] by Scholle et al. gives excellent examples and illustrations on how to solve the Navier-Stokes equations numerically exactly with particular boundary conditions.

2. The null Navier-Stokes equations

In this section, most equations will not be numbered since every equation follows from the previous in a logical way, and almost every step of the proof will be provided so anyone with the understanding of calculus, Laplace transform, and some knowledge of the Dirac delta function (see reference [7]) will easily follow these steps.

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are many claims of solving the Navier-Stokes equations, and it is amazing that not a single one of them will correctly test the solution into the Navier-Stokes equations to see that it does indeed satisfy the Navier-Stokes equations. I do recommend seeing the other claimed solutions in the literature.

The divergence of the incompressible fluid velocity is given as (Eq. 1),

$$\sum_k \frac{\partial u_k\big(t, x_k(t, x_{ok})\big)}{\partial X_k} = 0$$

By setting time to zero, we obtain the divergence of the initial fluid velocity,

$$\sum_k \frac{\partial u_k(0, x_{ok})}{\partial X_k} = \sum_k \frac{\partial u_{ok}}{\partial X_k} = 0$$

Subtracting the bottom equation from the top, results in

$$\sum_k \frac{\partial (u_k - u_{ok})}{\partial X_k} = 0$$

Define $U_k = u_k - u_{ok}$ so the initial conditions are null, $U_k|_{t=0} = u_{ok} - u_{ok} = 0$. The divergence equation for the null initial conditions fluid velocity is

$$\sum_k \frac{\partial U_k}{\partial X_k} = 0 \tag{10}$$

The incompressible Navier-Stokes momentum equations are given by (Eq. 2)

$$\frac{\partial u_k}{\partial t} + \sum_j u_j \frac{\partial u_k}{\partial X_j} - \nu \Delta u_k = -(\frac{\partial \phi}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_k})$$

In particular if we apply to the Navier-Stokes momentum equation to initial conditions, but using the same flow field u_j in the material derivative, we obtain

$$\frac{\partial u_{ok}}{\partial t} + \sum_j u_{oj} \frac{\partial u_{ok}}{\partial X_j} - \nu \Delta u_{ok} = -(\frac{\partial \phi_o}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_k})$$

Subtracting both sets of equations[4], after making some algebraic manipulation, the added cross terms below keeps the equation balanced,

$$\frac{\partial (u_k - u_{ok})}{\partial t} + \sum_j (u_j - u_{oj}) \frac{\partial (u_k - u_{ok})}{\partial X_j} + \sum_j (u_{oj} \frac{\partial u_k}{\partial X_j} + u_j \frac{\partial u_{ok}}{\partial X_j}) - 2 \sum_j (u_{oj} \frac{\partial u_{ok}}{\partial X_j})$$
$$- \nu \Delta (u_k - u_{ok}) = -\left(\frac{\partial \phi}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_k}\right) + (\frac{\partial \phi_o}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_k})$$

Neglect the terms, $\sum_j (u_{oj} \frac{\partial u_k}{\partial X_j} + u_j \frac{\partial u_{ok}}{\partial X_j}) - 2 \sum_j (u_{oj} \frac{\partial u_{ok}}{\partial X_j})$, in order to simplify the equations.

Define the null Navier-Stokes equation as the Navier-Stokes equations with null initial conditions as shown below in (Eq. 11).

$$\frac{\partial U_k}{\partial t} + \sum_j U_j \frac{\partial U_k}{\partial X_j} - \nu \Delta U_k = -\left(\frac{\partial \phi}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_k}\right) + (\frac{\partial \phi_o}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_k}) \tag{11}$$

The null Navier-Stokes momentum equation are the incompressible Navier-Stokes equations if $u_{ok} \equiv 0, and\ p_o, \phi_o$ are constantsor zero. The null Navier- Stokes equations are given as

- Fluid velocity null initial conditions

$$U_k\big(0, x_k(0, x_{ok})\big) = 0$$

- Incompressibility of the fluid

$$\sum_k \frac{\partial U_k\big(t, x_k(t, x_{ok})\big)}{\partial X_k} = 0$$

- Null Navier-Stokes Momentum Equations

[4] Serrin employed a similar difference of Navier-Stokes equations in Section 72 Uniqueness of viscous flows page 251 [1].

$$\frac{\partial U_k\big(t,x_k(t,x_{ok})\big)}{\partial t}+\sum\nolimits_j U_j\big(t,x_k(t,x_{ok})\big)\frac{\partial U_k\big(t,x_k(t,x_{ok})\big)}{\partial X_j}-\nu\Delta U_k\big(t,x_k(t,x_{ok})\big)$$
$$=-\left(\frac{\partial\tilde{\phi}}{\partial X_k}+\frac{1}{\rho_o}\frac{\partial\tilde{p}}{\partial X_k}\right)$$

Where$U_k\big(t,x_k(t,x_{ok})\big)=u_k\big(t,x_k(t,x_{ok})\big)-u_{ok},\tilde{\phi}=\phi\big(x_k(t,x_{ok})\big)-\phi_o$ and $\tilde{p}=p\big(x_k(t,x_{ok})\big)-p_o$.

3.0 The Duhamel's solutions for the null equations of incompressible fluids

Two theorems and a lemma, will prove the Duhamel's solutions do indeed, solves the null Navier-Stokes &null Euler equations with the understanding the nonlinear time dependent terms are nulled out for incompressible fluids, although, at the expense of practical applications. This article shows a clear path of how the solution is obtained via Laplace transform of the null Navier-Stokes equations. Therefore, the null equations resemble the incompressible Navier-Stokes equations with zero initial conditions and the understanding the nonlinear time dependent terms are nulled out for incompressible fluids.

Theorem 1. Duhamel's solution of the incompressible null Navier-Stokes equations

If the Duhamel's formula is defined as the following convolution[5] integral

$$U_k\big(t,x_k(t,x_{ok})\big)=u_k\big(t,x_k(t,x_{ok})\big)-u_{ok}=-\int_0^t(e^{(t-\tau)\nu\Delta}-1)\frac{\partial W_k\big(\tau,x_k(\tau,x_{ok})\big)}{\partial\tau}d\tau$$

then modified Duhamel's formula solves the null incompressible Navier-Stokes equations (Eq. 10 and 11) and remain valid solutions for time, t, in the following approximated finite time interval $[0,\frac{R^2}{3\nu})$. The vector function $W_k\big(\tau,x_k(\tau,x_{ok})\big)$are not unique since adding any arbitrary vector function, $\psi_k\big(\tau,x_k(\tau,x_{ok})\big)$, with the property that it's partial time derivative satisfies the Laplace equation, will also be a solution since $\Delta\frac{\partial}{\partial t}\psi_k=0$.

$$W_k\big(\tau,x_k(\tau,x_{ok})\big)$$
$$=\iiint\limits_{V(\tau)}\left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k)+\frac{p(Y_k)}{\rho_o}\right)\right\}\frac{-1}{4\pi\sqrt{\sum_n(x_n(\tau,x_{on})-Y_n)^2}}\frac{dY}{\nu}$$
$$+\psi_k\big(\tau,x_k(\tau,x_{ok})\big)$$

And$dY=dY_1dY_2dY_3$are the sides of the differential cube volume.Additionally, the spatial vector argument of the flow field,$x_k(\tau,x_{ok})$, is the Lagrangian coordinates of the fluid parcel's center at time,τ,within an arbitrary sized spherical control volume, $V(\tau)$, at fixed time, τ.

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[5] The solution represents a time convolution of a volume integral of a fluid parcel of volume, $V(\tau)$, with time Kernel operator $(e^{(t-\tau)\nu\Delta}-1)$. But the alternative representation due to the convolution theorem may not provide an easy way to verify the solution to the null Navier-Stokes equations (probably will need integration by parts, etc.), therefore the kernel will be treated as a function of two variables, $K(t,\tau)=(e^{(t-\tau)\nu\Delta}-1)$, rather than a function of a single variable, $(t-\tau)$.

Proof of Theorem 1. **Duhamel's solution of the incompressible null Navier-Stokes equations**
Part 0: Set up, finding Duhamel's solution via Laplace Transforms
The null Navier-Stokes equations solution will hopefully resemble the full Navier-Stokes equation solution. The Navier-Stokes equation for the null initial conditions for the fluid velocity is

$$\frac{\partial U_i}{\partial t}+\sum_j U_j\frac{\partial U_i}{\partial X_j}-\nu\Delta U_i=-\left(\frac{\partial\phi}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial p}{\partial X_i}\right)+\left(\frac{\partial\phi_o}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i}\right)$$

Notice, now at time zero, both sides of the equal sign are null or zero. Treat the $\nu\Delta$ operator as a constant as Oliver Heaviside would have done. Given the solution satisfies this requirement, take the Laplace transform to obtain,

$$\int_0^\infty e^{-st}\frac{\partial U_i}{\partial t}dt+\int_0^\infty e^{-st}\sum_j U_j\frac{\partial U_i}{\partial X_j}dt-\int_0^\infty e^{-st}\,\nu\Delta U_i dt$$
$$=\int_0^\infty e^{-st}[-\left(\frac{\partial\phi}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial p}{\partial X_i}\right)+(\frac{\partial\phi_o}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i})]dt$$

where the Laplace transforms are given by,

- $\int_0^\infty e^{-st}\frac{\partial U_i}{\partial t}dt=s\widehat{U}_i$With null initial conditions
- $\int_0^\infty e^{-st}\sum_j U_j\frac{\partial U_i}{\partial X_j}dt=\sum_j\frac{\partial}{\partial X_j}\int_0^\infty e^{-st}U_jU_i\,dt=\sum_j\frac{1}{2\pi i}\frac{\partial}{\partial X_j}\int_{\gamma-i\infty}^{\gamma+i\infty}\widehat{U}_j(s-\beta,x_k(s-\beta,x_{ok}))\,\widehat{U}_i\,(\beta,x_k(\beta,x_{ok}))d\beta$
- $-\int_0^\infty e^{-st}\,\nu\Delta U_i dt=-\nu\Delta\int_0^\infty e^{-st}\,U_i dt=-\nu\Delta\widehat{U}_i$
- $-\int_0^\infty e^{-st}\left(\frac{\partial\phi}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial p}{\partial X_i}\right)dt=-(\frac{\partial\vartheta}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial P}{\partial X_i})$
- $\int_0^\infty e^{-st}\,(\frac{\partial\phi_o}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i})dt=\frac{1}{s}(\frac{\partial\phi_o}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i})$

where

- $-\int_0^\infty e^{-st}\frac{\partial\phi}{\partial X_i}dt=-\frac{\partial}{\partial X_i}\int_0^\infty e^{-st}\,\phi dt=-\frac{\partial\vartheta}{\partial X_i}$
- $-\int_0^\infty e^{-st}\frac{1}{\rho_o}\frac{\partial p}{\partial X_i}dt=-\frac{1}{\rho_o}\frac{\partial}{\partial X_i}\int_0^\infty e^{-st}\,pdt=-\frac{1}{\rho_o}\frac{\partial P}{\partial X_i}$

Therefore, we have a nonlinear integral partial differential equation with respect to the Eulerian components in (Eq. 12). Solving this equation and inverting will provide the methods to solve the incompressible Navier-Stokes momentum equation (Eq. 2)[6].

$$(s-\nu\Delta)\widehat{U}_i+\sum_j\frac{1}{2\pi i}\frac{\partial}{\partial X_j}\int_{\gamma-i\infty}^{\gamma+i\infty}\widehat{U}_j\big(s-\beta,x_k(s-\beta,x_{ok})\big)\widehat{U}_i\big(\beta,x_k(\beta,x_{ok})\big)d\beta$$
$$=-\frac{1}{s}\{s\left(\frac{\partial\vartheta}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial P}{\partial X_i}\right)-\left(\frac{\partial\phi_o}{\partial X_i}+\frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i}\right)\}\qquad(12)$$

6 The complex convolution $\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\widehat{U}_j(s-\beta,x_k(s-\beta,x_{ok}))\,\widehat{U}_i\,(\beta,x_k(\beta,x_{ok}))d\beta$ can be evaluated with the help of the Bromwich contour (not shown) and the complex residue theorem using perturbation or asymptotic methods[8].

To simplify this equation even further we require the solution, $U_j \neq 0$, and $\frac{\partial U_i}{\partial X_j} \neq 0$, to satisfy $\sum_j U_j \frac{\partial U_i}{\partial X_j} = 0$, i.e. U_j is in the null space of $\frac{\partial U_i}{\partial X_j}$(see Appendix A for proof.)

$$\sum_j U_j \frac{\partial U_i}{\partial X_j} = 0 \xrightarrow{LT} \sum_j \frac{1}{2\pi i}\frac{\partial}{\partial X_j} \int_{\gamma - i\infty}^{\gamma + i\infty} \widehat{U}_j(s-\beta, x_k(s-\beta, x_{ok}))\, \widehat{U}_i\,(\beta, x_k(\beta, x_{ok}))d\beta = 0.$$

Thus,

$$(s - \nu\Delta)\widehat{U}_i = -\frac{1}{s}1\{s\left(\frac{\partial\vartheta}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial P}{\partial X_i}\right) - \left(\frac{\partial\phi_o}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i}\right)\}$$

Dividing by $(s - \nu\Delta)$ and multiplying by $1\left(=\frac{\nu\Delta}{\nu\Delta}\right)$.

$$\widehat{U}_i = -\frac{\nu\Delta}{s(s-\nu\Delta)}\frac{1}{\nu\Delta}\{s\left(\frac{\partial\vartheta}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial P}{\partial X_i}\right) - \left(\frac{\partial\phi_o}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i}\right)\}$$

Notice the Laplace transform of the exponential operator $e^{t\nu\Delta}$is $\frac{1}{(s-\nu\Delta)}$.

$$\widehat{U}_i = -\frac{\nu\Delta}{s(s-\nu\Delta)}\{s\frac{1}{\nu\Delta}\left(\frac{\partial\vartheta}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial P}{\partial X_i}\right) - \frac{1}{\nu\Delta}\left(\frac{\partial\phi_o}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i}\right)$$

The Laplace transform of the kernel operator is factored out by partial fractions below.

$$\frac{\nu\Delta}{s(s-\nu\Delta)} = \left(\frac{1}{(s-\nu\Delta)} - \frac{1}{s}\right)$$

Plugging the expansion in

$$\widehat{U}_i\big(s, x_k(s, x_{ok})\big) = -\left(\frac{1}{(s-\nu\Delta)} - \frac{1}{s}\right)\{s\frac{1}{\nu\Delta}\left(\frac{\partial\vartheta}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial P}{\partial X_i}\right) - \frac{1}{\nu\Delta}\left(\frac{\partial\phi_o}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_i}\right)\}$$

Inverting, using the convolution theorem, to obtain, now complex variable s is a partial time derivative, to easily obtain,

$$U_i\big(t, x_k(t, x_{ok})\big) = u_i\big(t, x_k(t, x_{ok})\big) - u_{oi} = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial}{\partial\tau}\{\frac{1}{\nu\Delta}\left(\frac{\partial\phi}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_i}\right)\}d\tau$$

Or

$$u_i\big(t, x_k(t, x_{ok})\big) = u_{oi} - \int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_i\big(\tau, x_i(\tau, x_{ok})\big)}{\partial\tau}d\tau$$

Where,

$$W_k\big(\tau, x_k(\tau, x_{ok})\big) = \frac{1}{\nu\Delta}\left\{\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right)\right\} = \frac{1}{\nu\Delta}f\big(x_k(\tau, x_{ok})\big) = P\big(\tau, x_k(\tau, x_{ok})\big)$$

The meaning of the inverse Laplacian operator is the well-known Newton's potential integral operator, $P\big(\tau, x_k(\tau, x_{ok})\big)$, as Professor Terrence Tao pointed out in equation 15 of [3] which has been modified to integrate only over the moving fluid parcel or moving fluid volume, $V(\tau)$, with Lagrangian coordinate center, $x_k(\tau, x_{ok})$, at time τ, rather than through all three dimensional space, $\mathbb{R}^3$, which would cause a cancellation of the terms since, $\mathbb{R}^3$, is independent of time and the origin is arbitrary.

The moving volume or fluid parcel depends on a dummy integration time variable, τ, but the spatial coordinates, Y_k, are the dummy integration coordinate points inside the volume, $V(\tau)$, with center at $x_k(\tau, x_{ok})$ as shown in (Eq. 13).

$$P\big(\tau, x_k(\tau, x_{ok})\big) = \frac{1}{\nu\Delta} f\big(x_k(\tau, x_{ok})\big) = \iiint_{V(\tau)} f(Y_k) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on}) - Y_n)^2}} \frac{dY}{\nu} \tag{13}$$

Where for simplicity and space consideration, $dY = dY_1 dY_2 dY_3$ are the sides of the differential cube volume.

$$u_k\big(t, x_k(t, x_{ok})\big) = u_{ok} - \int_0^t (e^{(t-\tau)\nu\Delta} - 1) \frac{\partial W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} d\tau$$

Thus, $\frac{1}{\nu\Delta}$ is the Newtonian potential given in [3] with some minor modifications already discussed. The application of (Eq. 13) to set $f\big(x_k(\tau, x_{ok})\big)$ to $\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right)$ gives rise to $W_k\big(\tau, x_k(\tau, x_{ok})\big)$ (i.e. the potential *P*).

$$W_k\big(\tau, x_k(\tau, x_{ok})\big) = \iiint_{V(\tau)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu} + \psi_k\big(\tau, x_k(\tau, x_{ok})\big)$$

The vector function $W_k\big(\tau, x_k(\tau, x_{ok})\big)$ is not unique since adding any arbitrary vector function, $\psi_k\ \big(\tau, x_k(\tau, x_{ok})\big)$, with the property that its partial time derivative satisfies the Laplace equation, will also be a solution since $\Delta_{\vec{X}} \frac{\partial}{\partial t} \psi_k = 0$.

Although, the kernel of the solution can be obtained via Laplace transforms, there are many other representations that could have been chosen. Alternatives kernels could have been obtained such as different products of $1 = \frac{s}{s} = \frac{\sqrt[2]{s}}{\sqrt[2]{s}} = \frac{\sqrt[3]{s}}{\sqrt[3]{s}}, ect.$,

$$\frac{\nu\Delta}{s(s-\nu\Delta)} s \frac{1}{\nu\Delta} = \frac{\nu\Delta}{\sqrt[2]{s}(s-\nu\Delta)} \sqrt[2]{s} \frac{1}{\nu\Delta} = \frac{\nu\Delta}{\sqrt[3]{s}(s-\nu\Delta)} \sqrt[3]{s} \frac{1}{\nu\Delta} = \frac{1}{s^2\left(\frac{1}{\nu\Delta} - \frac{1}{s}\right)} s \frac{1}{\nu\Delta}\ ect.$$

The Inverse Laplace Transform (ILT) of these operators is shown below by use of fractional calculus techniques [9] and tables found in [8]. These operator kernels are quite complicated and will not be pursued further other than the simplest one below (Note that $\frac{1}{\Delta}$ is the Newton potential operator [3]).

$$\left(\frac{1}{(s-\nu\Delta)} - \frac{1}{s}\right)\left(\frac{s}{\nu\Delta} F\big(x_k(s, x_{ok})\big) - \frac{1}{\nu\Delta} f(x_{ok})\right) \xrightarrow{ILT} \int_0^t d\tau (e^{(t-\tau)\nu\Delta} - 1) \frac{\partial}{\partial \tau} \left\{\frac{1}{\nu\Delta} f\big(x_k(\tau, x_{ok})\big)\right\}$$

Bessel representation, kernel operator, is intriguing where J_1 is a Bessel function order 1 found in formula 18, pg. 244 of [8]. Recall Δ is the Laplacian operator.

$$\frac{1}{s^2\left(\frac{1}{\nu\Delta}-\frac{1}{s}\right)}\left(\frac{s}{\nu\Delta}F\big(x_k(s,x_{ok})\big)\right.$$
$$\left.-\frac{1}{\nu\Delta}f(x_{ok})\right) \xrightarrow{ILT} \sqrt{\nu\Delta t}\int_0^\infty \frac{dq}{\sqrt{q}}J_1(2\sqrt{q\nu\Delta t})e^{q\nu\Delta t}\frac{\partial}{\partial q}\ \{\frac{1}{\nu\Delta}f\big(x_k(q,x_{ok})\big)\};$$

The fractional integral and derivative of ½ order [9] representation, kernel operator.

$$\frac{\nu\Delta}{\sqrt[2]{s}(s-\nu\Delta)}\sqrt[2]{s}\frac{1}{\nu\Delta}F\big(x_k(s,x_{ok})\big) \xrightarrow{ILT} \nu\Delta\int_0^t \frac{d\tau e^{(t-\tau)\nu\Delta}}{\sqrt{\pi(t-\tau)}}\frac{\partial}{\partial\tau}\int_0^\tau \frac{dq}{\sqrt{\pi(\tau-q)}}\{\frac{1}{\nu\Delta}f\big(x_k(q,x_{ok})\big)\}$$

And fractional integral of 1/3 order convolve with a fractional derivative of 1/3 order [9] representation, kernel operator.

$$\frac{\nu\Delta}{\sqrt[3]{s}(s-\nu\Delta)}\sqrt[3]{s}\frac{1}{\nu\Delta}F\big(x_k(s,x_{ok})\big) \xrightarrow{ILT} \frac{\nu\Delta}{\Gamma(\frac{1}{3})}\int_0^t d\tau(t-\tau)^{\frac{1}{3}-1}e^{(t-\tau)\nu\Delta}\frac{\partial}{\partial\tau}\int_0^\tau \frac{dq}{\Gamma(\frac{1}{3})}(\tau$$
$$-q)^{\frac{1}{3}-1}\{\frac{1}{\nu\Delta}f\big(x_k(q,x_{ok})\big)\}.$$

The first and simplest kernel was selected since it would be easier to show it satisfies the null Navier-Stokes equations. The kernel contains a positive exponential Laplacian operator; therefore, it will be referred as the Laplacian exponential operator. We will derive some properties of this exponential operator using the Taylor series representation.

The infinite series represents the Taylor series of the exponential operator, $e^{t\nu\Delta}$, inside the kernel is given as

$$e^{t\nu\Delta}=\sum_{i=0}^{\infty}\frac{1}{\Gamma(i+1)}t^i\nu^i\Delta^i$$

The Laplace transform of the exponential operator $e^{t\nu\Delta}$is $\frac{1}{(s-\nu\Delta)}=\frac{1}{s}\frac{1}{\left(1-\frac{\nu\Delta}{s}\right)}$. The ratio $\frac{1}{\left(1-\frac{\nu\Delta}{s}\right)}$ can be expanded as a series of $\frac{\nu\Delta}{s}$ using the geometric series representation as long as $\left|\frac{\nu\Delta}{s}\right|<1$. The Heaviside integral operator, $T=\frac{d^{-1}}{dt^{-1}}, is\ the\ Laplace\ Inverse\ of\ \frac{1}{s}$. If Heaviside integral operator, T, and the Laplacian operator need to satisfy the following inequality $|\nu T\Delta|<1$, then the inequality $|T|<\frac{1}{\nu|\Delta|}$ means the solution representation is only valid for time periods $T=\frac{d^{-1}}{dt^{-1}}1=\int_0^t d\tau\ 1=t<\frac{1}{\nu|\Delta|}\cong\frac{R^2}{3\nu}$. Where $\frac{1}{\nu|\Delta|}$was modeled using Newton's potential[7] volume integral to be *physically* approximately proportional to spherical surface area of an equivalent spherical volume of the fluid due to the simple pole at 1 of $\frac{1}{\left(1-\frac{\nu\Delta}{s}\right)}$. Therefore, the trial solution is valid for finite times, $t\,\epsilon\ [0,\frac{R^2}{3\nu})$. So, for R= 1 m, ν for 20 ^{o}C water$\sim10^{-6}\frac{m^2}{s}$; $\frac{1}{\nu|\Delta|}\approx$ 90 $hours$, for R=10 m, then the time validity is 100 times greater ~ 9,000 hours > 1 year by using 1 significant digit in the calculation. This does not represent a blowout of the solution, but simply the accuracy of the solution goes away if time is beyond the specified limit.

7 $\frac{1}{\nu|\Delta|}=\frac{1}{4\pi\nu}|\iiint_{sphere}\frac{-1}{R}dV|\approx\frac{1}{4\pi}\frac{1}{\nu R}\frac{4\pi R^3}{3}=\frac{R^2}{3\nu}\ i.e. volume\ of\ a\ sphere\ with\ a\ constant\ radius, R;$

Additionally, it is easily seen the Laplacian operator and the exponential operator commute,

$$\nu\Delta e^{t\nu\Delta} = e^{t\nu\Delta}\nu\Delta$$

Thus, let time-spatial function $F_k = F_k\big(t, x_k(t, x_{ok})\big)$, be applied an operator as follows

$$\begin{aligned}
\nu\Delta e^{t\nu\Delta}F_k &= \nu\Delta\sum_{i=0}^{\infty}\frac{1}{\Gamma(i+1)}t^i\nu^i\Delta^iF_k \\
&= \nu\Delta\left(F_k + t^1\nu^1\Delta^1F_k + \frac{1}{2}t^2\nu^2\Delta^2F_k + \cdots + \frac{1}{\Gamma(i+1)}t^i\nu^i\Delta^iF_k + \cdots\right) \\
&= (\nu\Delta F_k + t^1\nu^2\Delta^2F_k + \frac{1}{2}t^2\nu^3\Delta^3F_k + \cdots + \frac{1}{\Gamma(i+1)}t^i\nu^{i+1}\Delta^{i+1}F_k + \cdots) \\
&= \left(1 + t^1\nu^1\Delta^1 + \frac{1}{2}t^2\nu^2\Delta^2 + \cdots + \frac{1}{\Gamma(i+1)}t^i\nu^i\Delta^i + \cdots\right)\nu\Delta F_k = e^{t\nu\Delta}\nu\Delta F_k
\end{aligned}$$

Using the same technique with the material derivative operators

$$\frac{d}{dt}(e^{t\nu\Delta}F_k) = \frac{\partial}{\partial t}(e^{t\nu\Delta})F_k + e^{t\nu\Delta}\frac{d}{dt}F_k$$

since the Laplacian operator, Δ, has no time dependence, i.e. a constant in time.

$$\begin{aligned}
\frac{\partial}{\partial t}(e^{t\nu\Delta}) = \frac{\partial}{\partial t}\sum_{i=0}^{\infty}\frac{1}{\Gamma(i+1)}t^i\nu^i\Delta^i &= \frac{\partial}{\partial t}\left(1 + t^1\nu^1\Delta^1 + \frac{1}{2}t^2\nu^2\Delta^2 + \cdots + \frac{1}{\Gamma(i+1)}t^i\nu^i\Delta^i + \cdots\right) \\
&= (0 + \nu^1\Delta^1 + t^1\nu^2\Delta^2 + \cdots + \frac{1}{\Gamma(k)}t^{i-1}\nu^i\Delta^i + \cdots) \\
&= (1 + t^1\nu^1\Delta^1 + \frac{1}{2}t^2\nu^2\Delta^2 + \cdots + \frac{1}{\Gamma(i+1)}t^i\nu^i\Delta^i + \cdots)\nu\Delta = e^{t\nu\Delta}\nu\Delta
\end{aligned}$$

Using the Leibniz's rule for differentiation [10], the Laplacian operator,$\Delta_{\vec{X}}$, can commute with the time integral, $\int_0^t K(t,\tau)F_k(\tau, x_k(\tau, x_{ok}))d\tau$ for any function $F_k(\tau, x_k(\tau, x_{ok}))$ and any kernel $K(t,\tau)$.

$$\Delta_{\vec{X}}\int_0^t K(t,\tau)F_k\big(\tau, x_k(\tau, x_{ok})\big)d\tau = \Delta_{\vec{X}}\text{Limit}_{n\to\infty}\sum_{i=0}^{n}K(t,\tau_i)F_k\big(\tau_i, x_k(\tau, x_{ok})\big)\,(\tau_{i+1} - \tau_i)$$

Laplacian $\Delta_{\vec{X}}$ operator commutes with the limit (it does not depend on n or time) and finite series of n

$$\Delta_{\vec{X}}\int_0^t K(t,\tau)F_k\big(\tau, x_k(\tau, x_{ok})\big)d\tau = \text{Limit}_{n\to\infty}\sum_{i=0}^{n}K(t,\tau_i)\Delta_{\vec{X}}F_k\big(\tau_i, x_k(\tau, x_{ok})\big)\,(\tau_{i+1} - \tau_i)$$

Taking the limit as n goes to infinity shows the Laplacian operator commutes

$$\Delta_{\vec{X}}\int_0^t K(t,\tau)F_k(\tau, x_k(\tau, x_{ok}))d\tau = \int_0^t K(t,\tau)\Delta_{\vec{X}}F_k(\tau, x_k(\tau, x_{ok}))d\tau.$$

We need also to review the Leibniz rule [10] for the field or material derivative operating on a time integral on the variable time t. (I forgot to include this property in examples section of [6]).

$$\frac{d}{dt}\int_0^t K(t,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau = \mathrm{Limit}_{h\to 0}\frac{1}{\mathrm{h}}\Big(\int_0^{t+h} K(t+h,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau - \int_0^t K(t,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau\Big)$$

Notice, the definition of the limit is on the variable time t only. In order to evaluate this limit add 0.

$$\mathrm{Limit}_{h\to 0}\frac{1}{\mathrm{h}}\Big(\int_0^{t+h} K(t+h,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau - \int_0^t K(t+h,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau\Big) + \mathrm{Limit}_{h\to 0}\frac{1}{\mathrm{h}}\Big(\int_0^{t} K(t+h,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau - \int_0^t K(t,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau\Big)$$

This simplifies to

$$\mathrm{Limit}_{h\to 0}\frac{1}{\mathrm{h}}\Big(\int_0^{t+h} K(t+h,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau - \int_0^t K(t+h,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau\Big) + \int_0^t \mathrm{Limit}_{h\to 0}\frac{K(t+h,\tau)-K(t,\tau)}{h}F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau$$

Thus, using l'Hospital's rule [10] on the first limit, and the definition of the partial time derivative [10] on the second limit to obtain (Eq. 14)

$$\frac{d}{dt}\int_0^t K(t,\tau)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau = K(t,t)F_k\big(t,x_k(\tau,x_{ok})\big) + \int_0^t \frac{\partial K(t,\tau)}{\partial t}F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau \quad (14)$$

Notice if there is no explicit time dependence of variable t in the integrand kernel, then $\frac{\partial K(t,\tau)}{\partial t} = 0$, as expected, this was used in reference [6] with $K(t,\tau) = 1$, but not explicitly stated.
In particular if the Kernel, $K(t,\tau)$, is given as $(e^{(t-\tau)\nu\Delta} - 1)$ then $K(t,t) = 0$ and $\frac{\partial K(t,\tau)}{\partial t} = e^{(t-\tau)\nu\Delta}\nu\Delta$

$$\frac{d}{dt}\int_0^t (e^{(t-\tau)\nu\Delta} - 1)F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau = (e^{(0)\nu\Delta} - 1)F_k\big(t,x_k(\tau,x_{ok})\big) + \int_0^t e^{(t-\tau)\nu\Delta}\nu\Delta F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau = \int_0^t e^{(t-\tau)\nu\Delta}\nu\Delta F_k\big(\tau,x_k(\tau,x_{ok})\big)d\tau$$

Part 1. Proof of (Eq. 1): Divergence of the Duhamel solution flow field is incompressible (Eq. 10)
Let $G(Y_k) = \phi(Y_k) + \frac{p(Y_k)}{\rho_o}$, then

$$U_k\big(t,x_k(t,x_{ok})\big) = u_k\big(t,x_k(t,x_{ok})\big) - u_{ok} = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial}{\partial\tau}\Big[\iiint_{V(\tau)} \frac{\partial}{\partial Y_k}G(Y_k)\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on}) - Y_n)^2}}\frac{dY}{\nu}\Big]d\tau$$

The divergence of the equation is given by

$$\sum_k \frac{\partial U_k}{\partial X_k} = \sum_k \frac{\partial u_k}{\partial X_k} - \frac{\partial u_{ok}}{\partial X_k}$$
$$= -\sum_k \frac{\partial}{\partial X_k} \int_0^t (e^{(t-\tau)\nu\Delta}$$
$$- 1)\{\frac{\partial}{\partial \tau}[\iiint_{V(\tau)} \frac{\partial}{\partial Y_k} G(Y_k) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu}]\}d\tau \quad = 0$$

It is well known that the divergence of the initial field must be zero, [by setting t=0 in (Eq. 1)].

$$\sum_k \frac{\partial u_{ok}}{\partial X_k} = 0$$

Thus,

$$-\int_0^t (e^{(t-\tau)\nu\Delta} - 1) \sum_k \frac{\partial}{\partial X_k} \{\frac{\partial}{\partial \tau}[\iiint_{V(\tau)} \frac{\partial}{\partial Y_k} G(Y_k) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu}]\}d\tau \quad = 0$$

Thus, notice the partials derivative operators commute, thus within the integrand curly brackets

$$\sum_k \frac{\partial}{\partial X_k} \frac{\partial}{\partial \tau} \iiint_{V(\tau)} \frac{\partial}{\partial Y_k} G(Y_k) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu}$$
$$= \frac{\partial}{\partial \tau} \sum_k \frac{\partial}{\partial X_k} \iiint_{V(\tau)} \frac{\partial}{\partial Y_k} G(Y_k) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu}$$

Thus, by integration by parts of the volume integral,

$$\sum_k \frac{\partial}{\partial X_k} \iiint_{V(\tau)} \frac{\partial}{\partial Y_k} G(Y_k) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu}$$
$$= \oint\!\!\!\oint_{S(\tau)} G(Y_k) \sum_k \frac{\partial}{\partial X_k} \frac{\partial}{\partial Y_k} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dS}{\nu}$$
$$- \iiint_{V(\tau)} G(Y_k) \sum_k \frac{\partial}{\partial X_k} \frac{\partial}{\partial Y_k} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu}$$

Notice, $\Delta_{\vec{Y}} = \sum_k \frac{\partial^2}{\partial Y_k^2}$

$$\sum_k \frac{\partial}{\partial X_k}\frac{\partial}{\partial Y_k}\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} = -\sum_k \frac{\partial^2}{\partial Y_k^2}\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}}$$
$$= -\Delta_{\vec{Y}}\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}}$$

Thus, $\frac{\partial}{\partial X_k}\iiint_{V(\tau)}\frac{\partial}{\partial Y_k}G(Y_k)\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on})-Y_n)^2}}\frac{dY}{\nu} =$

$$-\oiint_{S(\tau)} G(Y_k)\sum_k \frac{\partial^2}{\partial Y_k^2}\left\{\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on})-Y_n)^2}}\right\}\frac{dS}{\nu} + \iiint_{V(\tau)} G(Y_k)\sum_k \frac{\partial^2}{\partial Y_k^2}\{\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on})-Y_n)^2}}\}\frac{dY}{\nu}$$

Recall, $x_k(\tau, x_{ok})$ is a center Lagrangian coordinate of the finite spherical volume, $V\ (\tau)$.As time flows, the Lagrangian fluid parcels flow along the path-lines and cross the stream lines as time τ increases. To see how both Eulerian and Lagrangian coordinates and velocities relate to each other please refer to [6] in Section 3.3.Also, it is well known [7] that the Laplacian of $\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on})-Y_n)^2}}$ becomes a Dirac delta function.Since the control volume size is arbitrary; therefore we can shrink the control volume[8] to $V_\varepsilon(\tau) \ll 1$where epsilon is the radius of the sphere, during the evaluation of the volume integral of the delta function as spatial average specified in equation 16 of reference [7].

$As^9\ V\ (\tau)$
$\rightarrow V_\varepsilon(\tau)\ with\ both\ control\ volumes\ have\ Lagrangian\ center\ x_n(\tau, x_{on})\ as\ the\ limit\ \varepsilon$
$$\rightarrow 0, then\ \Delta_{\vec{Y}}\left\{\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}}\right\} \rightarrow \delta^3(x_n(\tau, x_{on}) - Y_n)\delta^3_{n,k} = \delta^3(x_k(\tau, x_{ok}) - Y_k).$$

The Lagrangian coordinate center for $V(\tau)\ or\ V_\varepsilon(\tau)$ is $x_k(\tau, x_{ok})$, since they both are co-centered spheres, thus, the Lagrangian coordinate center, $x_k(\tau, x_{ok})$, is syphon off to yield a time dependent Dirac delta function.

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[8]See section 3 of reference 7, which suggest a novel definition of the Laplacian of 1/r using the spatial average volume, $\frac{4\pi}{3}\varepsilon^3$, concept of the Laplacian where epsilon is the radius of the arbitrary small sized volume sphere of fluid. Although, the nomenclature for the Laplacian operator in [7] is not being followed exactly. Additionally, we will not describe the arguments of the epsilon limit; the reader can see it in [7].

[9]See section 3 of [7] for detailed arguments not being reproduced here in this article. Note, $\delta^3(x_n(\tau, x_{on}) - Y_n)\delta^3_{n,k} = \delta^3\ \ (x_k(\tau, x_{ok}) - Y_k) = \delta\ \ (x_1(\tau, x_{ok}) - Y_1)\delta\ \ (x_2(\tau, x_{ok}) - Y_2)\delta\ \ (x_3(\tau, x_{ok}) - Y_3) = \delta^3(\boldsymbol{x}(\tau, \boldsymbol{x_o}) - \boldsymbol{Y})$ is the style used the academic literature such as reference [7]. Note, $\delta^3_{n,k} = (\frac{\partial Y_n}{\partial Y_k})^3$

$$\sum_k \frac{\partial^2}{\partial Y_k^2} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on}) - Y_n)^2}} = \Delta_{\vec{Y}} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on}) - Y_n)^2}} = \delta^3(x_k(\tau,x_{ok}) - Y_k)$$

Thus, the original volume integral is given by the equation below,

$$\sum_k \frac{\partial}{\partial X_k} \iiint_{V(\tau)} \frac{\partial}{\partial Y_k} G(Y_k) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{on}) - Y_n)^2}} \frac{dY}{\nu}$$

$$= - \oiint_{S_\varepsilon(\tau)} G(Y_k)\delta^3(x_k(\tau,x_{ok}) - Y_k)\frac{dS}{\nu} + \iiint_{V_\varepsilon(\tau)} G(Y_k)\delta^3(x_k(\tau,x_{ok}) - Y_k)\}\frac{dY}{\nu}$$

The surface integral does not contribute since the coordinate center of the fluid parcel, $X_k(\tau) \notin S_\varepsilon(\tau)$, thus $\delta^3(x_k(\tau,x_{ok}) - Y_k) = 0$ because $Y_k \epsilon\, S_\varepsilon(\tau)$.
Thus,

$$- \oiint_{S_\varepsilon(\tau)} G(Y_k)\delta^3(x_k(\tau,x_{ok}) - Y_k)\frac{dS}{\nu} = 0$$

Results in,

$$\frac{\partial}{\partial \tau}\sum_k \frac{\partial}{\partial X_k}\left[\iiint_{V(\tau)} \frac{\partial}{\partial Y_k} G(Y_k) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{ok}) - Y_n)^2}} \frac{dY}{\nu}\right]$$

$$= \frac{\partial}{\partial \tau}[\iiint_{V_\varepsilon(\tau)} G(Y_k)\delta^3(x_k(\tau,x_{ok}) - Y_k)\}\frac{dY}{\nu}] = \frac{1}{\nu}\frac{\partial}{\partial \tau} G\big(x_k(\tau,x_{ok})\big)$$

$$= \frac{1}{\nu}\frac{\partial}{\partial \tau}(G(x_k(\tau,x_{ok}))) = \frac{1}{\nu}\frac{\partial}{\partial \tau}(\phi(x_k(\tau,x_{ok})) + \frac{p\big(x_k(\tau,x_{ok})\big)}{\rho_o}) = 0$$

Since $\phi\big(x_k(\tau,x_{ok})\big)$ and $p\big(x_k(\tau,x_{ok})\big)$ are not explicit functions of time, τ.Therefore, we have shown $\sum_k \frac{\partial}{\partial X_k}\frac{\partial}{\partial \tau}\iiint_{V(t)} \left\{\frac{\partial}{\partial Y_k}\left(\phi + \frac{p}{\rho_o}\right)\right\}\left\{\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{ok}) - Y_n)^2}}\right\}\frac{dY}{\nu} = 0$ as expected, and thus,

$$\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\sum_k \frac{\partial}{\partial X_k}\frac{\partial}{\partial \tau}[\iiint_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau,x_{ok}) - Y_n)^2}}\frac{dY}{\nu} d\tau\,]$$
$$= 0$$

Thus, the divergence of the fluid velocity is zero so the fluid is incompressible, satisfying (Eq. 10), $\sum_k \frac{\partial U_k(t,x_k(\tau,x_{ok}))}{\partial X_k} = 0$ and (Eq. 1) $\sum_k \frac{\partial u_k(t,x_k(\tau,x_{ok}))}{\partial X_k} = 0$.

Part 2: Proof the Duhamel solution satisfies the null Navier-Stokes momentum equations (Eq.11)
The incompressible null Navier-Stokes momentum equations is given by (Eq. 11) and repeated below.

$$\frac{\partial U_k}{\partial t} + \sum_j U_j \frac{\partial U_k}{\partial X_j} - \nu\Delta U_k = \left(\frac{d}{dt} - \nu\Delta\right) U_k\big(t, x_k(t, x_{ok})\big)$$
$$= -\left(\frac{\partial \phi}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_k}\right) + (\frac{\partial \phi_o}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_k})$$

where

$$U_k = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k}{\partial \tau} d\tau$$

Appendix A demonstrates the nonlinear terms add up to zero,
$\sum_j U_j \frac{\partial U_k}{\partial X_j} = 0$ for $U_k = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k}{\partial \tau} d\tau$.
Therefore, the equation can be represented as (including the null time dependent nonlinear terms)

$$\left(\frac{d}{dt} - \nu\Delta\right) U_k\big(t, x_k(t, x_{ok})\big) = -\left(\frac{\partial \phi}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_k}\right) + (\frac{\partial \phi_o}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_k})$$

Plugging in $U_k = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k}{\partial \tau} d\tau$

$$(\frac{d}{dt} - \nu\Delta)\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k}{\partial \tau} d\tau$$
$$= -\int_0^t e^{(t-\tau)\nu\Delta}\nu\Delta\frac{\partial W_k}{\partial \tau} d\tau + \int_0^t (e^{(t-\tau)\nu\Delta}\nu\Delta - \nu\Delta)\frac{\partial W_k}{\partial \tau} d\tau = -\nu\Delta\int_0^t \frac{\partial W_k}{\partial \tau} d\tau$$

By the Leibniz Rule [10] the material derivative becomes a partial time derivative inside the **time** integral as explained in Part 0 (Eq. 14). Although, not noticeable here, the solution nulls out the nonlinear operator terms since the material derivative becomes a partial time derivative as a result of the Leibniz Rule (see Eq. 14 in Part 0 and Appendix A).

Thus, the equation finally simplifies to

$$\left(\frac{d}{dt} - \nu\Delta\right) U_k\big(t, x_k(t, x_{ok})\big) = -\nu\Delta\int_0^t \frac{\partial W_k}{\partial \tau} d\tau = -\left(\frac{\partial \phi}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_k}\right) + (\frac{\partial \phi_o}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p_o}{\partial X_k})$$

Need to prove$-\nu\Delta\int_0^t \frac{\partial W_k}{\partial \tau} d\tau = -\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right) + \frac{\partial}{\partial X_k}\left(\phi_o + \frac{p_o}{\rho_o}\right)$, via by **Lemma 1**.

Lemma 1. If Δ is the Laplacian operator of the spatial coordinates, X_k, then we conjecture

$$-\nu\Delta\int_0^t \frac{\partial W_k}{\partial \tau} d\tau = -\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right) + \frac{\partial}{\partial X_k}\left(\phi_o + \frac{p_o}{\rho_o}\right).$$

Proof:

Given$W_k\big(t, x_k(t, x_{ok})\big) = \frac{1}{\nu\Delta}\left\{\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right)\right\}$ found in Part 0 above.

$$-\nu\Delta\int_0^t \frac{\partial W_k}{\partial \tau} d\tau = -\nu\Delta\int_0^t \frac{\partial}{\partial \tau}[\frac{1}{\nu\Delta}\left\{\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right)\right\}] d\tau$$

Integrating both time integrals, where $x_k(0, x_{ok}) = x_{ok}$

$$-\nu\Delta\Big(W_k\big(t, x_k(t, x_{ok})\big) + \varphi_k\big(x_k(t, x_{ok})\big) - (W_k(0, x_{ok}) + \varphi_k(0, x_{ok})\big)\Big)$$
$$= [-\nu\Delta\frac{1}{\nu\Delta}\left\{\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right)\right\} - \varphi_k\big(x_k(t, x_{ok})\big)]|_{\tau=0}^{t}$$

Thus, $W_k\big(t, x_k(t, x_{ok})\big) = \left\{\frac{1}{\nu\Delta}\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right)\right\}_{\tau=t}$, and $W_k(0, x_{ok}) = \left\{\frac{1}{\nu\Delta}\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right)\right\}_{\tau=0}$ since the arbitrary terms φ_k, cancel out, because they are on both sides of the equal sign. Thus,

$$-\nu\Delta(W_k\big(t, x_k(t, x_{ok})\big) - W_k(0, x_{ok})) = -\nu\Delta\left\{\frac{1}{\nu\Delta}\frac{\partial}{\partial X_k}\left(\phi + \frac{p}{\rho_o}\right)\right\}_{\tau=0}^{t}$$

Thus,

$$\begin{aligned} -\nu\Delta(W_k\big(t, x_k(t, x_{ok})\big) - W_k(0, x_{ok})) \\ = -\nu\Delta \iiint\limits_{V(t)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\}\left\{\frac{-1}{4\pi\sqrt{\sum_n (x_n(t, x_{on}) - Y_n)^2}}\right\}\frac{dY}{\nu} \\ + \nu\Delta \iiint\limits_{V(0)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\}\left\{\frac{-1}{4\pi\sqrt{\sum_n (x_{on} - Y_n)^2}}\right\}\frac{dY}{\nu} \end{aligned}$$

Since $\Delta = \Delta_{\vec{X}}$, moving the Laplacian inside the volume integral,

$$\begin{aligned} -\nu\Delta_{\vec{X}} W_k\big(t, x_k(t, x_{ok})\big) + \nu\Delta_{\vec{X}} W_k(0, x_{ok}) \\ = -\iiint\limits_{V(t)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\}\frac{\nu\Delta_{\vec{X}}}{4\pi}\left\{\frac{-1}{\sqrt{\sum_n (x_n(t, x_{on}) - Y_n)^2}}\right\}\frac{dY}{\nu} \\ + \iiint\limits_{V(0)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\}\frac{\nu\Delta_{\vec{X}}}{4\pi}\left\{\frac{-1}{\sqrt{\sum_n (x_{on} - Y_n)^2}}\right\}\frac{dY}{\nu} \end{aligned}$$

It is well known [7] that $\frac{\Delta_{\vec{X}}}{4\pi}\left\{\frac{-1}{\sqrt{\sum_n (x_n(t, x_{on}) - Y_n)^2}}\right\} = \delta^3(x_k(t, x_{ok}) - Y_k)$, where $x_k(t, x_{ok})$ is the Lagrangian coordinate center of the fluid parcel volume (See Part 1, near Eq. 13).

$$\begin{aligned} -\nu\Delta_{\vec{X}} W_k\big(t, x_k(t, x_{ok})\big) + \nu\Delta_{\vec{X}} W_k(0, x_{ok}) \\ = -\iiint\limits_{V_\varepsilon(t)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\}\delta^3(x_k(t, x_{ok}) - Y_k)dY \\ + \iiint\limits_{V_\varepsilon(0)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\}\delta^3(x_{ok} - Y_k)dY \end{aligned}$$

Thus,

$$\begin{aligned} -\iiint\limits_{V_\varepsilon(t)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\}\delta^3(x_k(t, x_{ok}) - Y_k)dY \\ = -\frac{\partial}{\partial X_k}\left(\phi(x_k(t, x_{ok})) + \frac{p(x_k(t, x_{ok}))}{\rho_o}\right) \end{aligned}$$

and

$$\iiint\limits_{V_\varepsilon(0)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k)+\frac{p(Y_k)}{\rho_o}\right)\right\}\delta^3(x_{ok}-Y_k)dY=\frac{\partial}{\partial X_k}\left(\phi(x_{ok})+\frac{p(x_{ok})}{\rho_o}\right)=\frac{\partial}{\partial X_k}\left(\phi_o+\frac{p_o}{\rho_o}\right)$$

Thus, **Lemma 1** has been proven to be a correct conjecture, thus

$$-\nu\Delta\int_0^t\frac{\partial W_k}{\partial\tau}d\tau=-\nu\Delta_{\vec{X}}W_k\big(t,x_k(t,x_{ok})\big)+\nu\Delta_{\vec{X}}W_k(0,x_{ok})$$
$$=-\frac{\partial}{\partial X_k}\left(\phi\big(x_k(t,x_{ok})\big)+\frac{p\big(x_k(t,x_{ok})\big)}{\rho_o}\right)+\frac{\partial}{\partial X_k}\left(\phi_o+\frac{p_o}{\rho_o}\right)$$

The null Navier-Stokes momentum equations are satisfied (Eq. 11) above.

$$\left(\frac{d}{dt}-\nu\Delta\right)U_k\big(t,x_k(t,x_{ok})\big)=-\nu\Delta\int_0^t\frac{\partial W_k}{\partial\tau}d\tau=$$
$$=-\frac{\partial}{\partial X_k}\left(\phi\big(x_k(t,x_{ok})\big)+\frac{p\big(x_k(t,x_{ok})\big)}{\rho_o}\right)+\frac{\partial}{\partial X_k}\left(\phi_o+\frac{p_o}{\rho_o}\right)$$

Thus far, we have thus shown the divergence of the flow field is incompressible (Eq. 1), $\frac{\partial U_k}{\partial X_k}=0$, the null Navier-Stokes momentum equations (Eq. 11) have been satisfied although by nulling out the nonlinear time dependent terms (see Appendix A). Thus the field velocity which solves the null Navier-Stokes equations is given by $U_k\big(t,x_k(t,x_{ok})\big)=u_k\big(t,x_k(t,x_{ok})\big)-u_{ok}=-\int_0^t(e^{(t-\tau)\nu\Delta}-1)\frac{\partial W_k(\tau,x_k(\tau,x_{ok}))}{\partial\tau}d\tau$ and these equations are approximately valid for finite times, t, $in\left[0,\frac{R^2}{3\nu}\right)$. The vector function $W_k\big(\tau,x_k(\tau,x_{ok})\big)$ is not uniquely identified since $W_k\big(\tau,x_k(\tau,x_{ok})\big)$ are given within any arbitrary vector function, $\psi_k\big(\tau,x_k(\tau,x_{ok})\big)$, which it's partial time derivative satisfies the Laplace equation, $\Delta\frac{\partial}{\partial\tau}\psi_k=0$.

$$W_k\big(\tau,x_k(\tau,x_{ok})\big)$$
$$=\iiint\limits_{V(\tau)}\left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k)+\frac{p(Y_k)}{\rho_o}\right)\right\}\frac{-1}{4\pi\sqrt{\sum_n(x_n(\tau,x_{on})-Y_n)^2}}\frac{dY}{\nu}$$
$$+\psi_k\big(\tau,x_k(\tau,x_{ok})\big)$$

Therefore, the solution is not unique, unless we can make the ***assumption*** to set all components of ψ_k to zero, to obtain after moving the constant kinematic viscosity outside the volume integral,

$$U_k\big(t,x_k(t,x_{ok})\big)$$
$$=-\int_0^t\frac{(e^{(t-\tau)\nu\Delta}-1)}{\nu}\frac{\partial}{\partial\tau}\Bigg[\iiint\limits_{V(\tau)}\left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k)+\frac{p(Y_k)}{\rho_o}\right)\right\}\frac{-1}{4\pi\sqrt{\sum_n(x_n(\tau,x_{on})-Y_n)^2}}dY\Bigg]d\tau$$

Setting time to zero yields

$$U_k\big(0,x_k(0,x_{ok})\big)$$

$$= -\int_0^0 \frac{(e^{(0-\tau)\nu\Delta} - 1)}{\nu} \frac{\partial}{\partial \tau}\left[\iiint_{V(\tau)} \left\{\frac{\partial}{\partial Y_k}\left(\phi + \frac{p}{\rho_o}\right)\right\} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY\right] d\tau = 0$$

Therefore, the initial condition is satisfied trivially.
QED.
An added benefit of the incompressible null Navier-Stokes solution is to obtain the solution to the incompressible null Euler fluid momentum equation by setting the kinematic viscosity to zero as shown in the proof of Theorem 2.

Theorem 2.Duhamel's solution for the incompressible Euler equation
If the null incompressible Navier-Stokes Duhamel solution where the kinematic viscosity, ν, is set to zero via a limit,

$$U_k\big(t, x_k(t, x_{ok})\big)$$
$$= -\int_0^t \frac{(e^{(t-\tau)\nu\Delta} - 1)}{\nu} \frac{\partial}{\partial \tau}[\iiint_{V(\tau)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY] d\tau$$

Then the solution is given as $U_k^{Euler}\big(t, x_k(t, x_{ok})\big) = u_k^{Euler}\big(t, x_k(t, x_{ok})\big) - u_{ok}$then a solution[10] of the incompressible null Euler equations for all times, $t \geq 0$, is obtained, as

$$U_k^{Euler}\big(t, x_k(t, x_{ok})\big)$$
$$= -\int_0^t \left[(t-\tau)\Delta_{\vec{X}} \frac{\partial}{\partial \tau} \iiint_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY\right] d\tau$$

where $dY = dY_1 dY_2 dY_3$ are the sides of the differential cube volume.

Proof of Theorem 2.Duhamel's solution for the null incompressible Euler equation
By taking the limit as the kinematic viscosity goes to zero, $\nu \to 0$, in the Duhamel's solution,

$$U_k\big(t, x_k(t, x_{ok})\big)$$
$$= -\int_0^t \{Limit_{\nu\to 0} \frac{(e^{(t-\tau)\nu\Delta} - 1)}{\nu}\} \frac{\partial}{\partial \tau}[\iiint_{V(\tau)} \frac{\partial}{\partial Y_k}\Big(\phi(Y_k)$$
$$+ \frac{p(Y_k)}{\rho_o}\Big) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY] d\tau$$

By L'Hospital's rule [10], differentiating with respect to ν on the numerator and denominator yields,

$$Limit_{\nu\to 0} \frac{(e^{(t-\tau)\nu\Delta} - 1)}{\nu} = (t-\tau)\Delta_{\vec{X}}$$

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10 Obviously the solution is not unique, therefore the additional arbitrary vector functions $\psi_k^{Euler}\big(\tau, x_k(\tau, x_{on})\big)$ terms are neglected or set to zero.

Plug in the results of the limit, obtains the desired solution

$$U_k^{Euler}\big(t, x_k(t, x_{ok})\big)$$
$$= -\int_0^t \left[(t-\tau)\Delta_{\vec{X}} \frac{\partial}{\partial \tau} \iiint\limits_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY\right] d\tau$$

Part 1. Divergence of Euler Flow Field $\sum_k \frac{\partial}{\partial X_k} U_k^{Euler}$

$$= -\int_0^t \left[(t-\tau)\Delta_{\vec{X}} \frac{\partial}{\partial \tau} \sum\nolimits_k \frac{\partial}{\partial X_k} \iiint\limits_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY\right] d\tau$$
$$= 0$$

By Theorem 1 Part 1-Proof of (Eq. 1): Divergence of the flow field is incompressible, we proved

$$\frac{\partial}{\partial \tau} \sum\nolimits_k \frac{\partial}{\partial X_k} \left[\frac{1}{4\pi} \iiint\limits_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY\right] = 0.$$

Therefore $\sum_k \frac{\partial}{\partial X_k} U_k^{Euler} = 0$.

Part 2. The Duhamel's solution to Null Incompressible Euler Momentum Equation

Since the solution is given as $U_k^{Euler}\big(t, X_k(t)\big) = u_k^{Euler}\big(t, X_k(t)\big) - u_{ok}$

$$U_k^{Euler}\big(t, x_k(t, x_{ok})\big)$$
$$= -\int_0^t \left[(t-\tau)\Delta_{\vec{X}} \frac{\partial}{\partial \tau} \iiint\limits_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY\right] d\tau$$

The null Euler momentum equation is obtained by setting kinematic viscosity to zero, $\nu = 0$, in (Eq. 11)

$$\frac{dU_k^{Euler}\big(t, x_k(t, x_{ok})\big)}{dt} = -\frac{\partial}{\partial X_k}\left(\phi\big(x_k(t, x_{ok})\big) + \frac{p\big(x_k(t, x_{ok})\big)}{\rho_o}\right) + \frac{\partial}{\partial X_k}\left(\phi_o + \frac{p_o}{\rho_o}\right)$$

Plug in the solution into above equation,

$$\frac{dU_k^{Euler}\big(t, x_k(t, x_{ok})\big)}{dt}$$
$$= -\frac{d}{dt}\int_0^t \left[(t-\tau)\Delta_{\vec{X}} \frac{\partial}{\partial \tau} \iiint\limits_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right) \frac{1}{4\pi} \frac{-1}{\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY\right] d\tau$$

By the Leibniz Rule [10] the material derivative becomes a partial time derivative inside the **time** integral. Since $\frac{\partial(t-\tau)}{\partial t} = 1$

$$\frac{dU_k^{Euler}\big(t,x_k(t,x_{ok})\big)}{dt}=-\int_0^t\left[\frac{\partial}{\partial\tau}\iiint\limits_{V(\tau)}\frac{\partial}{\partial Y_k}\left(\phi(Y_k)+\frac{p(Y_k)}{\rho_o}\right)\frac{\Delta_{\vec{X}}}{4\pi}\frac{-1}{\sqrt{\sum_n\left(x_n(\tau,x_{on})-Y_n\right)^2}}dY\right]d\tau$$

From Theorem 1 part 2, $\frac{\Delta_{\vec{X}}}{4\pi}\frac{-1}{\sqrt{\sum_n(x_n(\tau,x_{on})-Y_n)^2}}=\delta^3(x_k(\tau,x_{ok})-Y_k)$. Recall the explanation of this fact in Part 1& 2 and see reference [7].

$$\frac{dU_k^{Euler}\big(t,x_k(t,x_{ok})\big)}{dt}=-\int_0^t\left[\frac{\partial}{\partial\tau}\iiint\limits_{V_\varepsilon(\tau)}\frac{\partial}{\partial Y_k}\left(\phi+\frac{p}{\rho_o}\right)\delta^3(x_k(\tau,x_{ok})-Y_k)dY\right]d\tau$$

Integration the partial derivative, and recall the arbitrary integration functions cancel out,

$$\frac{dU_k^{Euler}\big(t,x_k(t,x_{ok})\big)}{dt}$$

$$=-[\iiint\limits_{V_\varepsilon(t)}\frac{\partial}{\partial Y_k}\left(\phi+\frac{p}{\rho_o}\right)\delta^3(x_k(t,x_{ok})-Y_k)dY-\iiint\limits_{V_\varepsilon(0)}\frac{\partial}{\partial Y_k}\left(\phi+\frac{p}{\rho_o}\right)\delta^3(x_{ok}-Y_k)dY]$$

Using the well known properties of Dirac delta function [7],

$$\frac{dU_k^{Euler}\big(t,x_k(t,x_{ok})\big)}{dt}=-\frac{\partial}{\partial X_k}\left(\phi\big(x_k(t,x_{ok})\big)+\frac{p\big(x_k(t,x_{ok})\big)}{\rho_o}\right)+\frac{\partial}{\partial X_k}\left(\phi_o+\frac{p_o}{\rho_o}\right)$$

Thus, the null Euler Equation is satisfied. The solution of the null Euler equation has the validity for all time t in $\left[0, Limit_{\nu\to 0}\frac{R^2}{3\nu}\right)$ $i.e.\;\; t\in[0,\infty)$.

QED.

4. Does the convolution kernel, $e^{(t-\tau)\nu\Delta}-1$, makes the function W_k null?

It's complicated, although the short answer is no for t > 0. The Taylor series of the operator kernel is given by

$$e^{(t-\tau)\nu\Delta}-1=\sum_{i=0}^{\infty}\frac{1}{\Gamma(i+1)}(t-\tau)^i\nu^i\Delta^i-1$$

$$=\sum_{i=1}^{\infty}\frac{1}{\Gamma(i+1)}(t-\tau)^i\nu^i\Delta^i=((t-\tau)^1\nu^1\Delta^1+\frac{(t-\tau)^2\nu^2\Delta^2}{2}+\cdots$$

$$+\frac{1}{\Gamma(i+1)}(t-\tau)^i\nu^i\Delta^i+\cdots)$$

Therefore, $U_k\big(t,x_k(t,x_{ok})\big)=u_k\big(t,x_k(t,x_{ok})\big)-\;u_{ok}=-\int_0^t(e^{(t-\tau)\nu\Delta}-1)\frac{\partial W_k(\tau,x_k(\tau,x_{ok}))}{\partial\tau}d\tau$

Since

$$W_k\big(\tau,x_k(\tau,x_{ok})\big)$$

$$=\iiint\limits_{V(\tau)}\left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k)+\frac{p(Y_k)}{\rho_o}\right)\right\}\frac{-1}{4\pi\sqrt{\sum_n\left(x_n(\tau,x_{on})-Y_n\right)^2}}\frac{dY}{\nu}\;+\psi_k\big(\tau,x_k(\tau,x_{ok})\big)$$

Plugging in the Taylor series into the integral

$$U_k\big(t, x_k(t, x_{ok})\big) = -\int_0^t ((t-\tau)^1 \nu^1 \Delta^1 + \frac{(t-\tau)^2 \nu^2 \Delta^2}{2} + \cdots + \frac{1}{\Gamma(i+1)} (t-\tau)^i \nu^i \Delta^i + \cdots) \frac{\partial W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} d\tau$$

Note that the $\psi_i\big(\tau, x_k(\tau, x_{ok})\big)$ will cancel out so it will not be included.

$$U_k\big(t, x_k(t, x_{ok})\big) = -\int_0^t ((t-\tau)^1 \nu^1 \Delta^1 \frac{\partial W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} + \frac{(t-\tau)^2 \nu^2 \Delta^2}{2} \frac{\partial W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} + \cdots + \frac{1}{\Gamma(i+1)} (t-\tau)^i \nu^i \Delta^i \frac{\partial W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} + \cdots) \, d\tau$$

Interchanging the Laplacian operator with the partial time derivative on a term-by-term basis to obtain $U_k\big(t, x_k(t, x_{ok})\big) = -\int_0^t ((t-\tau)^1 \nu^1 \frac{\partial \Delta^1 W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} + \frac{(t-\tau)^2 \nu^2}{2} \frac{\partial \Delta^1 \Delta^1 W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} + \cdots + \frac{1}{\Gamma(i+1)} (t-\tau)^i \nu^i \frac{\partial \Delta^{i-1} \Delta^1 W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} + \cdots) \, d\tau$

Taking the Laplacian inside the volume integral (recall Theorem 1 Part 1 & 2), then the ith terms reads (see [7])

$$\frac{1}{\Gamma(i+1)} (t-\tau)^i \nu^i \frac{\partial}{\partial \tau} \Delta^{i-1} \iiint\limits_{V(\tau)} \left\{ \frac{\partial}{\partial Y_k} \left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o} \right) \right\} \frac{\nu \Delta_{\vec{X}}}{4\pi} \left\{ \frac{-1}{\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \right\} \frac{dY}{\nu}$$

$$= \frac{1}{\Gamma(i+1)} (t-\tau)^i \nu^i \frac{\partial}{\partial \tau} \Delta^{i-1} \iiint\limits_{V_\varepsilon(t)} \left\{ \frac{\partial}{\partial Y_k} \left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o} \right) \right\} \delta^3 (x_k(\tau, x_{ok}) - Y_k) dY = \frac{1}{\Gamma(i+1)} (t-\tau)^i \nu^i \frac{\partial}{\partial \tau} \frac{\partial}{\partial X_k} \Delta^{i-1} \left(\left(\phi(x_k(\tau, x_{ok})) + \frac{p(x_k(\tau, x_{ok}))}{\rho_o} \right) \right)$$

The first term, i=1, so $\Delta^{i-1} = 1$

$$\frac{1}{\Gamma(2)} (t-\tau)^1 \nu^1 \frac{\partial}{\partial X_k} \left(\frac{\partial}{\partial \tau} \phi(x_k(\tau, x_{ok})) + \frac{\partial}{\partial \tau} \frac{p(x_k(\tau, x_{ok}))}{\rho_o} \right) = 0$$

There is no explicit time dependence for pressure and external potentials. But the later terms are nonzero. By taking the divergence operator on the Navier-Stokes momentum equations (Eq. 2),

$$\frac{\partial}{\partial t} (\sum_k \frac{\partial u_k}{\partial X_k}) + \sum_{k,j} \frac{\partial u_j}{\partial X_k} \frac{\partial u_k}{\partial X_j} + \sum_j u_j \frac{\partial}{\partial X_j} (\sum_k \frac{\partial u_k}{\partial X_k}) = \nu \Delta \sum_k \frac{\partial u_k}{\partial X_k} - \Delta \left(\phi\big(x_k(t, x_{ok})\big) + \frac{p\big(x_k(t, x_{ok})\big)}{\rho_o} \right)$$

Since, the fluid is incompressible (Eq. 1)

$$\sum_k \frac{\partial u_k}{\partial X_k} = 0$$

Thus, relabeling the dummy sum index k to n,

$$\Delta\left(\phi\big(x_k(t,x_{ok})\big)+\frac{p\big(x_k(t,x_{ok})\big)}{\rho_o}\right)=-\sum\nolimits_{n,j}\frac{\partial u_j}{\partial X_n}\frac{\partial u_n}{\partial X_j}$$

For i$\geq$ 2, the i[th] term reads

$$\frac{1}{\Gamma(i+1)}(t-\tau)^i\nu^i\frac{\partial}{\partial\tau}\frac{\partial}{\partial X_k}\Delta^{i-2}\Delta^1\left(\phi(x_k(\tau,x_{ok}))+\frac{p(x_k(\tau,x_{ok}))}{\rho_o}\right)$$
$$=\frac{-1}{\Gamma(i+1)}(t-\tau)^i\nu^i\frac{\partial}{\partial\tau}\frac{\partial}{\partial X_k}\Delta^{i-2}\sum\nolimits_{n,j}\frac{\partial u_j}{\partial X_n}\frac{\partial u_n}{\partial X_j}\neq 0$$

Plugging it into the solution,

$$U_k\big(t,x_k(t,x_{ok})\big)=\int_0^t\sum_{i=2}^{\infty}\frac{1}{\Gamma(i+1)}(t-\tau)^i\nu^i\frac{\partial}{\partial\tau}\frac{\partial}{\partial X_k}\Delta^{i-2}\sum\nolimits_{n,j}\frac{\partial u_j}{\partial X_n}\frac{\partial u_n}{\partial X_j}d\tau\neq 0$$

Moving the partials inside the finite, n & j sums, notice we change the component index from i to n in the left and right part of the equation, to avoid confusion,

$$U_k\big(t,x_k(t,x_{ok})\big)=\int_0^t\sum_{i=2}^{\infty}\frac{1}{\Gamma(i+1)}(t-\tau)^i\nu^i\Delta^{i-2}\sum\nolimits_{n,j}\frac{\partial^2}{\partial\tau\partial X_k}\{\frac{\partial u_j}{\partial X_n}\frac{\partial u_n}{\partial X_j}\}d\tau\neq 0 \text{ for t } > 0.$$

Results in a *nonlinear* integral equation.

$$U_k\big(t,x_k(t,x_{ok})\big)=u_k\big(t,x_k(t,x_{ok})\big)-u_{ok}$$
$$=\int_0^t\left[e^{(t-\tau)\nu\Delta}-(1+(t-\tau)\nu\Delta)\right]\frac{1}{\Delta^2}\sum\nolimits_{n,j}\frac{\partial^2}{\partial\tau\partial X_k}\left\{\frac{\partial u_j}{\partial X_n}\frac{\partial u_n}{\partial X_j}\right\}d\tau\neq 0 \text{ for t } > 0.$$

5.0 The meaning of the exponential Laplacian operator

The exponential Laplacian operator, $e^{t\Delta}$, operating on some function is equivalent to the solution of the heat equation with the initial condition being the function, according to Professor Terrence Tao in reference [3] in page 39 reads

> "In either $\mathbb{R}^3$ or $\mathbb{R}^3/L\mathbb{Z}^3$, we let $e^{t\Delta}$ for t > 0 be the usual semigroup associated to the heat equation $u_t=\Delta u$. On $\mathbb{R}^3$ this takes the explicit form
>
> $$e^{t\Delta}f(x)=\frac{1}{(4\pi t)^{3/2}}\int_{\mathbb{R}^3}e^{-|x-y|^2/4t}f(y)dy$$
>
> for $f\in L_x^p(\mathbb{R}^3)$ for some $1\leq p\leq\infty$."

Remark: A argument which may hold true, is as follows:

Let $u(t,x)=e^{t\Delta}f(x)$ then $u_t-\Delta u=e^{t\Delta}\Delta f(x)-\Delta e^{t\Delta}f(x)=e^{t\Delta}\Delta f(x)-e^{t\Delta}\Delta f(x)=0$ and $u(0,x)=e^{0\Delta}f(x)=f(x)$, so indeed $e^{t\Delta}f(x)$ behaves as if it's the unique solution of the Heat equation, since it solves it. We will not pursue this connection any further.

6. Conclusion

In Section 2, the null Navier-Stokes equations was developed from the incompressible Navier-Stokes equations by subtracting the incompressible Navier-Stokes equations evaluated at the initial time, 0, from itself at some future time, t.

Section 3 consists of finding the solutions to the null Navier-Stokes equations via Laplace transform. Although we checked the Duhamel's solution indeed satisfies the null Navier-Stokes equations with the understanding the time dependent nonlinear terms are nulled out (see Appendix A). Section 3 contains two theorems and a lemma, which proves the Duhamel's solutions do indeed, solves both Navier-Stokes equations and Euler equations with the understanding the time dependent nonlinear terms are nulled out for incompressible fluids. We have shown the Duhamel's function satisfies the divergence equation for incompressible fluids (Eq. 10) and null incompressible Navier-Stokes momentum equations (Eq. 11) for finite times, t, $in \left[0, \frac{R^2}{3\nu}\right)$. Also the Duhamel's solution nulls out the nonlinear terms of the null Navier-Stokes (see Appendix A) and is given by the following formula,

$$U_k\big(t, x_k(t, x_{ok})\big) = u_k\big(t, x_k(t, x_{ok})\big) - u_{ok} = -\int_0^t (e^{(t-\tau)\nu\Delta_{\vec{X}}} - 1)\frac{\partial W_k\big(\tau, x_k(\tau, x_{ok})\big)}{\partial \tau} d\tau$$

where$x_k(\tau, x_{ok})$is the Lagrangian coordinate center of fluid parcel of volume, $V(\tau)$.

$$W_k\big(\tau, x_k(\tau, x_{ok})\big) = \iiint\limits_{V(\tau)} \left\{\frac{\partial}{\partial Y_k}\left(\phi + \frac{p}{\rho_o}\right)\right\} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu} + \psi_k\big(\tau, x_k(\tau, x_{ok})\big)$$

The vector function $W_k\big(\tau, x_k(\tau, x_{ok})\big)$ is not uniquely identified, since $W_k\big(\tau, x_k(\tau, x_{ok})\big)$ are given within any arbitrary vector function, $\psi_k\big(\tau, x_k(\tau, x_{ok})\big)$, which it's partial time derivative satisfying the Laplace equation. Therefore, the solution is not unique, unless we can make the assumption to set all components of $\psi_k\big(\tau, x_k(\tau, x_{ok})\big)$ to zero. Theorem 1 and Lemma 1 shows the methodology to prove Duhamel's solution satisfies both the null divergence equation for incompressible fluids and the incompressible null Navier-Stokes momentum equations. Theorem 2 shows how to obtain the incompressible Euler solution by taking the limit of the kinematic viscosity to zero on the Duhamel's solution. The Euler solution is given by

$$U_k^{Euler}\big(t, x_k(t, x_{ok})\big) = u_k^{Euler}\big(t, x_k(t, x_{ok})\big) - u_{ok}$$
$$= -\int_0^t \left[(t-\tau)\Delta_{\vec{X}} \frac{\partial}{\partial \tau} \iiint\limits_{V(\tau)} \frac{\partial}{\partial Y_k}\left(\phi + \frac{p}{\rho_o}\right) \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} dY\right] d\tau$$

The Euler solution is valid for all times, $t \geq 0$,where the additional arbitrary vector functions terms, $\psi_k^{Euler}\big(\tau, x_k(\tau, x_{ok})\big)$, are neglected or set to zero.

Due to null initial conditions the field derivative operator in Navier-Stokes momentum equation becomes linearized and the remaining linear operator is as Professor Terrence Tao states "the usual semigroup associated to the heat equation $u_t = \Delta u$" [3]. Therefore, the methods of linear partial differential equations such as Laplace transforms worked even though the field derivative is a nonlinear operator in terms of the field velocities, i.e. $\frac{d}{dt} u_k\big(t, x_k(t, x_{ok})\big) \neq \frac{d}{dt} u_{ok} + \frac{d}{dt} U_k\big(t, x_k(t, x_{ok})\big)$, which is why $u_k\big(t, x_k(t, x_{ok})\big) = u_{ok} + U_k\big(t, x_k(t, x_{ok})\big)$ does not solve the full Navier-Stokes momentum equations.

Sections 4 showed the kernel operator does not zero out the vector function $\frac{\partial W_k(\tau, x_k(\tau, x_{ok}))}{\partial \tau}$.

Section 5described the meaning of the kernel operating on vector function $\frac{\partial W_k(\tau, x_k(\tau, x_{ok}))}{\partial \tau}$ according to Professor Terence Tao.

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7. Appendix A: Solution has null nonlinear terms, $\sum_j U_j \frac{\partial U_k}{\partial X_j} = 0$

Let $U_k\big(t, x_k(t, x_{ok})\big) = u_k\big(t, x_k(t, x_{ok})\big) - u_{ok} = -\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k(\tau, x_k(\tau, x_{ok}))}{\partial \tau} d\tau$ where $W_k\big(\tau, x_k(\tau, x_{ok})\big) = \frac{1}{\nu\Delta}\left(\frac{\partial \phi}{\partial X_i} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_i}\right) = \iiint_{V(\tau)} \left\{\frac{\partial}{\partial Y_i}\left(\phi(Y_i) + \frac{p(Y_i)}{\rho_o}\right)\right\} \frac{-1}{4\pi\sqrt{\sum_n (x_n(\tau, x_{on}) - Y_n)^2}} \frac{dY}{\nu}$.

By taking the material or field derivative [6] of the nonlinear terms,

$$\frac{d}{dt}\sum_j U_j\big(t, x_k(t, x_{ok})\big)\frac{\partial U_k(t, x_k(t, x_{ok}))}{\partial X_j}$$

$$= \sum_j \frac{d}{dt}\{U_j\big(t, x_k(t, x_{ok})\big)\}\frac{\partial U_k(t, x_k(t, x_{ok}))}{\partial X_j}$$

$$+ \sum_j U_j\big(t, x_k(t, x_{ok})\big)\frac{d}{dt}\frac{\partial U_i(t, x_k(t, x_{ok}))}{\partial X_j}$$

$$= \sum_j \frac{d}{dt}\Big(-\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_j}{\partial \tau} d\tau\Big)\frac{\partial}{\partial X_j}\Big(-\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k}{\partial \tau} d\tau\Big)$$

$$+ \sum_j \Big(-\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_j}{\partial \tau} d\tau\Big)\frac{d}{dt}\Big(-\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial}{\partial X_j}\frac{\partial W_k}{\partial \tau} d\tau\Big)$$

Cancelling the negative signs, and simplifying, by Leibniz rule of differentiation [10] (see Part 0 Eq. 14) inside the time integral the material derivative becomes the partial time derivative of t, $\frac{d}{dt}\Big(\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial}{\partial X_j}\frac{\partial W_k}{\partial \tau} d\tau\Big) = \int_0^t \frac{\partial}{\partial t}\{e^{(t-\tau)\nu\Delta} - 1\}\frac{\partial}{\partial X_j}\frac{\partial W_k}{\partial \tau} d\tau = \int_0^t e^{(t-\tau)\nu\Delta}\nu\Delta\frac{\partial}{\partial X_j}\frac{\partial W_k}{\partial \tau} d\tau$, rearranging the order of partial differentiation and Laplacian operator,

$$\sum_j \Big(\int_0^t e^{(t-\tau)\nu\Delta}\frac{\partial}{\partial \tau}\nu\Delta W_j d\tau\Big)\frac{\partial}{\partial X_j}\Big(\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k}{\partial \tau} d\tau\Big)$$

$$+ \sum_j \Big(\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_j}{\partial \tau} d\tau\Big)\frac{\partial}{\partial X_j}\int_0^t e^{(t-\tau)\nu\Delta}\frac{\partial}{\partial \tau}\nu\Delta W_k\, d\tau = 0$$

since $\frac{\partial}{\partial \tau}\nu\Delta W_k = \frac{\partial}{\partial \tau}\left(\frac{\partial \phi}{\partial X_k} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_k}\right) = 0$ and $\frac{\partial}{\partial \tau}\nu\Delta W_j = \frac{\partial}{\partial \tau}\left(\frac{\partial \phi}{\partial X_j} + \frac{1}{\rho_o}\frac{\partial p}{\partial X_j}\right) = 0$ by proof of Theorem 1 Part 1 in this article. Therefore, $\sum_j U_j \frac{\partial U_k}{\partial X_j}$ is a constant along the stream line. Thus, assume the constant is $C = \sum_j U_j \frac{\partial U_k}{\partial X_j} \neq 0$. But, by setting time to zero, notice $U_j\big(0, x_k(t, x_{ok})\big) = 0$. Therefore, $C = 0$, a contradiction has been obtained, so $\sum_j U_j \frac{\partial U_k}{\partial X_j}$ is zero along a streamline. Thus, $\sum_j U_j \frac{\partial U_k}{\partial X_j} = 0$.

For t > 0, this does not imply $\frac{\partial U_k}{\partial X_j} = 0$, thus

$$\frac{\partial U_k}{\partial X_j} = \frac{\partial}{\partial X_j}\Big(-\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial W_k}{\partial \tau} d\tau\Big) = \Big(-\int_0^t (e^{(t-\tau)\nu\Delta} - 1)\frac{\partial}{\partial \tau}\frac{\partial W_k}{\partial X_j} d\tau\Big) \neq 0$$

Since upon differentiating,

$$\frac{\partial}{\partial X_j} W_k\big(\tau, x_k(\tau, x_{ok})\big) = -\iiint_{V(\tau)} \left\{\frac{\partial}{\partial Y_i}\left(\phi(Y_i) + \frac{p(Y_i)}{\rho_o}\right)\right\} \frac{(x_j(\tau, x_{ok}) - Y_j)}{4\pi(\sum_n (x_n(\tau, x_{on}) - Y_n)^2)^{\frac{3}{2}}} \frac{dY}{\nu} \neq 0$$

For t > 0, since $U_j \neq 0$ (See Section 6) and $\frac{\partial U_i}{\partial X_j} \neq 0$, then this implies U_j is in the null space of matrix $\frac{\partial U_k}{\partial X_j}$.

An alternate method to the demonstration above is shown below field derivative methods found in [6] and using the Leibniz rule [10].

$$\left[\sum_j U_j\big(t, x_k(t, x_{ok})\big) \frac{\partial U_k\big(t, x_k(t, x_{ok})\big)}{\partial X_j}\right]_{fixed\ t} \equiv$$

$$\left(\frac{d}{dt} - \left[\frac{\partial}{\partial t}\right]_{fixed\ \vec{X},\vec{u}} - \left[\sum_i \frac{\partial u_i}{\partial t}\frac{\partial}{\partial u_i}\right]_{fixed\ t}\right) U_k\big(t, x_k(t, x_{ok})\big) = -\frac{d}{dt}\int_0^t \frac{[\![(e]\!]^{(t-\tau)\nu\Delta} - 1)(\partial W_k)}{\partial \tau} d\tau +$$

$$\left[\frac{\partial}{\partial t}\int_0^t [\![\frac{[\![(e]\!]^{(t-\tau)\nu\Delta} - 1)(\partial W_k)}{\partial \tau} d\tau\right]_{fixed\ \vec{X},\vec{u}} + \left[\sum_i \frac{\partial u_i}{\partial t}\frac{\partial}{\partial u_i}\int_0^t \frac{[\![(e]\!]^{(t-\tau)\nu\Delta} - 1)(\partial W_k)}{\partial \tau} d\tau\right]_{fixed\ t}]\!] =$$

$$-\int_0^t e^{(t-\tau)\nu\Delta} \nu\Delta \frac{\partial W_k}{\partial \tau} d\tau + \int_0^t e^{(t-\tau)\nu\Delta} \nu\Delta \frac{\partial W_k}{\partial \tau} d\tau + 0 \equiv 0.$$

This method does not depend on the streamline assumption, but on the definition of the field derivative, therefore this proof is stronger than the previous method.

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