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On the Solvability of the Diophantine Equation $p^x + (p+8)^y = z^2$ when $p > 3$ and $p + 8$ are Primes

Fernando Neres de Oliveira

Department of Science and Technology, Federal Rural Semi-Arid University
Caraúbas Campus, RN, Brazil. E-mail: fernandoneres@ufersa.edu.br

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Abstract. In this paper we prove that the diophantine equation $p^x + (p+8)^y = z^2$ has no solution (x, y, z) in positive integers when $p > 3$ and $p + 8$ are primes.

Keywords: diophantine equation; pairs of prime of the form $(p, p + 8)$; positive integer solutions

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The study of the solvability of diophantine equations is one of the classic problems in elementary number theory and algebraic number theory. In recent years, the authors have published several papers on the solvability of diophantine equations of the type

$$p^x + q^y = z^2 \tag{1}$$

where p and q are distinct prime numbers. In 2016, Rabago [9] proved that the equation $2^x + 17^y = z^2$ has exactly five solutions (x, y, z) in positive integers. The only solutions are $(3,1,5)$, $(5,1,7)$, $(6,1,9)$, $(7,3,71)$ and $(9,1,23)$. In 2017, Asthana and Singh [1] proved that the equation $3^x + 13^y = z^2$ has exactly four solutions (x, y, z) in non-negative integers. The solutions are $(1,0,2)$, $(1,1,4)$, $(3,2,14)$ and $(5,1,16)$. Recently, Burshtein [2] proved that the diophantine equation $p^x + (p+4)^y = z^2$ has no solution (x, y, z) in positive integers when $p > 3$ and $p + 4$ are primes.

In this paper, we investigated the solvability of the diophantine equation (1) when $(p, q) = (p, p + 8)$. We do not know if the list of pairs of primes of the form $(p, p + 8)$ is finite or infinite. The first five pairs of such list are (3,11), (5,13), (11,19), (23,31) and (29,37). Other pairs of primes from this list can be found in A156320 of the OEIS (Online Encyclopedia of Integer Sequences). Using only some basic results on prime numbers and divisibility of integers, we will prove that the diophantine equation $p^x + (p+8)^y = z^2$ (with $p > 3$ and $p + 8$ primes) has no solution (x, y, z) in positive integers. In addition, we show as an immediate consequence of this result, that if $k \geq 2$ is a fixed integer (and arbitrary) then the diophantine equation $p^x + (p+8)^y = w^{2k}$ (with $p > 3$ and $p + 8$ primes) has no solution (x, y, w) in positive integers.

2. Preliminaries

In this section, we will present some basic results that will be useful in the next section.

Lemma 2.1. Suppose that $p > 3$ and $p + 8$ are primes. If (x, y, z) is a solution in positive integers of the diophantine equation $p^x + (p + 8)^y = z^2$ then $4|z^2$.

Proof: We have $p^x + (p + 8)^y = z^2$. Since $p > 3$ and $p + 8$ are primes then p^x and $(p + 8)^y$ are odd. Thus, $p^x + (p + 8)^y$ is even, i.e., z^2 is even. Therefore, $4|z^2$. □

Lemma 2.2. Any power of an integer of the form $4L + 1$ ($L > 0$ integer) is of the form $4A + 1$ ($A > 0$ integer), i.e., for every integer $k \geq 1$, we have $(4L + 1)^k = 4A + 1$, where $A > 0$ is an integer.

Proof: For $k = 1$, we have $(4L + 1)^1 = 4L + 1$. Suppose that for some integer $k \geq 1$, we have $(4L + 1)^k = 4A + 1$, where $A > 0$ is an integer. Hence,

$$(4L + 1)^{k+1} = (4L + 1)^k(4L + 1) = (4A + 1)(4L + 1) = 4(4AL + A + L) + 1.$$

Therefore, it follows from the principle of induction that for every integer $k \geq 1$, we have $(4L + 1)^k = 4A + 1$, where $A > 0$ is an integer. □

Lemma 2.3. Any even power of an integer of the form $4L + 3$ ($L > 0$ integer) is of the form $4A + 1$ ($A > 0$ integer) and any odd power of an integer of the form $4L + 3$ ($L > 0$ integer) is of the form $4B + 3$ ($B > 0$ integer), i.e., for every integer $k \geq 1$, we have

$$(4L + 3)^k = \begin{cases} 4A + 1, \text{ if } k \text{ is even} \\ 4B + 3, \text{ if } k \text{ is odd} \end{cases}$$

where $A > 0$ is an integer and $B > 0$ is an integer.

Proof: For $k = 2$ we have $(4L + 3)^2 = 16L^2 + 24L + 9 = 4(4L^2 + 6L + 2) + 1$. Suppose that for some integer $k = 2q$ ($q \geq 1$) we have $(4L + 3)^{2q} = 4A + 1$, where $A > 0$ is an integer. Hence, we obtain that

$$(4L + 3)^{2q+2} = (4A + 1)(16L^2 + 24L + 9) = 4[A(4L + 3)^2 + 4L^2 + 6L + 2] + 1,$$

and, therefore, follows from the principle of induction that for every integer $k \geq 2$ even, we have $(4L + 3)^k = 4A + 1$, where $A > 0$ is an integer. On the other hand, for $k = 1$ we get $(4L + 3)^1 = (4L + 3)$. Suppose that for some integer $k = 2q + 1$ ($q \geq 0$) we have $(4L + 3)^{2q+1} = 4B + 3$, where $B > 0$ is an integer. Hence, we obtain that

$$(4L + 3)^{2q+3} = (4B + 3)(16L^2 + 24L + 9) = 4[B(4L + 3)^2 + 12L^2 + 18L + 6] + 3,$$

and, therefore, follows from the principle of induction that for every integer $k \geq 1$ odd, we have $(4L + 3)^k = 4B + 3$, where $B > 0$ is an integer. □

3. Main results

Theorem 3.1. If p is any prime of the form $p = 4M + 1$ ($M > 0$) and $p + 8$ is prime then the equation $p^x + (p + 8)^y = z^2$ has no solution in positive integers x, y and z.

Proof: Suppose there are positive integers x, y and z such that $p^x + (p + 8)^y = z^2$. Hence, it follows from Lemma 2.2 that there are positive integers A and B, such that,

$$z^2 = p^x + (p + 8)^y = (4A + 1) + (4B + 1) = 4(A + B) + 2.$$

But, this implies that $4 \nmid z^2$, contradicting the Lemma 2.1. □

Theorem 3.2. Suppose that p is any prime of the form $p = 4M + 3$ $(M > 0)$ and $p + 8$ is prime. If x and y have the same parity then the diophantine equation $p^x + (p + 8)^y = z^2$ has no solution in positive integers x, y and z.
Proof: Suppose there are positive integers $x = 2K$ $(K \geq 1)$, $y = 2S$ $(S \geq 1)$ and z, such that, $p^{2K} + (p + 8)^{2S} = z^2$. Hence, it follows from Lemma 2.3 that there are positive integers A and B, such that,

$$z^2 = p^{2K} + (p + 8)^{2S} = (4A + 1) + (4B + 1) = 4(A + B) + 2.$$

But, this implies that $4 \nmid z^2$, contradicting the Lemma 2.1. Suppose now that there are positive integers $x = 2K + 1$ $(K \geq 0)$, $y = 2S + 1$ $(S \geq 0)$ and z, such that, $p^{2K+1} + (p + 8)^{2S+1} = z^2$. Hence, it follows from Lemma 2.3 that there are positive integers A and B, such that,

$$z^2 = p^{2K+1} + (p + 8)^{2S+1} = (4A + 3) + (4B + 3) = 4(A + B + 1) + 2.$$

But, this implies that $4 \nmid z^2$, contradicting the Lemma 2.1. □

Theorem 3.3. Suppose that p is any prime of the form $p = 4M + 3$ $(M > 0)$ and $p + 8$ is prime. If x is even and y is odd then the diophantine equation $p^x + (p + 8)^y = z^2$ has no solution in positive integers x, y and z.
Proof: Suppose there are positive integers $x = 2K$ $(K \geq 1)$, $y = 2S + 1$ $(S \geq 0)$ and z, such that, $p^{2K} + (p + 8)^{2S+1} = z^2$, or equivalently, such that,

$$(p + 8)^{2S+1} = (z + p^K)(z - p^K).$$

Thus, it follows from the primality of $p + 8$ that $z + p^K = (p + 8)^\alpha$ and $z - p^K = (p + 8)^\beta$, where $\alpha > \beta \geq 0$ are integers such that $\alpha + \beta = 2S + 1$. Hence,

$$2p^K = (p + 8)^\alpha - (p + 8)^\beta = (p + 8)^\beta[(p + 8)^{\alpha-\beta} - 1]. \qquad (2)$$

Note that, $\beta = 0$ (otherwise, we would conclude from (2) that $p + 8$ divides 2 or $p + 8$ divides p (both impossible)). This implies that

$$2p^K = (p + 8)^\alpha - 1 = (p + 8)^{2S+1} - 1. \qquad (3)$$

For $S = 0$, it follows from (3) that $p = 7$ and $p + 8 = 15$, but this contradicts the primality of $p + 8$. For $S \geq 1$, we have

$$2p^K = (p + 8)^{2S+1} - 1 = (p + 7)[(p + 8)^{2S} + (p + 8)^{2S-1} + \cdots + (p + 8)^1 + 1]. \quad (4)$$

It follows then from (4) that $p + 7$ is an even positive divisor (and non-trivial) of $2p^K$, i.e., $p + 7 = 2p^j$ where j is an integer such that $0 \leq j < K$. For $j = 0$, we obtain that $p + 7 = 2$, which is impossible. For $1 \leq j < K$, we obtain that $p = 7$ and $p + 8 = 15$, which contradicts the primality of $p + 8$. □

Theorem 3.4. Suppose that p is any prime of the form $p = 4M + 3$ $(M > 0)$ and $p + 8$ is prime. If x is odd and y is even then the diophantine equation $p^x + (p + 8)^y = z^2$ has no solution in positive integers x, y and z.
Proof: Suppose there are positive integers $x = 2K + 1$ $(K \geq 0)$, $y = 2S$ $(S \geq 1)$ and z, such that, $p^{2K+1} + (p + 8)^{2S} = z^2$, or equivalently, such that,

$$p^{2K+1} = [z + (p + 8)^S][z - (p + 8)^S].$$

Thus, it follows from the primality of p that $z + (p + 8)^S = p^\alpha$ and $z - (p + 8)^S = p^\beta$, where $\alpha > \beta \geq 0$ are integers such that $\alpha + \beta = 2K + 1$. Hence,

$$2(p + 8)^S = p^\alpha - p^\beta = p^\beta(p^{\alpha-\beta} - 1). \quad (5)$$

Note that, $\beta = 0$ (otherwise, we would conclude from (5) that p divides 2 or p divides $p + 8$ (both impossible)). This implies that

$$2(p + 8)^S = p^\alpha - 1 = p^{2K+1} - 1. \quad (6)$$

For $K = 0$, we obtain from (6) that $2(p + 8)^S = p - 1 = (p + 8) - 9$. But, this implies that $p + 8 = 3$, which is impossible. For $K \geq 1$, we have

$$2(p + 8)^S = p^{2K+1} - 1 = (p - 1)[p^{2K} + p^{2K-1} + \cdots + p^1 + 1]. \quad (7)$$

It follows then from (7) that $p - 1$ is an even positive divisor (and non-trivial) of $2(p + 8)^S$, i.e., $p - 1 = 2(p + 8)^j$ where j is an integer such that $0 \leq j < S$. For $j = 0$, we obtain that $p = 3$ (contradiction!). For $1 \leq j < S$, we obtain that $p + 8 = 3$, which is impossible. □

Corollary 3.1. Let $k \geq 2$ be a fixed integer (and arbitrary). If $p > 3$ and $p + 8$ are primes then the diophantine equation $p^x + (p + 8)^y = w^{2k}$ has no solution in positive integers x, y and w.
Proof: Suppose there are positive integers a, b and c such that $p^a + (p + 8)^b = c^{2k}$. Hence, it follows that $(x, y, z) = (a, b, c^k)$ is an solution in positive integers of the diophantine equation $p^x + (p + 8)^y = z^2$ (contradiction!). □

Final remark: The main focus of this paper is to study the solvability of the class of diophantine equations $p^x + (p + 8)^y = z^2$ for $p > 3$ and $p + 8$ primes. The case

$(p, p+8) = (3{,}11)$ was not considered in this work, but through a brief investigation it is possible to make some considerations. All possible solutions in positive integers of the diophantine equation $3^x + 11^y = z^2$ occur when y is even, since $3^x + 11^{2K} \equiv 1 (\text{mod } 3)$, $3^x + 11^{2S+1} \equiv 2 (\text{mod } 3)$ and $z^2 \equiv 0{,}1 (\text{mod } 3)$. Moreover, if x is odd and y is even, we can easily verify that $(x, y, z) = (5{,}4{,}122)$ is an solution of the diophantine equation $3^x + 11^y = z^2$. However, some issues remain open. Is this the only solution in positive integers? If there are other solutions, what are they?

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