Annals of Pure and Applied Mathematics
Vol. 10, No.2, 2015, 147-151
ISSN: 2279-087X (P), 2279-0888(online)
Published on 9 September 2015
www.researchmathsci.org

Decomposable and Strongly Decomposable Nearlattices

Y.S.Pawar[1] **and *M.A.Gandhi***[2]

Ex.Professor, Shivaji University, Kolhapur, India
email: [1]yspawar1950@gmail.com

[2]Department of Mathematics
N.K.Orchid College of Engineering and Technology, India
E-mail: madhavigandhi17@gmail.com

Received 19 July 2015; accepted 10 August 2015

Abstract. In this paper the concepts of decomposable and strongly decomposable nearlattices are introduced and some properties of these nearlattices are furnished.

Keywords: distributive nearlattice, prime ideals, decomposable and strongly decomposable nearlattice

AMS Mathematics Subject Classification (2010): 06A12, 06A99, 06B10

1. Introduction

Motivated by the characterizations of Stone lattices Cornish [2] and Pawar [5] characterized distributive lattices with minimum element 0 in which every prime ideal contains a unique minimal prime ideal and call such lattices normal lattices. This work Inspired Lu et al. [4] to introduce the concept of decomposable lattices by replacing the word normality by decomposability. A distributive lattice L with minimum element 0 is said to be decomposable if for any incomparable elements $a, b \in L$, there exist $x, y \in L$ such that $a = x \vee (a \wedge b)$ and b $= y \vee (a \wedge b)$ such that$x \wedge y = 0$. Further prime, minimal prime and special ideals in decomposable lattices are studied explicitly in [4] In order to describe the posets by algebraic structures which are similar to lattices the author of[3] introduce the near lattices with two binary operations denoted by $\wedge$ and $\vee$. In comparison with lattices only weaker forms of the associative and the commutative laws hold and also the correspondence between poset and nearlattices is not unique. Note that nearlattice is a lattice if and only if the commutative laws hold. Many researchers have developed the theory of nearlattices (see []6,7,8,9]).

Motivated by the work of Lu et al. [4].We introduce the concept of decomposable and strongly decomposable nearlattices. Some properties of decomposable and strongly decomposable nearlattices are proved.A Nearlattice is a meet semilattice (lower semilattice) together with the property that any two elements possessing a common upper bound have a supremum. This property is known as the upper bound property. A Nearlattice S is called distributive if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $(y \vee z)$ exists in S. For the basic concepts in nearlattices we refer [1] and [3]. The nearlattice depicted by the Hasse diagram of Fig.1 is a distributive nearlattice while the Hasse diagram of Fig. 2 represents nearlattice which is not distributive.

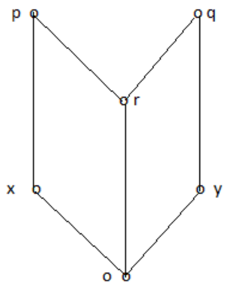


Figure 1:

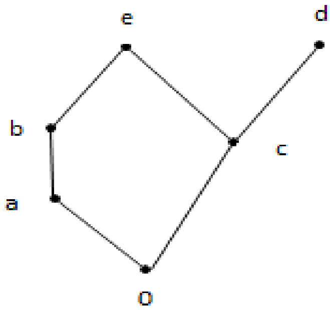


Figure 2:

A nonempty subset of S is called an ideal if i) For $x, y \in I$, $x \vee y \in I$,provided $x \vee y$ exists, and ii) For $x \in I, t \leq x (t \in S)$ implies $t \in I$. An ideal I of a nearlattice S is proper if $I \neq$ S. A proper ideal I of a nearlattice S is called a prime ideal if $x \wedge y \in p \Rightarrow$ either $x \in P$ or $y \in P$. The set of all prime ideals in a nearlattice with 0 is denoted by $\wp$ (S).A proper ideal M of a nearlattice is called maximal if for any ideal Q $\supseteq$ M implies either Q = M or Q = S. The set of all maximal ideals in a nearlattice with 0 is denoted by $\mathcal{M}$ (S). A prime ideal of a nearlattice S is said to be minimal if it does not contain properly any other prime ideal. The set of all minimal prime ideals in a nearlattice S with zero is denoted by$\mathfrak{M}(S)$.For A $\subseteq S$ we define $A^* = \{x \in S / x \wedge a = 0 \text{ for all } a \in A\}$ If S is distributive then clearly A^* is an ideal of S.Moreover $\{a\}^* = \{x \in S / x \wedge a = 0\}$ and$A^* = \bigcap_{a \in A}\{\{a\}^*\}$.An ideal I in anearlattice S with 0 is called a normal ideal if $I = I^{**}$.The set of all normal ideals in a nearlattice S with 0 is denoted by N (S).Let K be a subset of a nearlattice S then (K] denote ideal of S generated by K. If K = {a/ a∈ S} then we write ({a}] = (a].

2. Some results

Now onwards S will denote a distributive nearlattice with 0. For$I, J \in \mathcal{I}(S)$, we write $I \parallel J$ when I and J are incomparable in the poset $(\mathcal{I}(S), \subseteq)$.We begin with the following definition.

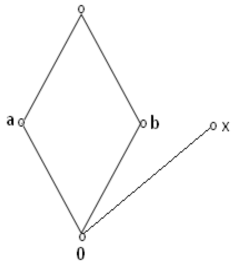
Definition 2.1. A distributive nearlattice S with 0 is called decomposable if for any $I \parallel J$, $I, J \in \mathcal{I}(S)$, there exist $x \in I \setminus J$ and $y \in J \setminus I$ such that $x \wedge y = 0$.The nearlattice depicted by the Hasse diagram of Fig.3 is an example of distributive nearlattice which is decomposable while the nearlattice depicted by the Hasse diagram of Fig.4 represents adistributive nearlattice which is not decomposable.

Theorem 2.2. In a decomposable nearlattice S, a proper ideal P is prime iff the set $\{I \in \mathcal{I}(S) / I \supseteq P\}$ is totally ordered.

Proof. Let a proper ideal P in S be such that the set $\{I \in \mathcal{I}(S) / I \supseteq P\}$ is a totally ordered subset of $\mathcal{I}(S)$**.** Let P be not prime. Then there exist $a, b \in S$ such that $(a \wedge b) \in P$ with $a \notin P$ and $b \notin P$. As $P \vee (a] \supset P$ and $P \vee (b] \supset P$. By assumption $P \vee (a] \subseteq P \vee (b]$ or $P \vee (b] \subseteq P \vee (a]$. Let us assume without loss of generality that $P \vee (a] \subseteq P \vee (b]$.
As$(a \wedge b) \in P$, we get

$P = P \vee (a \wedge b]\quad = P \vee [(a] \wedge (b]]$.
$= [P \vee (a] \wedge [(P \vee (b]$ (Since $\mathcal{I}(S)$ is distributive.)
$= P \vee (a]$ (Since $P \vee (a] \subseteq P \vee (b]$)

This shows that a$\in P$; a contradiction. Hence P must be a prime ideal. For converse, let $\{ I \in \mathcal{I}(S)/I \supseteq P\}$ be not totally ordered. Then there exist $I, J \in \mathcal{I}(S)$ containing P and $I \parallel J$. As S is decomposable there exists $x \in I \backslash J$ and $y \in J \setminus I$ such $x \wedge y = 0$. As $x \wedge y = 0 \in P$ and P is prime $x \in P$ or $\in P$ .But then $x \in J$ or $y \in I$, leading to a contradiction. Hence the set $\{I \in \mathcal{I}(S)/I \supseteq P\}$ is totally ordered.■



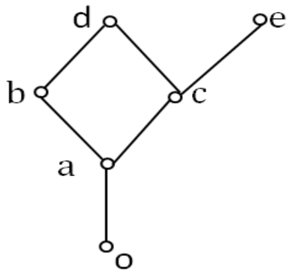


Figure 3: **Figure 4:**

Theorem 2.3. Let S be decomposable nearlattice. If $P_1 \parallel P_2$, for prime ideals P_1 , P_2 of S and $a \in S$, such that $a \notin P_1 \cup P_2$, then there exists $a_1 \in P_1 \setminus P_2$ and $a_2 \in P_2 \setminus P_1$ such that $0 < a_1, a_2, < a$ and $a_1 \wedge a_2 = 0$.

Proof. Let $P_1 \parallel P_2$ and $a \notin P_1 \cup P_2$ .As S is decomposable, there exist $x_1 \in P_2 \backslash P_1$ and $x_2 \in P_1 \backslash P_2$ such that $x_1 \wedge x_2 = 0$. Define $a_1 = a \wedge x_1$ and $a_2 = a \wedge x_2$ .Then a$\notin P_1$ and $x_1 \notin P_1 \Rightarrow a \wedge x_1 = a_1 \notin P_1$ .Again $a \notin P_2$ and $x_2 \notin P_2 \Rightarrow a \wedge x_2 = a_2 \notin P_2$.Clearly $0 < a_1$ and $0 < a_2$. Since, $a_1 = a \Rightarrow a \wedge x_1 = a \Rightarrow a \leq x_1 \in P_2 \Rightarrow a \in P_2$ .Hence $a_1 < a$ .Similarly we get $a_2 < a$. Thus $0 < a_1 < a$ and $0 < a_2 < a$. Further $a_1 \wedge a_2 = (a \wedge x_1) \wedge (a \wedge x_2) = a \wedge (x_1 \wedge a \wedge x_2) = a \wedge (a \wedge x_1 \wedge x_2)$
$= a \wedge (a \wedge 0) = a \wedge 0 = 0$. Hence the proof.■

Let $\mathfrak{M}$ denote the set of minimal prime ideals in S. For a prime ideal P in S define $S_P = \bigcap \{M \in \mathfrak{M} \mid M \subseteq P\}$. A necessary and sufficient condition for any two prime ideals to be comparable in a decomposable nearlattice is proved in the following theorem.

Theorem 2.4. In a decomposable nearlattice, two prime ideals P and Q are comparable iff $S_P \subseteq Q$ or $S_Q \subseteq P$.

Proof. The proof of only if part being obvious, we prove if part only .Let $S_P \subseteq Q$ and $P \| Q$. As S is decomposable, there exist $x \in P \backslash Q$ and $y \in Q \backslash P$ such that $x \wedge y = 0$. As for $M \in \mathfrak{M}$ contained in P, $0 = x \wedge y \in M$ implies $x \in M$ as $y \notin P$. But then $x \in \bigcap\{M \in \mathfrak{M} \mid M \subseteq P\} = S_P$ implies $x \in Q$; a contradiction. Hence P and Q must be comparable. Similarly we prove if $S_Q \subseteq P$,then prime ideals P and Q are comparable.■

Definition 2.5. A distributive nearlattice S is called strongly decomposable if for any $a \parallel b, a, b \in S$, there exist $x, y \in S$ such that $a = x \vee (a \wedge b)$ and $b = y \vee (a \wedge b)$ such that $x \wedge y = 0$.Nearlattice depicted by Hasse diagram of Fig.5 is a distributive nearlattice which is decomposable but not strongly decomposable.

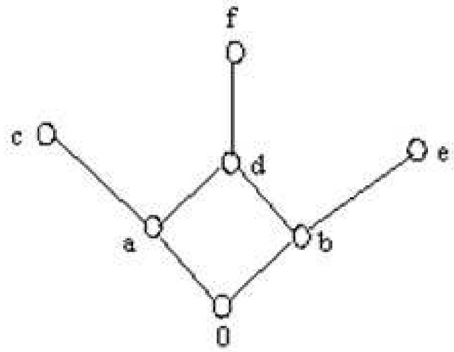


Figure 5:

Theorem 2.6. Every strongly decomposable nearlattice S is decomposable.

Proof. Let S be strongly decomposable. To prove that S is decomposable. Let $I \parallel J\ in\ \mathcal{J}(S)$.select $x \in I \setminus J$ and $y \in J \setminus I$. Then $x \parallel y$. As S is strongly decomposable, there exist $a, b \in S$such that $x = a \vee (x \wedge y)$ and $y = b \vee (x \wedge y)$, with $a \wedge b = 0. a \leq x = a \vee (x \wedge y)$, gives $a = x \wedge a \in I$ as $x \in I$. $y \in I \Rightarrow y \wedge x \in J \Rightarrow x \wedge y \in J$. Similarly $b \leq b \vee (x \wedge y) \Rightarrow b \leq y \Rightarrow b = b \wedge y$. As$y \in J \Rightarrow y \wedge b \in J \Rightarrow b \wedge y \in J \Rightarrow b \in J$.Thus $for\ I \parallel J$in $\mathcal{J}(S)$there exist $a \in I$ and $b \in J$ such that $a \wedge b = 0$.This proves that S is decomposable.■

Theorem 2.7. If $(a] \vee (a]^* = S$ for any $a \in$ S, then S is a strongly decomposable nearlattice.

Proof. Let $x \parallel y$ in S . By assumption $(x \wedge y] \vee (x \wedge y]^* = S$. Hence $x = a \vee b$ and $y = c \vee d$ for some a, c $\in (x \wedge y]$ and b, d $\in (x \wedge y]^*$.

Now, $x \wedge y = (x \wedge y) \wedge (a \vee b) = (x \wedge y \wedge a) \vee (x \wedge y \wedge b) = (x \wedge y \wedge a)$ (since b$\in (x \wedge y]^*$). So $x \wedge y \leq a$, which implies that $x \wedge y = a$. Similarly$x \wedge y = c$.Then= $x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) = (x \wedge y) \vee (x \wedge b)$ .Similarly $y = y \wedge (c \vee d) = (y \wedge c) \vee (y \wedge d) = (x \wedge y) \vee (y \wedge d)$.Further$(x \wedge b) \wedge (y \wedge d) = (x \wedge y) \wedge (b \wedge d) = 0$.This shows that S is a strongly decomposable nearlattice.■

Acknowledgement: The authors gratefully acknowledge the referee's valuable comments.

REFERENCES

1. M.Ayub Ali, A.S.A.Noor and A.K.M.S.Islam, Annulets in a distributive nearlattice, *Annals of Pure and Applied Mathematics*, 1(2012) 91-96.
2. W.H.Cornish, Normal lattices, *J. Australian Math. Soc.*, 14 (1972) 200-215.
3. D.Schweigert, Near lattices, *Math. Slovaca*, 32 (1982) 313-317.
4. X.M.Lu, D.S.Liu, Z.N.Qi and H.R.Qin, Prime ideals in decomposable lattices, *arXiv* 19 (2010) 1-21.
5. Y.S.Pawar, Characterizations of normal lattices, *Indian J. Pure and Appl. Math.,* 24 (1993) 651-656.
6. M.Rahman and A.S.A.Noor, Finitely generated n-Ideals of a nearlattice, *Annals of Pure and Applied Mathematics*, 2 (2012) 40-44.

7. Md.Zaidur Rahman, Md.Bazlar Rahman and A.S.A.Noor, 0-distributive nearlattice, *Annals of Pure and Applied Mathematics*, 2 (2012) 177-184.
8. Md.Zaidur Rahman,Md.Bazlar Rahman and A.S.A.Noor, Some properties of 0-distributive Nearlattice, *Annals of Pure and Applied Mathematics*, 2 (2012) 185-193.
9. Md.Zaidur Rahman, Md.Bazlar Rahman and A.S.A.Noor, Some characterizations of properties of 0-distributive nearlattice, *Annals of Pure and Applied Mathematics*, 2 (2012) 194-199.