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Algebraic Structures of Certain Lie Algebras of Polynomial Fields

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Abstract. We study some Lie algebra of polynomial vector fields on R^n that contains all constant fields and the Euler field in order to find some algebraic structures. We then show that any Lie algebra polynomial fields on R^n admits a corresponding Lie group.

Keywords:Lie algebras of polynomials fields, Lie groups, Polynomial fields, Euler field, Derivation.

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1. Introduction

In [12], R-subalgebras of Lie polynomial vector fields P on R that contain all the constant fields and Euler field was studied by giving a kind of classification. The classification of complex simple Lie algebras (and real) is well known. This is partly due to the work of Elie Cartan, Dynkin and Killing. In this paper, our work is to study the algebraic structures of the Lie algebra of the polynomial (vectors) fields. To do this, we classify the Lie subalgebras polynomial fields P on R^n after giving the properties of sets of polynomial vector fields following to graduation of P and Euler field. We find that the Lie algebra of affine fields on R^n has a centralizer zero, the Lie algebra coincides with its normalizer, all its derivation is inner compared to a vector field and its first Chevalley-Eilenberg cohomology space is zero. Knowing all abelian Lie subalgebras of P, we obtain some nilpotent and also solvable Lie algebras of polynomial vectors fields. Moreover, we find that every Lie algebra polynomial fields P on R^n, $n \geq 1$ (integer), containing all the constant fields, diagonal linear fields and the diagonal fields of degree 1 is semisimple. Then we identify the Lie subalgebras P of denombrable infinite-dimensional. Finally, we note that any Lie algebra of polynomial vector fields P admits a corresponding Lie group connected and simply connected on R^n. Examples were given to illustrate the results. In the next paper, we will examine some geometric aspects of the Lie algebras of the polynomial fields as valid connections to Lie algebras of affine fields cf. [6, 7,8,9] as well as those defined by a vector form cf. [5]. We adopt the convention of Einstein on the index summation unless otherwise stated.

2. Notations and preliminary

We denote by $\chi(\mathbb{R}^n)$ the Lie algebra of vector fields on $\mathbb{R}^n$. Throughout this section, we consider a $\mathbb{R}$-Lie subalgebra $\mathcal{P}$ (consisting) of polynomial vector fields on $\mathbb{R}^n$ that contains all constant fields and the Euler field E, where $E = x^i \frac{\partial}{\partial x^i}$ in local coordinates, $\mathcal{N}$ its normalizer in $\mathbb{R}^n$; H_i the vector space of homogeneous fields of degree i with $i \in \mathrm{N} \cup \{-1\}$. We will denoted H_0^d the set of all diagonal linear fields, H_1 the set of quadratic fields, H_i^{jk} (resp. $\widetilde{H}_i^{jk}$) the set of mixed fields (resp.the mixed fields not containing simultaneously of vector fields) generated in local coordinates by $\left(x^j\right)^i \frac{\partial}{\partial x^k}$ with $j \neq k$ (resp. $\left(x^a\right)^r \frac{\partial}{\partial x^b}, \left(x^c\right)^s \frac{\partial}{\partial x^d}$ where $(a \neq b, c \neq d, r \geq 2, s \geq 1)$ with $(a = d \ where \ b = c)$; H_i^c the set of compound fields of degree i generated by $\left(x^1\right)^{\alpha_1}\left(x^2\right)^{\alpha_2} \ldots \left(x^n\right)^{\alpha_n} \frac{\partial}{\partial x^k}$ with $\sum_j \alpha_j = i, i \geq 2$. In particular, constant fields are homogeneous of degree -1, linear fields of degree 0 and the quadratic fields are homogeneous of degree 1. We denote L_X the Lie derivative with respect to $X \in \chi(\mathbb{R}^n)$. The bracket of two vector fields $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ of $\chi(\mathbb{R}^n)$ in coordinates $\left(x^i\right)_{1 \leq i \leq n}$ is given by:

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \quad (1)$$

Lie subalgebras Lie $\mathcal{P}$ of polynomial fields are finite-dimensional or denombrable infinite-dimensional. They are made of fields whose components are polynomials on $\mathbb{R}^n$. They decompose in subspaces formed by their constant, linear and quadratic fields and this decomposition is respected by the Lie bracket. We can now define the notion of graded subalgebra $\chi(\mathbb{R}^n)$ using the formula (1):

Definition 2.1.[12] A subalgebra $\mathcal{P}$ of $\chi(\mathbb{R}^n)$ is graduated if it admits the graduation $\mathcal{P} = \underset{i \geq -1}{\oplus} \mathcal{P}_i$, where every $\mathcal{P}_i$ is a finite-dimensional subspace of H_i such that $[\mathcal{P}_{-1}, \mathcal{P}_{-1}] = \{0\}$ and $\forall i, j \geq -1,$ where $i + j \geq -1,$ $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j}$ (2)

Proposition 2.2. [12]A polynomial vector fields X of $\mathcal{P}$ is homogeneous of degree p if and only if $[E, X] = pX$.

Definition 2.3. We say that an element X of a Lie algebra $\mathfrak{g}$ is separable if it can be written in terms of element components of the "usual" basis of $\mathfrak{g}$.

Example 2.4. In R^2, the vector fields $X \in \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y} \rangle$, that is $X \in H_{-1} \oplus H_0$ is separable while $X \in \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \rangle = H_{-1} \oplus \langle E \rangle$ is not.

Remark 2.5. The Euler field has an important role in algebra and geometry.

We give some of the properties of the following Euler field:

Properties 2.6.

- The Euler field E is the commutant of H_0. It commutes to all vector fields of homogeneous degree 0, that is, $[E, H_0] = \{0\}$,
- The radial field E is a stabilizer of H_k with $k = 0$, that is, $[E, H_0] = \{0\}$,
- For all $X \in \mathsf{P}$, we can not have $[E, X] = E$ except in the case where X is separable.

Proof. Immediate, by using the graduation defined in Definition 2.1 or by taking the polynomial fields of the set of vector fields considered, in local coordinates, the only use of the Lie brackets is enough.

Example 2.7. In local coordinates, $(x^a)^{k+1} \frac{\partial}{\partial x^b} \in H_k^{ab}$, $a \neq b$, $E = x^i \frac{\partial}{\partial x^i}$ we have

$$\left[E, (x^a)^k \frac{\partial}{\partial x^b} \right] = (k-1)(x^a)^k \frac{\partial}{\partial x^b} \in H_k^{ab}.$$

In the following, we suppose that the Euler field is separable.

Remark 2.8. On R^n, mixed fields of H_k^{ij} and compound fields of H_k^c homogenous of degree k exist only for $n \geq 2$. For example, in local coordinates, $x\frac{\partial}{\partial y}$ is a mixed field and, $(x)(y)^3 \frac{\partial}{\partial y}$ is a compound field on R^2.

The sets of mixed fields H_k^{ij} and $\widetilde{H}_k^{ij}$ admit the following properties:

Properties 2.9.

$$[H_k^{ij}, H_l^{ij}] = H_{k+l}^{ij},$$
$$[\widetilde{H}_k^{ij}, \widetilde{H}_l^{ij}] = \{0\},$$
$$[E, H_k^{ij}] = H_k^{ij}, \quad [E, \widetilde{H}_k^{ij}] = \widetilde{H}_k^{ij},$$
$$[H_{-1}, H_k^{ij}] = H_{k-1}^{ij}, \quad [H_{-1}, \widetilde{H}_k^{ij}] = \widetilde{H}_{k-1}^{ij},$$

$H_k^{ij} = \widetilde{H}_k^{ij} = H_{-1}$ if $k = -1$,

$H_k^{ij} = \widetilde{H}_k^{ij} = \{0\}$ if $k < -1$.

Proof. Reasoning analogous to the previous one.

For linear diagonal fields, we give some properties:

Properties 2.10.

$H_{-1}^d = H_{-1}$ and $[H_{-1}, H_{-1}] = 0$,

$[H_1, H_0^d] = H_{-1}$,

$[H_1, H_1^d] = H_0^d = [H_0^d, H_0^d]$,

$[H_0^d, H_1^d] = H_1^d$,

$[H_1^d, H_1^d] = H_2^d$,

$[H_{-1}, H_k^d] = H_{k-1}^d$ for all $k \geq 1$

$[H_i^d, H_j^d] = H_{i+j}^d$, for any $i, j \geq -1$.

Proof. Immediate.

Definition 2.11. [12] Let $\mathfrak{g}$ be a Lie algebra.

- The normalizer $\mathfrak{g}$ denoted by $\mathcal{N}(g)$ is defined by $\mathcal{N}(g) = \{X \in \chi(\mathcal{P}^n) / [X, g] \subset g\}$.
- The centralizer $\mathcal{C}(g)$ of $\mathfrak{g}$ is defined by $\mathcal{C}(g) = \{X \in \chi(\mathcal{R}^n) / [X, g] = \{0\}\}$.
- An ideal (resp. characteristic ideal) of $\mathfrak{g}$ is a stable subspace by inner derivations (resp any derivation.)

Example 2.12. The ideal $\Delta(g) = [g, g]$ and the center $z(g)$ are the characteristic ideals of $\mathfrak{g}$.

3. Study of certain Lie algebra of polynomial fields

Definition 3.1. [12] We denote by λ_X the degree of homogeneity of an homogeneous field X of $\mathcal{P}$, λ_S is the maximum degree of homogeneity of any homogeneous fields of $S \subset \mathcal{P}$. The latter can be infinite.

Theorem 3.2. [12] Any derivation of $\mathcal{P}$ is inner compared to normalizer $\mathcal{N}$ of $\mathcal{P}$. Moreover $D = L_{(F+X)}$ with $X \in H_0$ and $F \in \mathcal{P} - H_0$.

Definition 3.3. [2] Let $\mathfrak{g}$ be a Lie algebra on K. Pose, for all $j \geq 0, \Delta^{j+1}\mathfrak{g} = [\Delta^j \mathfrak{g}, \Delta^j \mathfrak{g}]$ the derived algebra of $j+1$ order of Lie algebra $\mathfrak{g}$ with $\Delta^0 \mathfrak{g} = \mathfrak{g}$. The lower sequence of ideals $\Delta^0\mathfrak{g} \supseteq \Delta^1\mathfrak{g} \supseteq \ldots \supseteq \Delta^j\mathfrak{g}\ldots$ is called the derived series of $\mathfrak{g}$. A Lie algebra nontrivial $\mathfrak{g}$ on K is said to be solvable if the sequence of commutators cancel each other from a certain rank, that is, if there exists an integer $k \geq 1$ such that $\Delta^k \mathfrak{g} = \{0\}$.

Theorem 3.4. Let $H_{-1} \oplus H_0$ be a Lie algebra of affine fields on $\mathbb{R}^n$. Its centralizer is zero, this Lie algebra coincides to its normalizer, all its derivation is inner compared to a vector fields and its first cohomology space of Chevalley-Eilenberg is zero.

Proof. Note A the Lie algebra $H_{-1} \oplus H_0$, in local coordinates $(x^i)_{1 \leq i \leq n}$, generated by $\frac{\partial}{\partial x^i}, x^j \frac{\partial}{\partial x^i}$, N its normalizer and, $\mathrm{H}^1(\mathrm{A})$ its first cohomology space of Chevalley-Eilenberg. With the simple calculations, we show that the centralizer of A is zero and this Lie algebra coincides with its normalizer N. By Theorem 2.9, p 90 cf. [12], any derivation of the Lie algebra of affine fields A is inner compared with a vector field of A. And by Theorem 2.12, p 91 cf.[12], the first cohomology space of Chevalley-Eilenberg $\mathrm{H}^1(\mathrm{A})$ is zero. Where the evidence.

Remark 3.5. The Lie algebra of polynomial vector fields $H_{-1} \oplus H_0 \oplus H_1$ forms a Lie triple system then by [4] this Lie algebra admits a geometrical subject.

Proposition 3.6. The Lie algebra of polynomial fields $H_{-1} \oplus H_0^d \oplus \widetilde{H}_i^{jk}$, $i \in \mathrm{N}$, is solvable of order less than or equal to i.

Proof. Using a definition of solvability of Lie algebra and under the Properties 2.9, we get the result.

Example 3.7. Let $\mathsf{P} = H_{-1} \oplus H_0^d \oplus \widetilde{H}_i^{jk}$ where $i > 0$ be a Lie algebra of polynomial fields does not contains simultaneously the fields $(x^a)^r \frac{\partial}{\partial x^b}$ and $(x^b)^s \frac{\partial}{\partial x^a}$ with $r \geq 2, s \geq 1$. The Lie algebra P is solvable of order 3, the inner derivation with respect to its normalizer, centralizer void, and its first cohomology space of Chevalley-Eilenberg is $\mathrm{H}^1(P) = H_0/_{\{E\}}$.

Definition 3.8. We define the descending central series of $\mathfrak{g}$ by :
$\mathrm{X}^1(\mathfrak{g}) = \mathfrak{g}, \mathrm{X}^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ and for all $i, j \geq 1, [\mathrm{X}^i(\mathfrak{g}), \mathrm{X}^j(\mathfrak{g})] \subset \mathrm{X}^{i+j}(\mathfrak{g})$, as well as for all $k \geq 2, \mathrm{X}^{k+1}(\mathfrak{g}) = [\mathfrak{g}, \mathrm{X}^k(\mathfrak{g})]$. We see that for $i \geq 1$, every $\mathrm{X}^1(\mathfrak{g})$ is an characteristic ideal of $\mathfrak{g}$, and $\mathrm{X}^i(\mathfrak{g})/_{\mathrm{X}^{i+1}(\mathfrak{g})}$ is a central ideal (i.e, contained in the center) of $\mathfrak{g}/_{\mathrm{X}^{i+1}(\mathfrak{g})}$.

Definition 3.9. We say that $\mathfrak{g}$ is nilpotent of order n if there exists n such that $\mathrm{X}^n(\mathfrak{g}) = (0)$. Note that any abelian Lie algebra is nilpotent.

Theorem 3.10. Any Lie algebras of diagonal (vector) fields H_i^d on $\mathbb{R}^n$, with $i \in \mathbb{N} \cup \{-1\}$ are commutatives.

Proof. Immediate, by using the Lie bracket of H_i^d with $i \in \mathbb{N} \cup \{-1\}$, we find that $\left[H_i^d, H_i^d\right] = \{0\}$.

Example 3.11. The Lie algebra of polynomial fields on $\mathbb{R}^n$ generated by constant fields is abelian so nilpotent.

Proposition 3.12. [2] Let $\mathfrak{g}$ be a Lie algebra, I and J its two ideals. The quotient algebras $\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}, \mathfrak{g}/_{(I+J)}, \mathfrak{g}/_{(I\cap J)}$ and $\mathfrak{g}/_{[I,J]}$ are quotient abelian Lie algebras.

Remark 3.13. The previous Proposition 3.12 of [2] is not verified by virtue of a result of [10] in general for the Lie algebra of polynomial fields. For example, on $\mathbb{R}^2$, consider the Lie algebra g generated by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, (y)^2\frac{\partial}{\partial x}$. There are any two ideals I and J of g such that their Lie bracket is an ideal generated by $\frac{\partial}{\partial x}$. Then the quotient algebra $g/_{[I,J]}$ is not commutative.

Definition 3.14. [2] Let K be a field of characteristic 0, $\mathfrak{g}$ a Lie algebra over a field K of finite-dimensional. We say that $\mathfrak{g}$ is semisimple if the only commutative ideal of g is $\{0\}$. Moreover, if $\mathfrak{g}$ is a semisimple Lie algebra of finite-dimensional, then it coincides with its derived algebra.

Theorem 3.15. Any Lie algebra of polynomial fields $\mathcal{P}$ on $\mathbb{R}^n$, $n \geq 1$ (integer) containing all the constant fields, diagonal linear fields and the diagonal fields of degree 1 is semisimple.

Proof. On $\mathbb{R}$, the Lie algebra $\mathcal{P}$ is generated by $\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, (x)^2\frac{\partial}{\partial x}$. Using the multiplication tables (Lie bracket) of $\mathcal{P}$, we find that $[\mathcal{P}, \mathcal{P}] = \mathcal{P}$. It's immediate to check that the only commutative ideal of $\mathcal{P}$ is $\{0\}$. Analogously on $\mathbb{R}^2$ we find the same result. In making a reasoning by recurrence on $\mathbb{R}^n$, we obtain the result.

Example 3.16. In $\mathbb{R}^2$, the Lie algebra $H_{-1} \oplus H_i^d$ with $i \in \{0,1\}$, in local coordinates, generated by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, (x)^2\frac{\partial}{\partial x}, (y)^2\frac{\partial}{\partial y}$ is semisimple, all its derivation is inner, it coincides with its derived algebra and the first cohomology space of

Chevalley-Eilenberg of this Lie algebra is zero.

Definition 3.17. [1] A Lie algebra is said to be sympathetic if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, $Der(\mathfrak{g}) = ad(\mathfrak{g})$ and $z(\mathfrak{g}) = \{0\}$.

Theorem 3.18. Any Lie algebra of polynomials fields verifying the Theorem 3.15 is a sympathetic Lie algebra.
Proof. Immediate.

Remark 3.19. The semisimple Lie algebras of polynomial fields verify results on the sympathy and the local sympathy in [1].

Theorem 3.20. The Lie subalgebra $H_{-1} \oplus H_0^d \oplus H_i^d$ of $\mathfrak{P}$ on $\mathbb{R}^n$ with $n \geq 2, i \in \mathbb{N}^*$ coincides to its derived algebra, all its derivation is inner compared with a vector fields and its first cohomology space of Chevalley-Eilenberg is zero yet it is not semisimple.
Proof. Let $H_{-1} \oplus H_0^d \oplus H_i^d$ be a Lie subalgebra of $\mathfrak{P}$ on $\mathbb{R}^n$ with $n \geq 2, i \in \mathbb{N}$. By making successive calculations of Lie brackets on the elements of $H_{-1} \oplus H_0^d \oplus H_i^d$, successive applications properties 2.10 assure us that the degrees of homogeneity of $\left[H_j^d, H_k^d\right] = H_{j+k}^d$, for all $j, k \geq 2$ run over $\mathbb{N}$, thus the Lie algebra coincides to its derived algebra and has a denombrable infinite-dimensional. Therefore, it is not semisimple. Finally, the Corollary 2.13 and the Theorem 2.12 in [12] lead us to finish the proof.

Remark 3.21. Any semisimple Lie algebra is reductive cf.[2] thus all semisimple Lie algebras of polynomial fields $\mathfrak{P}$ are reductives.

Corollary 3.22. *[12] In local coordinates $(x_i)_{1 \leq i \leq n}$ of $\mathbb{R}^n$, the values of $\lambda_{\mathfrak{P}}$ is infinite if and only if*

$$\left(\exists i \neq j \,/\, x^j \frac{\partial}{\partial x^i}, x^i x^j \frac{\partial}{\partial x^j}\right) \quad or \quad \left(\exists i \neq j \,/\, x^j \frac{\partial}{\partial x^i}, \left(x^j\right)^2 \frac{\partial}{\partial x^j}\right) \quad or \quad \left(\exists i \,/\, \left(x^i\right)^3 \frac{\partial}{\partial x^i}\right)$$

appearing in the expression of a nonzero element of $\mathfrak{P}$.

Therefore, any Lie algebra $\mathfrak{P}$ satisfying the Corollary 3.22 is a Lie algebra of denombrable infinite-dimensional whose inner derivation, centralizer and first cohomology space of Chevalley-Eilenberg are zero.

Lemma 3.23. Let $\mathfrak{g}$ be a $\mathbb{R}$-vector space containing all the constant fields, the Euler field and a compound field of degree $r \geq 1$ (finite). In order that $\mathfrak{g}$ is a $\mathbb{R}$-Lie algebra of polynomial fields, all compound fields of lower degree and mixed fields obtained from of initial compound field must be in $\mathfrak{g}$.
Proof. With a rather long calculation of the successive Lie brackets and given the role of constant fields and field Euler $\mathfrak{g}$ would be stable by the bracket as soon as all the

compound fields of degree s with $s \le r-1$ belong to $\mathfrak{g}$. But these compound fields of lower order can be mixed fields when all variables in the head of the vector fields have exponent zero except for one and only one variable. Which completes our proof.

Proposition 3.24. In $\mathbb{R}^2$, any Lie algebra of polynomial fields $\mathcal{P}$ who contains a compound fields is of denombrable infinite-dimensional, of inner derivation compared with its normalizer $\mathcal{N}$ and of nul centralizer. In $\mathbb{R}^n$ with $n \ge 3$, any Lie algebra of polynomial fields $\mathcal{P}$ who contain simultaneously at least a mixed field and a compound field, the foot of one being in the head of the other, is of denombrable infinite-dimensional, its derivation is inner and its centralizer is zero. Otherwise, this Lie algebra is solvable of order 3.

Proof. This follows from Lemma 3.23, of the Corollary 3.22, Proposition 2.11 in [12] and of the Definition 3.3.

Example 3.25.

• The Lie algebra of polynomial vector fields generated, in local coordinates by $\frac{\partial}{\partial x^j}$, $x^j \frac{\partial}{\partial x^j}, \ldots \left(x^j\right)^r \frac{\partial}{\partial x^j}, \ldots$ with $j \in \{1, \ldots, n\}$ and r an integer greater than or equal to 3 is denombrable infinite-dimensional, coincides with its derived ideal, do not admit any other commutative ideal than $\{0\}$, of inner derivation, centralizer zero and of first cohomology space of Chevalley-Eilenberg zero.

• The Lie algebra of polynomial vector fields of $\mathcal{P}$ of denombrable infinite-dimensional generated, in local coordinates by $\frac{\partial}{\partial x^j}, x^j \frac{\partial}{\partial x^j}, \left(x^a\right)^r \frac{\partial}{\partial x^b}, \ldots, \left(x^c\right)^s \frac{\partial}{\partial x^d}$ with $a, b, c, d, j \in \{1, \ldots, n\}$ and $r \ge 1, s \ge 2$ (nonnegative integers) such that $a = d$ or $c = b$ admits a nontrivial commutative ideal generated by $\frac{\partial}{\partial x^i}, y^j \frac{\partial}{\partial x^i}, \left(y^j\right)^r \frac{\partial}{\partial x^i}, \ldots$, where $i, j \in \{1, \ldots, n\}$ and r nonnegative integers. Its derivation is inner, its centralizer is zero and, its derived ideal is strictly included in this algebra.

• In $\mathbb{R}^3$, let $\mathfrak{g}$ be the $\mathbb{R}$-vector space generated, in local coordinates by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, E, xy\frac{\partial}{\partial z}$. In order that $\mathfrak{g}$ is a Lie algebra, all the mixed fields formed by the compound fields $xy\frac{\partial}{\partial z}$ and constant fields of $\mathfrak{g}$ must be in $\mathfrak{g}$. Then the Lie algebra $\mathfrak{g}$ is solvable of order 3, of zero centralizer, of inner derivation from its normalizer and of first cohomology space of Chevalley-Eilenberg $\mathrm{H}^1(\mathfrak{g}) = H_0/_{\{E\}}$.

4. Link of Lie group and Lie algebra of polynomial vector fields

Definition 4.1.[3] We call X category a collection of points or objects $X, Y, \ldots$ (Resp arrows or morphisms $f, g, \ldots$) in such a way that two points are associated to each arrow f (resp. point X): its source (or domain) X and goal (or codomain) Y, what note that f : domain) X and goal (or codomain) Y, noted that $f : X \rightarrow Y$ (resp is associated an arrow: its identity $id_X : X \rightarrow X$); and each pair of consecutive arrows $f : X \rightarrow Y, g : Y \rightarrow Z$ is associated with an arrow : its composed $g \circ f : X \rightarrow Z$. It satisfies the following properties:

- for all $f : X \rightarrow Y, g : Y \rightarrow Z, h : Z \rightarrow T$, we have $(h \circ g) \circ f = h \circ (g \circ f) : X \rightarrow T$,
- for all $f : X \rightarrow Y$, we have $f \circ id_X = f$ and $id_Y \circ f = f$.

We note $Hom_X(X, Y)$, or $\mathrm{X}(X, Y)$, the set of arrows of X of source X and of goal Y.

Definition 4.2.[3] Let two categories X and Δ. A functor $F : X \rightarrow \Delta$ is a function F_p (resp. a function F_f) or simply F of points (resp. arrows) de X to the points (resp. arrows) of Δ such that the following properties are verified :

- for all arrow $f : X \rightarrow Y$ in X, we have $F(f) : F(X) \rightarrow F(Y)$ in Δ,
- for all point X in X, we have $F(id_X) = id_F(X)$ in Δ,
- for all pair of consecutive arrows $f : X \rightarrow Y, g : Y \rightarrow Z$ in X, we have $F(g \circ f) = F(g) \circ F(f)$ in Δ.

Proposition 4.3. Let P and Q two real Lie groups of Lie algebras of polynomial fields P and Q respectively. If $f : P \rightarrow Q$ is a morphism of Lie groups then the differential df of f defined by $df(1) : \mathsf{P} \rightarrow \mathsf{Q}$ is a morphism of Lie algebras of polynomial fields.

Proof. Just check that there exists a functor noted T of the category of real Lie groups de Lie to the category of Lie R-algebras of polynomial fields. Moreover, we can also show that for Lie groups P and Q of Lie algebras P and Q respectively, if $f : P \rightarrow Q$ is a morphism of Lie groups then $T_e f : \mathsf{P} \rightarrow \mathsf{Q}$ is a morphism of Lie algebras of polynomials fields, where e is a neutral element of Lie group G and $T_x f$ denotes tangente application of f to the point e.

Proposition 4.4. Let P be a real Lie group of R^n. There is a connected and simply connected Lie group $\widetilde{P}$ whose application π of $\widetilde{P}$ to P is a covering; then the tangential application $d\pi$ of Lie algebras $Lie(\widetilde{P})$ to $Lie(P)$ is an universal covering of P.

Proof. Let P be a Lie group. On R^n, we know that there exists another connected and simply connected Lie group $\widetilde{P}$ such that the application $\pi : \widetilde{P} \rightarrow P$ is a morphism of

Lie groups. So π is a covering. Taking the differential $d\pi$ of π and under Theorems and Propositions (3.22-26) of [13], we obtain that $d\pi$ is a isomorphism of Lie algebras of polynomial fields $Lie(\widetilde{P})$ to $Lie(P)$. As $Ker\pi$ is a central normal subgroup of $\widetilde{P}$ such that $\widetilde{P}/_{Ker\pi}$ is isomorphic to P then $\widetilde{P}$ is unique to up isomorphism. Hence $d\pi$ is an universal covering.

Theorem 4.5. Let P and Q two real Lie groups of Lie algebras P and Q respectively and $\varphi : \mathsf{P} \to \mathsf{Q}$ a morphisme of Lie algebras. If P is simply connected then there exists a unique morphism of Lie groups associated $f : P \to Q$ such that $df = \varphi$.
Proof. This follows from Theorem 3.27 of [13].

Corollary 4.6. Any connected and simply connected P of R^n is completely determined by its Lie algebra of polynomial fields P on R^n.
Proof. It's immediate, because P and Q are connected and simply connected spaces of R^n then φ is an unique and global isomorphism.

Theorem 4.7. There is an algebraic correspondence between the properties of Lie algebras of polynomial vector fields and the associated Lie groups.
Proof. Let P and Q two Lie groups on R^n, an application C^∞ noted $\psi : P \times Q \to Q$ such that $p \mapsto \psi(p,.)$ is a morphism of P in the automorphism groups Q, so $P \times_\psi Q$ is a Lie group. The differential $\overline{\psi}(p)$ of $\psi(p,.)$ in identity of Q is a morphism C^∞ of P in the automorphism of Q. Its differential $d\overline{\psi}$ is a homomorphism of P in $Der_{\mathsf{R}}(Q)$ and the Lie algebra of $P \times_\psi Q$ is $\mathsf{P} \oplus_{d\overline{\psi}} \mathsf{Q}$. Conversely, as P and Q are connected and simply connected Lie groups and considering an algebra homomorphisms $\varphi : \mathsf{P} \to Der(\mathsf{Q})$ then there exists an unique action ψ of P on Q such that $d\overline{\psi} = \varphi$ and $P \times_\psi Q$ is a connected and simply connected Lie group and of Lie algebra $\mathsf{P} \oplus_{d\overline{\psi}} \mathsf{Q}$. This finishes our proof.

Corollary 4.8. A Lie group P is said solvable, nilpotent, semisimple or infinite-dimensional if its Lie algebra of polynomial fields on R^n is solvable, nilpotent, semi-simple or denombrable infinite-dimensional.
Proof. This follows from Theorem 4.7.

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