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Ruscheweyh Derivative and a New Generalized Multiplier Differential Operator

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Abstract. A new subclass of analytic functions, containing the linear operator, obtained as a linear combination of Ruscheweyh derivative and a new generalized multiplier transformation, is introduced. Characterization and other properties of this class are studied. Coefficient estimates, distortion theorems of functions with negative coefficients belonging to this class are also determined.

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1. Introduction

Denote by U the open unit disc of the complex plane, $U = \{z \in C : |z| < 1\}$. Let $H(U)$ be the space of holomorphic functions in U. Let A denote the family of functions in $H(U)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

The author in [14], has recently introduced a new generalized multiplier differential operator as follows.

Definition 1.1. Let $m \in N_0 = N \cup \{0\}, \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then for $f \in A$, a new generalized multiplier operator $I_{\alpha,\beta}^{m}$ was defined by

$$I_{\alpha,\beta}^{0} f(z) = f(z), I_{\alpha,\beta}^{1} f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha + \beta}, \ldots, I_{\alpha,\beta}^{m} f(z) = I_{\alpha,\beta}(I_{\alpha,\beta}^{m-1} f(z)).$$

Remark 1.2. Observe that for $f(z)$ given by (1.1), we have

$$I_{\alpha,\beta}^{m} f(z) = z + \sum_{k=2}^{\infty} A_k(\alpha, \beta, m) a_k z^k, \tag{1.2}$$

where

$$A_k(\alpha,\beta,m)=\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m. \tag{1.3}$$

We note that: i) $I^m_{1-\beta,\beta,0}f(z)=D^m_\beta f(z),\ \beta\geq 0$ (See Al-Oboudi [1]), ii) $I^m_{l+1-\beta,\beta,0}f(z)=I^m_{l,\beta}f(z),\quad l>-1,\quad \beta\geq 0$ (See A. Catas [4]) and iii) $I^m_{\alpha,1}f(z)=I^m_\alpha f(z)$ (See Cho and Srivastava [5]) and Cho and Kim [6])

Remark 1.3. $D^m_1 f(z)$ was introduced by Salagean [9] and was considered for $m\geq 0$ in [3].

Definition 1.4. ([8]) For $m\in N_0$, $f\in A$, the operator R^m is defined by $R^m: A\to A$, $R^0 f(z)=f(z)$, $R^1 f(z)=zf'(z),\ldots,(m+1)R^{m+1}f(z)=z(R^m f(z))'+mR^m f(z)$, $z\in U$.

Remark 1.5. If $f(z)=z+\sum_{k=2}^{\infty}a_k z^k\in A$, then $R^m f(z)=z+\sum_{k=2}^{\infty}B_k(m)a_k z^k$, $z\in U$, where

$$B_k(m)=\frac{(m+k-1)!}{m!(k-1)!}. \tag{1.4}$$

The author in [15] has introduced the following operator:

Definition 1.6. Let $f\in A, m\in N_0=N\cup\{0\},\delta\geq 0,\beta\geq 0,\alpha$ a real number such that $\alpha+\beta>0$. Denote by $RI^m_{\alpha,\beta,,\delta}$, the operator given by $RI^m_{\alpha,\beta,\delta}: A\to A$, $RI^m_{\alpha,\beta,\delta}f(z)=(1-\delta)R^m f(z)+\delta I^m_{\alpha,\beta}f(z), z\in U$.

The operator $RI^m_{\alpha,\beta,,\delta}$ was studied in [11], [12] and [13] also. Clearly $RI^m_{\alpha,\beta,,0}=R^m$ and $RI^m_{\alpha,\beta,,1}=I^m_{\alpha,\beta}$.

Remark 1.7. i) If $f(z)=z+\sum_{k=2}^{\infty}a_k z^k\in A$, then we have

$$RI^m_{\alpha,\beta,\delta}f(z)=z+\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k,\quad z\in U,$$

where $A_k(\alpha,\beta,m)$ and $B_k(m)$ are as defined in (1.3) and (1.4), respectively.

By making use of the generalized operator $RI^m_{\alpha,\beta,\delta}$, we define the following new class.

Definition 1.8. Let $f \in A, m \in N_0 = N \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, \rho \in [0,1)$, $\beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$.Then $f(z)$ is in the class $\aleph^{m,\lambda,\mu}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\operatorname{Re}\left(\frac{z(RI^m_{\alpha,\beta,\delta}f(z))' + \mu z^2(RI^m_{\alpha,\beta,\delta}f(z))''}{(1-\lambda)RI^m_{\alpha,\beta,\delta}f(z) + \lambda z(RI^m_{\alpha,\beta,\delta}f(z))'}\right) > \rho, z \in U. \tag{1.5}$$

Taking $\lambda = 0$ in Definition 1.8 we have

Definition 1.9. Let $f \in A, m \in N_0 = N \cup \{0\}, \mu \geq 0, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z)$ is in the class $\Omega^{m,\mu}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\operatorname{Re}\left(\frac{z(RI^m_{\alpha,\beta,\delta}f(z))'}{RI^m_{\alpha,\beta,\delta}f(z)}\right)\left(1 + \mu\frac{z(RI^m_{\alpha,\beta,\delta}f(z))''}{(RI^m_{\alpha,\beta,\delta}f(z))'}\right) > \rho, z \in U. \tag{1.6}$$

In view of the above definition, we deem it worthwhile to note the relevance of the class $\aleph^{m,\lambda,\mu}_{\alpha,\beta,\delta}(\rho)$ with some known classes. Indeed we have i) $\aleph^{m,0,0}_{\alpha,\beta,\delta}(\rho) = S^m_{\alpha,\beta,\delta}(\rho)$, ii) $\aleph^{m,1/2,1/2}_{\alpha,\beta,\delta}(\rho) = K^m_{\alpha,\beta,\delta}(\rho)$, iii) $\aleph^{m,1,1}_{\alpha,\beta,\delta}(\rho) = C^m_{\alpha,\beta,\delta}(\rho)$, iv) $\aleph^{m,0,1/2}_{\alpha,\beta,\delta}(\rho) = \Im^m_{\alpha,\beta,\delta}(\rho)$,v) $\aleph^{m,0,1}_{\alpha,\beta,\delta}(\rho) = \Re^m_{\alpha,\beta,\delta}(\rho)$, vi) $\aleph^{m,1/2,1}_{\alpha,\beta,\delta}(\rho) = \ell^m_{\alpha,\beta,\delta}(\rho)$, vii) $\aleph^{m,0,0}_{1-\beta,\beta,\delta}(\rho) = S^m_{\beta,\delta}(\rho)$ and viii) $\aleph^{m,1,1}_{1-\beta,\beta,\delta}(\rho) = C^m_{\beta,\delta}(\rho)$. The new classes $S^m_{\alpha,\beta,\delta}(\rho)$, $K^m_{\alpha,\beta,\delta}(\rho)$ and $C^m_{\alpha,\beta,\delta}(\rho)$ are defined in [12], while the classes $\Im^m_{\alpha,\beta,\delta}(\rho)$, $\Re^m_{\alpha,\beta,\delta}(\rho)$ and $\ell^m_{\alpha,\beta,\delta}(\rho)$ are introduced in [11].Where as two classes $S^m_{\beta,\delta}(\rho)$ and $C^m_{\beta,\delta}(\rho)$ are investigated in [2].

In section 2 we study the characterization properties for the function $f \in A$ to belong to the class $\aleph^{m,\lambda,\mu}_{\alpha,\beta,\delta}(\rho)$ by obtaining the coefficient bounds. Distortion theorem of functions belonging to this class is obtained in section 3. Analytic functions with negative coefficients belonging to the above class are considered in section 4.

2. Gene1ral properties

In this section we study the characterization properties following the paper of M. Darus and R. Ibrahim [7].Unless otherwise mentioned we shall assume that $A_k(\alpha,\beta,m)$ and $B_k(m)$ are as defined in (1.3) and (1.4), respectively, throughout this paper.

Theorem 2.1. Let $f \in A, m \in N_0 = N \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, \rho \in [0,1)$, $\beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$.If

$$\sum_{k=2}^{\infty}\{(k-\rho) + (k-1)(k\mu - \rho\lambda)\}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho, \tag{2.1}$$

then $f(z)\in \aleph^{m,\lambda,\mu}_{\alpha,\beta,\delta}(\rho)$ .The results (2.1) is sharp.

Proof. It suffices to show that the values of

$\left(\dfrac{z(RI^m_{\alpha,\beta,\delta}f(z))' + \mu z^2(RI^m_{\alpha,\beta,\delta}f(z))''}{(1-\lambda)RI^m_{\alpha,\beta,\delta}f(z)+\lambda z(RI^m_{\alpha,\beta,\delta}f(z))'}\right)$ lie in a circle centred at 1 with radius $1-\rho$.

We have

$$\left|\frac{z(RI^m_{\alpha,\beta,\delta}f(z))' + \mu z^2(RI^m_{\alpha,\beta,\delta}f(z))''}{(1-\lambda)RI^m_{\alpha,\beta,\delta}f(z)+\lambda z(RI^m_{\alpha,\beta,\delta}f(z))'}-1\right| =$$

$$\left|\frac{[z(RI^m_{\alpha,\beta,\delta}f(z))' + \mu z^2(RI^m_{\alpha,\beta,\delta}f(z))''] - [(1-\lambda)RI^m_{\alpha,\beta,\delta}f(z)+\lambda z(RI^m_{\alpha,\beta,\delta}f(z))']}{(1-\lambda)RI^m_{\alpha,\beta,\delta}f(z)+\lambda z(RI^m_{\alpha,\beta,\delta}f(z))'}\right|$$

$$=\left|\frac{\sum_{k=2}^{\infty}\{(k-1)(1-\lambda+\mu k)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}{z+\sum_{k=2}^{\infty}(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}\right|$$

$$\le \frac{\sum_{k=2}^{\infty}\{(k-1)(1+\mu k-\lambda)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k||z|^{k-1}}$$

$$\le \frac{\sum_{k=2}^{\infty}\{(k-1)(1+\mu k-\lambda)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k|}{1-\sum_{k=2}^{\infty}(1+(k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k|}.$$

The last expression is bounded above by $1-\rho$ if

$$\sum_{k=2}^{\infty}\{(k-1)(1+\mu k-\lambda)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \le$$
$$(1-\rho)\left(1-\sum_{k=2}^{\infty}(1+k-1)\lambda)\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k|\right),$$

which is equivalent to (2.1). Hence

$\left|\dfrac{z(RI^m_{\alpha,\beta,\delta}f(z))' + \mu z^2(RI^m_{\alpha,\beta,\delta}f(z))''}{(1-\lambda)RI^m_{\alpha,\beta,\delta}f(z)+\lambda z(RI^m_{\alpha,\beta,\delta}f(z))'}-1\right| \le 1-\rho$ and the theorem is proved. The assertion (2.1) is sharp and extremal function is given by

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\rho}{\{(k-\rho)+(k-1)(k\mu-\rho\lambda)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k,\ z\in U.$$

Theorem 2.2. Let the hypothesis of the Theorem 2.1 be satisfied. Then

$$|a_k| \le \frac{1-\rho}{\{(k-\rho)+(k-1)(k\mu-\rho\lambda)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}},\ k\ge 2.$$

The following inclusion results can be proved using Theorem 2.1

Theorem 2.3. Let $0 \le \rho_1 \le \rho_2 < 1$.Then $\aleph_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho_2) \subseteq \aleph_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho_1)$.

Taking $\lambda = 0$ in Theorem 2.1 we obtain

Corollary 2.4. Let $f \in A, m \in N_0 = N \cup \{0\}, \mu \ge 0, \delta \ge 0, \rho \in [0,1), \beta \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty} \{(k-\rho)+(k-1)k\mu\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho, \qquad (2.2)$$

then $f(z) \in \Omega_{\alpha,\beta,\delta}^{m,\mu}(\rho)$ .The result (2.2) is sharp.

3. Distortion theorems

Theorem 3.1. Let $f \in A, m \in N_0 = N \cup \{0\}, \mu \ge 0, 0 \le \lambda \le 1, \lambda \le \mu, \delta \ge 0, \rho \in [0,1)$, $\beta \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty} \{(k-\rho)+(k-1)(k\mu-\rho\lambda)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho,$$

then

$$|z| - \frac{1-\rho}{(2-\rho)+(2\mu-\rho\lambda)}|z|^2 \le \left|RI_{\alpha,\beta,\delta}^{m} f(z)\right| \le |z| + \frac{1-\rho}{(2-\rho)+(2\mu-\rho\lambda)}|z|^2,$$

$z \in U$.

Proof. Note that $[(2-\rho)+(2\mu-\rho\lambda)]\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \le$

$$\sum_{k=2}^{\infty} \{(k-\rho)+(k-1)(k\mu-\rho\lambda)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho,$$

by Theorem 2.1. Thus

$\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \le \frac{1-\rho}{(2-\rho)+(2\mu-\rho\lambda)}$ .Hence we obtain

$$\begin{aligned}\left|RI_{\alpha,\beta,\delta}^{m} f(z)\right| &\le |z| + \sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k||z|^k \\ &\le |z| + |z|^2\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \\ &\le |z| + \frac{1-\rho}{(2-\rho)+(2\mu-\rho\lambda)}|z|^2.\end{aligned}$$

Similarly

$$\left|RI_{\alpha,\beta,\delta}^{m} f(z)\right| \ge |z| - \sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k||z|^k$$

$$\geq |z| - |z|^2 \sum_{k=2}^{\infty} \{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k|$$

$$\geq |z| - \frac{1-\rho}{(2-\rho)+(2\mu-\rho\lambda)}|z|^2.$$

This completes the proof of theorem.

.

Taking $\lambda = 0$ in Theorem 3.1 we get

Corollary 3.2. Let $f \in A, m \in N_0 = N \cup \{0\}, \mu \geq 0, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$ .If

$$\sum_{k=2}^{\infty} \{[(k-\rho)+(k-1)k\mu\}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho,$$

then**.**

$$|z| - \frac{1-\rho}{2-\rho+2\mu}|z|^2 \leq \left|RI^m_{\alpha,\beta,\delta} f(z)\right| \leq |z| + \frac{1-\rho}{2-\rho+2\mu}|z|^2, z \in U .$$

Theorem 3.3. Let $f \in A, m \in N_0 = N \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, \rho \in [0,1)$, $\beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty} \{(k-\rho)+(k-1)(k\mu-\rho\lambda)\}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho,$$

then

$$|f(z)| \geq |z| - \frac{1-\rho}{\{(2-\rho)+(2\mu-\rho\lambda)\}\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{\{(2-\rho)+(2\mu-\rho\lambda)\}\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2 \ z \in U$$

.

Proof. In virtue of Theorem 2.1, we have

$$\{(2-\rho)+(2\mu-\rho\lambda)\}\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}\sum_{k=2}^{\infty}|a_k| \leq$$

$$\sum_{k=2}^{\infty} \{(k-\rho)+(k-1)(k\mu-\rho\lambda)\}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho.$$

Thus $\sum_{k=2}^{\infty}|a_k| \leq \dfrac{1-\rho}{\{(2-\rho)+(2\mu-\rho\lambda)\}\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,\delta)\}}$.

So we get

$$|f(z)| \le |z| + |z|^2 \sum_{k=2}^{\infty} |a_k|$$

$$\le |z| + \frac{1-\rho}{\{(2-\rho)+(2\mu-\rho\lambda)\}\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2 .$$

On the other hand

$$|f(z)| \ge |z| - |z|^2 \sum_{k=2}^{\infty} |a_k|$$

$$\ge |z| - \frac{1-\rho}{\{(2-\rho)+(2\mu-\rho\lambda)\}\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2 .$$

.

Taking $\lambda = 0$ in Theorem 3.3 we get

Corollary 3.4. Let $f \in A, m \in N_0 = N \cup \{0\}, \mu \ge 0, \delta \ge 0, \rho \in [0,1), \beta \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$.

If $\sum_{k=2}^{\infty} [(k-\rho)+(k-1)k\mu]\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho,$ then

$$|f(z)| \le |z| + \frac{1-\rho}{(2-\rho+2\mu)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2 , z \in U ,$$

and

$$|f(z)| \ge |z| - \frac{1-\rho}{(2-\rho+2\mu)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} |z|^2 , z \in U .$$

4. Functions with negative coefficients

Let T denote the subclass of A consisting of functions of the form $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \ge 0$. We denote by $T\aleph_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho)$ and $T\Omega_{\alpha,\beta,\delta}^{m,\mu}(\rho)$, the classes of functions $f(z) \in T$ satisfying (1.5) and (1.6), respectively. We study the coefficient estimates, distortion theorems and other properties of the class $T\aleph_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho)$, following the paper of H. Silverman [10].For functions in T, the converse of Theorem 2.1 is also true.

Theorem 4.1. Let $m \in N_0 = N \cup \{0\}, \mu \ge 0, 0 \le \lambda \le 1, \lambda \le \mu, \delta \ge 0, \rho \in [0,1), \beta \ge 0, \alpha$ a real number such that $\alpha + \beta > 0$ .A function $f(z) \in T$ is in $T\aleph_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho)$ if and only if

$$\sum_{k=2}^{\infty} \{(k-\rho)+(k-1)(k\mu-\rho\lambda)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \le 1-\rho. \tag{4.1}$$

The result (4.1) is sharp.

Proof. In view of Theorem 2.1, it suffices to prove the only if part. Assume that

$$\operatorname{Re}\left(\frac{z(RI^{m}_{\alpha,\beta,\delta}f(z))' + \mu z^2 (RI^{m}_{\alpha,\beta,\delta}f(z))''}{(1-\lambda)RI^{m}_{\alpha,\beta,\delta}f(z) + \lambda z(RI^{m}_{\alpha,\beta,\delta}f(z))'}\right) =$$

$$\operatorname{Re}\left(\frac{z-\sum_{k=2}^{\infty}(1+\mu(k-1))\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}ka_k z^k}{z-\sum_{k=2}^{\infty}(1+\lambda(k-1))\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k z^k}\right) > \rho \quad (4.2)$$

clearing the denominator in (4.2) and letting $z \to 1^-$ through real values, we obtain

$$1-\sum_{k=2}^{\infty}(1+\mu(k-1))\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}ka_k \geq$$

$$\rho\left(1-\sum_{k=2}^{\infty}(1+\lambda(k-1))\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}a_k\right).$$

Hence we obtain (4.1), and the proof is complete. Finally, we note that assertion (4.1) of Theorem 4.1 is sharp, extremal functions being

$$f_k(z) = z - \frac{1-\rho}{\{(2-\rho)+(2\mu-\lambda\rho)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^k, k \geq 2, z \in U.$$

Taking $\lambda = 0$ in Theorem 4.1 we obtain

Corollary 4.2. Let $m \in N_0 = N \cup \{0\}, \mu \geq 0, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$ .A function $f(z) \in T$ is in $T\Omega^{m,\mu}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$\sum_{k=2}^{\infty}[(k-\rho)+(k-1)k\mu]\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}|a_k| \leq 1-\rho. \quad (4.3)$$

The result (4.3) is sharp.

Our coefficient bounds enable us to prove the following.

Theorem 4.3. Let
$m \in N_0 = N \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. If $f \in T\aleph^{m,\lambda,\mu}_{\alpha,\beta,\delta}(\rho)$, then

$$|f(z)| \geq |z| - \frac{1-\rho}{\{(2-\rho)+(2\mu-\rho\lambda)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{\{(2-\rho)+(2\mu-\rho\lambda)\}\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

with $\quad f(z) = z - \dfrac{1-\rho}{\{(2-\rho)+(2\mu-\lambda\rho)\}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}}z^2, (|z| = r),$

for equality.

Putting $\lambda = 0$ in Theorem 4.3 we have

Corollary 4.4. Let $m \in N_0 = N \cup \{0\}, \mu \geq 0, \delta \geq 0, \rho \in [0,1), \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$ .If $f \in T\Omega^{m,\mu}_{\alpha,\beta,\delta}(\rho)$, then

$$|f(z)| \geq |z| - \frac{1-\rho}{(2-\rho+2\mu)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{(2-\rho+2\mu)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}}|z|^2, z \in U,$$

with equality for $f(z) = z - \dfrac{1-\rho}{(2-\rho+2\mu)\{(m+1)(1-\delta)+\delta A_2(\alpha,\beta,m)\}} z^2, (|z| = r)$.

We conclude by mentioning some interesting consequences of our results.

i) Allowing $\lambda = \mu = 0$, our results are comparable with those results of the author in [12].

ii) Taking $\lambda = \mu = 1/2$, we see that our results agree with corresponding results of the author in [12].

iii) Putting $\lambda = \mu = 1$, we get results matching with those of the author in [12].

iv) Letting $\lambda = 0$ and $\mu = 1/2$, our results conform to results of the author in [11].

v) Setting $\lambda = 0$ and $\mu = 1$, we obtain corresponding results of the author in [11].

vi) For $\lambda = 1/2$ and $\mu = 1$, we have corresponding results of the author in [11].

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