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The *L(2, 1)*-Labeling on α-Product of Graphs

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Abstract. The *L(2, 1)*-labeling (or distance two labeling) of a graph G is an integer labeling of G in which two vertices at distance one from each other must have labels differing by at least 2 and those at distance two must differ by at least 1. The *L(2, 1)*-labeling numberλ(G) of G is the smallest number k such that G has an *L(2, 1)*-labeling $\max\{f(v): v \in V(G)\} = k$ with max (f(v): v ∈ V(G) = k. In this paper, upper bound for the *L(2, 1)*-labeling number for the α-product of two graphs has been obtained in terms of the maximum degrees of the graphs involved. Degrees of vertices, vertex of maximum degree and number of vertices of maximum degree have been discussed in the α-product of two graphs.

Keywords: Channel assignment, *L(2,1)*-labeling, *L(2,1)*-labeling number, graph α-product

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

The concept of *L(2, 1)*-labeling in graph come into existence with the solution of frequency assignment problem. In fact, in this problem a frequency in the form of a non-negative integer is to assign to each radio or TV transmitters located at various places such that communication does not interfere. Hale [6] was first person who formulated this problem as a graph vertex coloring problem.

Latter, Griggs and Yeh [5] introduced *L(2, 1)*-labeling on a simple graph. Let G be a graph with vertex set $V(G)$. A function $f : V(G) \to Z^+ \cup \{0\}$, where Z^+ is a set of positive integers, is called *L(2, 1)*-labeling or distance two labeling if $|f(u) - f(v)| \geq 2$ when $d(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ when $d(u,v) = 2$, where $d(u,v)$ is distance between u and v in G. A *k*- *L(2, 1)*-labeling is an *L(2, 1)*-labeling such that no labels is greater than k. The *L(2, 1)*-labeling number of G, denoted by $\lambda(G)$ or λ, is the smallest number k such that G has a *k*- *L(2, 1)*-labeling. The *L(2, 1)*-labeling has been extensively studied in recent past by many researchers [see, 1, 2, 4, 7, 8, 12-22]. The common trend in most of the research paper is either to determine the value of *L(2, 1)*-labeling number or to suggest bounds for particular classes of graphs.

Griggs and Yeh [5] provided an upper bound $\Delta^2 + 2\Delta$ for a general graph with the maximum degree Δ. Later, Chang and Kuo [1] improved the upper bound to $\Delta^2 + \Delta$, while Kral and Skrekarski [10] reduced the upper bound to $\Delta^2 + \Delta - 1$. Furthermore, recently Gonccalves [4] proved the bound $\Delta^2 + \Delta - 2$ which is the present best record. If G is a diameter 2 graph, then $\lambda(G) \leq \Delta^2$. The upper bound is attainable by Moore graphs (diameter 2 graphs with order $\Delta^2 + 1$). (Such graphs exist only if $\Delta = 2, 3, 7$ and possibly 57; [5]). Thus Griggs and Yeh [5] conjectured that the best bound is Δ^2 for any graph G with the maximum degree $\Delta \geq 2$. (This is not true for $\Delta = 1$. For example, $\Delta(K_2) = 1$ but $\lambda(K_2) = 2$).

Graph products play an important role in connecting many useful networks. Klavzar and Spacepan [9] have shown that Δ^2-conjecture holds for graphs that are direct or strong products of nontrivial graphs. After that Shao, et al. [13] have improved bounds on the *L(2, 1)*-labeling number of direct and strong product of nontrivial graphs with refined approaches. Shao and Shang [15] also consider the graph formed by the Cartesian sum of graphs and prove that the λ-number of *L(2, 1)*-labeling of this graph satisfies the Δ^2-conjecture (with minor exceptions).

In this paper, we have considered the graph formed by the α-product of graphs [3] and obtained a general upper bound for *L(2, 1)*-labeling number in term of maximum degree of the graphs. In the case of α-product of graphs, *L(2, 1)*-labeling number of graph holds Griggs and Yeh's conjecture [5] with minor exceptions.

2. A labeling algorithm

A subset X of $V(G)$ is called an i-stable set (or i-independent set) if the distance between any two vertices in X is greater than i, i.e. $\{d(u,v) > i, \ \forall \ u,v \in X\}$. A 1-stable set is a usual independent set. A maximal 2-stable subset X of a set Y is a 2-stable subset of Y such that X is not a proper subset of any 2-stable subset of Y.

Chang and Kuo [1] proposed the following algorithm to obtain an *L(2, 1)*-labeling and the maximum value of that labeling on a given graph.

Algorithm:
Input: A graph $G = (V, E)$.
Output: The value k is the maximum label.
Idea: In each step i, find a 2-maximal 2-stable set from the unlabeled vertices that are distance at least two away from those vertices labelled in the previous step. Then label all the vertices in that 2-stable with i in current stage. The label i starts from 0 and then increases by 1 in each step. The maximum label k is the final value of i.

Initialization: Set $X_{-1} = \phi; \ \ V = V(G); \ \ i = 0$.

Iteration:

1. Determine Y_i and X_i.
   - $Y_i = \{u \in V : u \text{ is unlabeled and } d(u,v) \geq 2 \ \ \forall\ v \in X_{i-1}\}$.
   - X_i is a maximal 2-stable subset of Y_i.
   - If $Y_i = \phi$ then set $X_i = \phi$.
2. Label the vertices of X_i (if there is any) with i.
3. $V \leftarrow V - X_i$.
4. If $V \neq \phi$, then $i \leftarrow i+1$, go to step 1.
5. Record the current i as k (which is the maximum label). Stop.

Thus k is an upper bound on $\lambda(G)$.

Let u be a vertex with largest label k obtained by above Algorithm. We have the following sets on the basis of Algorithm just defined above.

$I_1 = \{i \ : \ 0 \leq i \leq k-1 \ \text{ and } \ d(u,v) = 1 \ \text{ for some } \ v \in X_i\}$,

i.e. I_1 is the set of labels of the neighbourhood of the vertex u.

$I_2 = \{i \ : \ 0 \leq i \leq k-1 \ \text{ and } \ d(u,v) \leq 2 \ \text{ for some } \ v \in X_i\}$,

i.e. I_2 is the set of labels of the vertices at distance at most 2 from the vertex u.

$I_3 = \{i \ : \ 0 \leq i \leq k-1 \ \text{ and } \ d(u,v) \geq 3 \ \text{ for all } \ v \in X_i\}$

$\quad = \{0,1,\ldots,k-1\} - I_2$.

i.e. I_3 consists of the labels not used by the vertices at distance at most 2 from the vertex u.

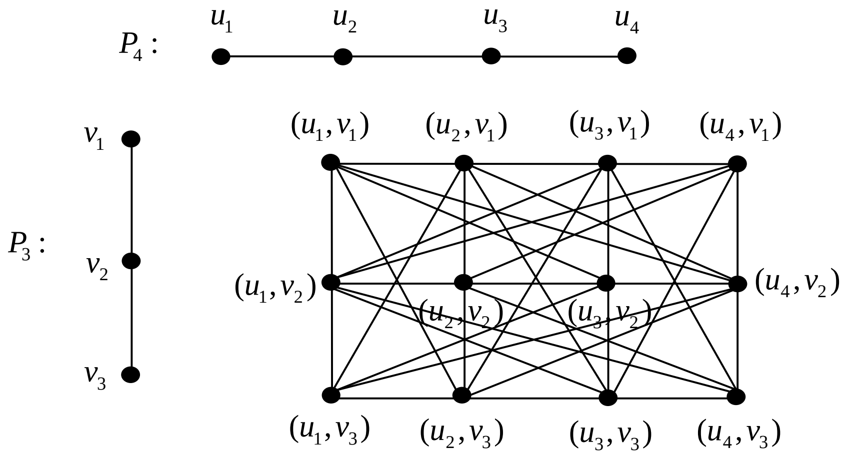
Then Chang and Kuo showed that $\lambda(G) \leq k \leq |I_2| + |I_3| \leq |I_2| + |I_1|$.

In order to find k, it suffices to estimate $B = |I_1| + |I_2|$ in terms of $\Delta(G)$. We will investigate the value B with respect to a particular graph (α-product of two graphs). The notations which have been introduced in this section will also be used in the following sections.

3. The α-product of graphs

The α-product $G \boxed{\alpha} H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which the vertex (u,v) is adjacent to the vertex (u',v') if and only if either u is adjacent to u' in G and v is not adjacent to v' in H or u is not adjacent to u' in G and v is adjacent to v' in H. i.e. either $uu' \in E(G)$ and $vv' \notin E(H)$ or $uu' \notin E(G)$ and $vv' \in E(H)$. For example, we consider the Figure 1.

Now, we state and prove the following corollary to find out the degree of any vertex of α-product of two graphs.



$P_4 \boxed{\alpha} P_3$

Figure1: α-product of P_4 and P_3

Corollary 3.1. Let $u_\alpha = (u,v) \in V(G \boxed{\alpha} H)$, where $u \in V(G)$, $v \in V(H)$, $|V(G)| = n_1$, $|V(H)| = n_2$ $\deg_G(u) = d_1$ and $\deg_H(v) = d_2$ then $\deg_{G\boxed{\alpha}H}(u_\alpha) = (n_1 - d_1)d_2 + (n_2 - d_2)d_1$.

Proof: By the definition of α- product, the vertex (u,v) is adjacent to the vertex (u',v') if and only if either $uu' \in E(G)$ and $vv' \notin E(H)$ or $uu' \notin E(G)$ and $vv' \in E(H)$.

Now number of adjacent vertices to u in G i.e. $uu' \in E(G) = d_1$

Number of non-adjacent vertices to v in H i.e. $vv' \notin E(H) = n_2 - d_2$

Number of adjacent vertices to v in H i.e. $vv' \in E(H) = d_2$

Number of non-adjacent vertices to u in G i.e. $uu' \notin E(G) = n_1 - d_1$

$\therefore$ Number of vertices where $uu' \in E(G)$ and $vv' \notin E(H) = d_1(n_2 - d_2)$

And Number of vertices where $uu' \notin E(G)$ and $vv' \in E(H) = d_2(n_1 - d_1)$

$\therefore$ Total number of vertices adjacent to $u_\alpha = (u,v)$ in $G \boxed{\alpha} H$

$= \deg_{G\boxed{\alpha}H}(u_\alpha) = (n_1 - d_1)d_2 - (n_2 - d_2)d_1$.

3.2. The maximum degree (largest degree) of $G \boxed{\alpha} H$

The maximum (largest) degree of $G \boxed{\alpha} H$ plays an important role in finding out the upper bound for the *L(2, 1)*-labeling. To find out the maximum degree of $G \boxed{\alpha} H$, we proceed as follows:

Let Δ_1, Δ_2 be the maximum degree of G, H and Δ'_1, Δ'_2 be the minimum degree of G, H respectively. Let Δ be the maximum degree of $G \boxed{\alpha} H$.

Case I: If $n_1 \geq n_2$, then

$$\Delta = \begin{cases} (n_1 - \Delta'_1)\Delta_2 + (n_2 - \Delta_2)\Delta'_1 & \text{when } \Delta_1 \leq \Delta_2 \\ (n_1 - \Delta_1)\Delta'_2 + (n_2 - \Delta'_2)\Delta_1 & \text{when } \Delta_1 > \Delta_2 \end{cases}$$

Case II: If $n_1 < n_2$, then

$$\Delta = \begin{cases} (n_1 - \Delta_1)\Delta'_2 + (n_2 - \Delta'_2)\Delta_1 & \text{when } \Delta_1 \geq \Delta_2 \\ (n_1 - \Delta'_1)\Delta_2 + (n_2 - \Delta_2)\Delta'_1 & \text{when } \Delta_1 < \Delta_2 \end{cases}$$

From the above two cases, it can be written as

$\Delta = \max[(n_1 - \Delta'_1)\Delta_2 + (n_2 - \Delta_2)\Delta'_1,\ (n_1 - \Delta_1)\Delta'_2 + (n_2 - \Delta'_2)\Delta_1]$.

3.3. The number of vertices of maximum degree in $G \boxed{\alpha} H$

Let $n_{1\min}$ and $n_{1\max}$ be the number of vertices of minimum and maximum degree in a graph G and $n_{2\min}$ and $n_{2\max}$ be the number of vertices of minimum and maximum degree in a graph H. Then

Case I: $n_1 \geq n_2$

If $\Delta_1 \leq \Delta_2$, then

The number of vertices of maximum degree in graph $G \boxed{\alpha} H = n_{1\min}.n_{2\max}$

If $\Delta_1 > \Delta_2$, then

The number of vertices of maximum degree in graph $G \boxed{\alpha} H = n_{1\max}.n_{2\min}$

Case II: $n_1 < n_2$

If $\Delta_1 \geq \Delta_2$, then

The number of vertices of maximum degree in graph $G \boxed{\alpha} H = n_{1\max}.n_{2\min}$

If $\Delta_1 < \Delta_2$, then

The number of vertices of maximum degree in graph $G \boxed{\alpha} H = n_{1\min}.n_{2\max}$

4. Upper bound for the *L(2, 1)*-labeling number in $G \boxed{\alpha} H$

In this section, general upper bound for the *L(2, 1)*-labeling number (λ-number) of α-product in term of maximum degree of the graphs has been established. In this regard, we state and prove the following theorem.

Theorem 4.1. Let $\Delta, \Delta_1, \Delta_2$ be the maximum degree of $G\boxed{\alpha}H, G, H$ and n, n_1, n_2 be the number of vertices of $G\boxed{\alpha}H, G, H$ respectively. If $\Delta_1, \Delta_2 \geq 2$, then $\lambda(G\boxed{\alpha}H) \leq \Delta^2 - \Delta_1\Delta_2(n_1 + n_2 - 6)$.

Proof: Let $u_\alpha = (u,v)$ be any vertex in the graph $G\boxed{\alpha}H$. Denote $d = \deg_{G\boxed{\alpha}H}(u_\alpha)$, $d_1 = \deg_G(u)$, $d_2 = \deg_H(v)$, $\Delta_1 = \max\deg(G)$, $\Delta'_1 = \min\deg(G)$, $\Delta_2 = \max\deg(H)$, $\Delta'_2 = \min\deg(H)$, $|v(G)| = n_1$, and $|v(H)| = n_2$. Hence $d = (n_1 - d_1)d_2 + (n_2 - d_2)d_1$ (from corollary 1) and

$\Delta = \Delta(G\boxed{\alpha}H) = \max[(n_1 - \Delta'_1)\Delta_2 + (n_2 - \Delta_2)\Delta'_1,\ (n_1 - \Delta_1)\Delta'_2 + (n_2 - \Delta'_2)\Delta_1]$.

Let us consider the Figure 2. For any vertex u' in G at distance 2 from u, there must be a path $u'u''u$ of length two between u' and u in G; but the degree of v in H is d_2, i.e. v has d_2 adjacent vertices in H, by the definition of α-product $G\boxed{\alpha}H$, there must be $2d_2 + 1$ internally-disjoint paths(two paths are said to be

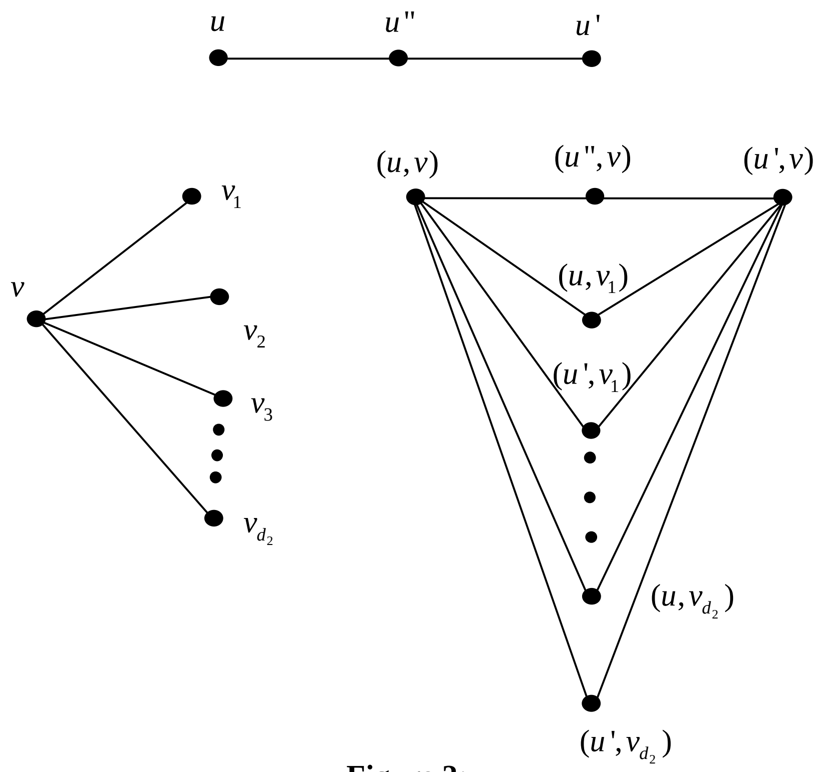


Figure 2:

internally-disjoint if they do not intersect each other) of length two between (u',v) and (u,v). Hence for any vertex in G at distance 2 from $u_\alpha = (u,v)$ which are coincided in

$G\boxed{\alpha}H$; on the contrary whenever there is no such a vertex in G at distance 2 from u in G, there exist no such corresponding $2d_2+1$ vertices at distance 2 from $u_\alpha=(u,v)$ which are coincided in $G\boxed{\alpha}H$. In the former case, since such $2d_2+1$ vertices at distance 2 from $u_\alpha=(u,v)$ are coincided in $G\boxed{\alpha}H$ and hence they should be counted once only and therefore we have to subtract $2d_2+1-1=2d_2$ from the value $d(\Delta-1)$ which is the best possible number of vertices at distance 2 from a vertex $u_\alpha=(u,v)$ in $G\boxed{\alpha}H$. Let the number of vertices in G at distance 2 from u be t, then $t\in[0,\ d_1(\Delta_1-1)]$. Now, if we take $t=d_1(\Delta_1-1)$ which is the best possible number of vertices at distance 2 from a vertex u in G, then to get the number of vertices at distance 2 from $u_\alpha=(u,v)$ in $G\boxed{\alpha}H$, we will have to subtract at least $2d_1d_2(\Delta_1-1)$ from the value $d(\Delta-1)$.

For H, we can proceed in similar way to get the number of vertices at distance 2 from $u_\alpha=(u,v)$ in $G\boxed{\alpha}H$ and in this case subtract $2d_1d_2(\Delta_2-1)$ from the value $d(\Delta-1)$. Hence, the number of vertices at distance 2 from $u_\alpha=(u,v)$ in $G\boxed{\alpha}H$ will decrease $2d_1d_2(\Delta_1-1)+2d_1d_2(\Delta_2-1)=2d_1d_2(\Delta_1+\Delta_2-2)$ from the value $d(\Delta-1)$ altogether. By the above analysis, the number $d(\Delta-1)-2d_1d_2(\Delta_1+\Delta_2-2)$ is now the best possible number of vertices at distance 2 from $u_\alpha=(u,v)$ in $G\boxed{\alpha}H$.

But some cases are remaining to be considered for finding out the best possible number of vertices at distance 2 from $u_\alpha=(u,v)$ in $G\boxed{\alpha}H$.

Let ε be the number of edges of the subgraph F induced by the neighbours of u_α. The edges of the subgraph F induced by the neighbours of u_α can be divided into the following two cases.

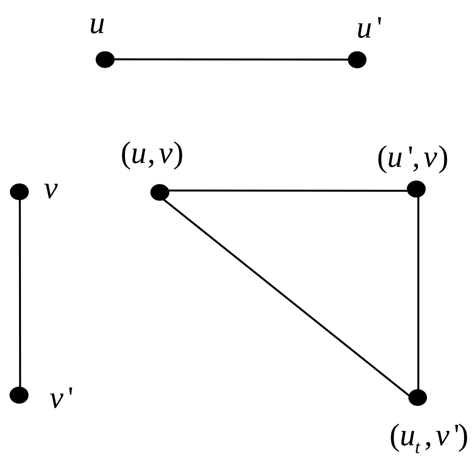


Figure 3:

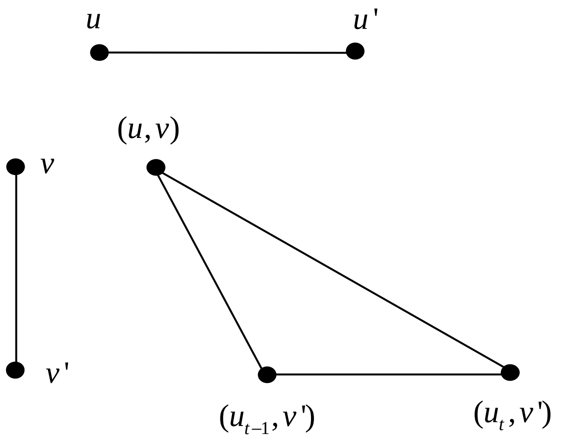


Figure 4:

Case I: Consider the Figure 3 for this case. For each neighbour vertex (u',v) (where u' is adjacent to u in G) of $u_\alpha = (u,v)$ and any vertex (u_t,v') (where v' is adjacent to v in H and u_t is any vertex of G which is not adjacent to u and u' in G). (u_t,v') must be the common neighbour vertex of (u',v) and (u,v), then there must be an edge between (u_t,v') and (u',v). But there are at least $(n_1 - d_1 - \Delta_1)d_2$ neighbour vertices like (u_t,v') of $x = (u,v)$ and there are d_1 neighbour vertices like (u',v) of $u_\alpha = (u,v)$. Hence the number of edges of the subgraph F induced by the neighbour vertices of u_α is at least $(n_1 - d_1 - \Delta_1)d_1 d_2$ i.e. $\varepsilon \geq (n_1 - d_1 - \Delta_1)d_1 d_2$. By a symmetric analysis, the neighbour vertices of u_α should again add at least $(n_2 - d_2 - \Delta_2)d_1 d_2$.

Case II: Consider the Figure 4 for this case. For each neighbour (u_{t-1},v') (where v' is adjacent to v in H and u_{t-1} is any vertex of G which is neither equal to u nor adjacent to u) and (u_t,v') (where u_t is adjacent to u_{t-1} in G) of $u_\alpha = (u,v)$ in $G\boxed{\alpha}H$. Obviously vertex (u_t,v') is adjacent to (u_{t-1},v') in $G\boxed{\alpha}H$, hence there is an edge between them. There are at least $(n_1 - d_1 - \Delta_1)d_2$ neighbour vertices like (u_t,v') of $u_\alpha = (u,v)$ in $G\boxed{\alpha}H$. So at least $(n_1 - d_1 - \Delta_1)d_2$ edges will exist between (u_t,v') and (u_{t-1},v'). Hence the number of edges of the subgraph F induced by the neighbours of u_α is at least $(n_1 - d_1 - \Delta_1)d_2$. By symmetric analysis, the neighbours of u_α should again add at least $(n_2 - d_2 - \Delta_2)d_1$.

By the analysis of the above two cases,
$$\varepsilon \geq (n_1 - d_1 - \Delta_1)d_1 d_2 + (n_2 - d_2 - \Delta_2)d_1 d_2 + (n_1 - d_1 - \Delta_1)d_2 + (n_2 - d_2 - \Delta_2)d_1 .$$

Whenever there is an edge in F, the number of vertices with distance 2 from u_α will decrease by 2, hence the number of vertices with distance 2 from $u_\alpha = (u,v)$ in $G\boxed{\alpha}H$ will still need at least a decrease $(n_1-d_1-\Delta_1)d_1d_2+(n_2-d_2-\Delta_2)d_1d_2+(n_1-d_1-\Delta_1)d_2+$ $(n_2-d_2-\Delta_2)d_1$ from the value $d(\Delta-1)-2d_1d_2(\Delta_1+\Delta_2-2)$ (the number $d(\Delta-1)-2d_1d_2(\Delta_1+\Delta_2-2)$ is now the best possible for the number of vertices with distance 2 from $u_\alpha=(u,v)$ in $G\boxed{\alpha}H$).

Hence for the vertex u_α, the number of vertices with distance 1 from u_α is no greater than Δ. The number of vertices with distance 2 from u_α is no greater than

$$d(\Delta-1)-2d_1d_2(\Delta_1+\Delta_2-2)-(n_1-d_1-\Delta_1)d_1d_2-(n_2-d_2-\Delta_2)d_1d_2-(n_1-d_1-\Delta_1)d_2$$
$$-(n_2-d_2-\Delta_2)d_1$$

Hence $|I_1|\le d$,

$$|I_2|\le d+d(\Delta-1)-2d_1d_2(\Delta_1+\Delta_2-2)-(n_1-d_1-\Delta_1)d_1d_2-(n_2-d_2-\Delta_2)d_1d_2$$
$$-(n_1-d_1-\Delta_1)d_2-(n_2-d_2-\Delta_2)d_1$$

Then

$$B=|I_1|+|I_2|$$
$$=d+d+d(\Delta-1)-2d_1d_2(\Delta_1+\Delta_2-2)-(n_1-d_1-\Delta_1)d_1d_2-(n_2-d_2-\Delta_2)d_1d_2$$
$$-(n_1-d_1-\Delta_1)d_2-(n_2-d_2-\Delta_2)d_1$$

$$=d(\Delta+1)-d_1d_2(n_1+n_2+\Delta_1+\Delta_2-d_1-d_2-4)-(n_1-d_1-\Delta_1)d_2-(n_2-d_2-\Delta_2)d_1$$

$$=((n_1-d_1)d_2+(n_2-d_2)d_1)(\Delta+1)-d_1d_2(n_1+n_2+\Delta_1+\Delta_2-d_1-d_2-4)-(n_1-d_1-\Delta_1)d_2$$
$$-(n_2-d_2-\Delta_2)d_1$$

Define

$$f(s,t)=((n_1-s)t+(n_2-t)s)(\Delta+1)-st(n_1+n_2+\Delta_1+\Delta_2-s-t-4)-(n_1-s-\Delta_1)t$$
$$-(n_2-t-\Delta_2)s$$

Then $f(s,t)$ has the absolute maximum at (Δ_1,Δ_2) on $[0,\Delta_1]\times[0,\Delta_2]$.

$$\therefore\ f(\Delta_1,\Delta_2)=((n_1-\Delta_1)\Delta_2+(n_2-\Delta_2)\Delta_1)(\Delta+1)-\Delta_1\Delta_2(n_1+n_2+\Delta_1+\Delta_2-\Delta_1-\Delta_2-4)$$
$$-(n_1-\Delta_1-\Delta_1)\Delta_2-(n_2-\Delta_2-\Delta_2)\Delta_1$$

$$=((n_1-\Delta_1)\Delta_2+(n_2-\Delta_2)\Delta_1)(\Delta+1)-\Delta_1\Delta_2(n_1+n_2-4)-(n_1-2\Delta_1)\Delta_2-(n_2-2\Delta_2)\Delta_1$$

$$=((n_1-\Delta_1)\Delta_2+(n_2-\Delta_2)\Delta_1)\Delta+(n_1-\Delta_1)\Delta_2+(n_2-\Delta_2)\Delta_1-\Delta_1\Delta_2(n_1+n_2-4)$$
$$-(n_1-2\Delta_1)\Delta_2-(n_2-2\Delta_2)\Delta_1$$

Since Δ is the maximum degree of graph $G\boxed{\alpha}H$ and $(n_1-\Delta_1)\Delta_2+(n_2-\Delta_2)\Delta_1$ is the degree of any vertices in graph $G\boxed{\alpha}H$. Therefore

$$(n_1-\Delta_1)\Delta_2+(n_2-\Delta_2)\Delta_1\le\Delta=\max[\,(n_1-\Delta_1^{'})\Delta_2+(n_2-\Delta_2)\Delta_1^{'},\,(n_1-\Delta_1)\Delta_2^{'}+(n_2-\Delta_2^{'})\Delta_1]$$
$$\therefore\ f(\Delta_1,\Delta_2)\le\Delta\Delta-n_1\Delta_2-\Delta_1\Delta_2+n_2\Delta_1-\Delta_1\Delta_2-\Delta_1\Delta_2(n_1+n_2-4)-n_1\Delta_2+2\Delta_1\Delta_2-n_2\Delta_1+2\Delta_1\Delta_2$$
$$=\Delta^2-\Delta_1\Delta_2(n_1+n_2-6)$$

Then,

$\lambda(G\boxed{\alpha}H)\le k\le B\le\Delta^2-\Delta_1\Delta_2(n_1+n_2-6)$,
where, $\Delta=\max[\,(n_1-\Delta_1^{'})\Delta_2+(n_2-\Delta_2)\Delta_1^{'},\,(n_1-\Delta_1)\Delta_2^{'}+(n_2-\Delta_2^{'})\Delta_1]$.

Corollary 4.2. Let Δ be the maximum degree of $G\boxed{\alpha}H$. Then $\lambda(G\boxed{\alpha}H)\le\Delta^2$ except for when one of $\Delta(G)$ and $\Delta(H)$ is 1.

Proof: If one of Δ_1 or Δ_2 is 1 then $G\boxed{\alpha}H$ is still a general graph, hence we can suppose that $\Delta_1\ge2$ and $\Delta_2\ge2$ (hence $n_1\ge3$ and $n_2\ge3$). Then
$\Delta^2-\Delta_1\Delta_2(n_1+n_2-6)\le\Delta^2-2.2(3+3-6)=\Delta^2$
This implies that $\lambda(G\boxed{\alpha}H)\le k\le B\le\Delta^2-\Delta_1\Delta_2(n_1+n_2-6)\le\Delta^2$.
Therefore the result follows.

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