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# Multi-stage Homotopy-perturbation Method for the Fractional Order Chen System

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***Abstract.*** In this paper, the multistage homotopy perturbation method is extended to solve the Chen fractional order systems. The multistage homotopy perturbation method is only a simple modification of the standard homotopy perturbation method, in which it is treated as an algorithm in a sequence of intervals for finding accurate approximate analytical solutions. The fractional derivatives are described in the Caputo sense. The solutions are obtained in the form of rapidly convergent infinite series with easily computable terms. Numerical results reveal that the multistage method is a promising tool for the Chen fractional order systems.

***Keywords:*** Fractional order differential equation, homotopy perturbation method, Chen system

***AMS Mathematics Subject Classification (2010):*** 93A30, 92B05

## 1. Introduction

Over the last decades, fractional order differential equations (FODEs) have been used to describe a variety of systems in interdisciplinary fields, such as viscoelasticity, biology, physiology, medicine, hydraulics geology and engineering. Based on the extension of applications of FODEs, the Chua's fractional order systems become a new topic due to its potential applications especially in secure communication, encryption and control processing.

For better understanding the dynamic behavior of a Chen fractional order system, the solution of the Chua's fractional order system is much involved. In general, it is difficult to obtain the exact solution for nonlinear FODEs. Finding accurate and efficient methods for solving FODEs has been an active research undertaking. Some analytical and numerical methods have been proposed for the solutions of FODEs. The classical approaches include Laplace transform method, Mellin transform method, fractional Green's function method and power series method.

In the literatures of fractional non-linear field, two approximation methods have been advised for the numerical solutions of the fractional order systems. One method is based on the approximation of the fractional order system behavior in the frequency domain and time domain. The other method is the well known predictor correctors scheme .According to the theory of factional calculus, our concern in this work is to extend the MHPM to consider the approximate numeric analytic solutions of the Chen fractional order systems. The MHPM is a very effective and simple method for the accurate approximate solutions of the Chen fractional order systems for a long time.

The paper is organized as follows. In Section 2, some definitions and properties of fractional calculus are introduced. Section 3 is devoted to describe the standard HPM and the MHPM. In Section 3, Numerical comparisons with the HPM show that the MHPM is a simple, yet powerful method to give the approximate solutions for the Chen fractional order system. Finally, conclusion is presented in section 4.

## 2. Basic definition and preliminary

We give some basic definitions and properties of fractional calculus which are used further in this paper.

**Definition 1.** A real function $h(t), t > 0$ is said to be in the space $C_\mu, \mu \in R$, if there exist a real number$p(> \mu)$, such that $h(t) = t^p h_1(t) \in [0, \infty)$, and it is said to be in the space $C_\mu^n$ $if\ and\ only\ if\ h^{(n)} \in C_\mu, n \in N.$

**Definition 2.** The Riemann-Liouville fractional integral operator $\left( {}_aJ_t^\alpha \right)$ of order $\alpha \geq 0$ of a function $h \in C_\mu, \mu \geq -1$ is defined as

$$ {}_aJ_t^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} h(\tau) d\tau, (\alpha > 0) $$

$$ {}_aJ_t^\alpha h(t) = h(t) $$

where $t \geq a \geq 0$, $\Gamma(.)$ is the well-known Gamma function. Some of the properties are given as follows: For $h \in C_\mu, \mu \geq -1, a, \alpha, \beta \geq 0, \gamma \geq -1$

$$ (i)\ {}_aJ_t^\alpha\ {}_aJ_t^\beta h(t) = {}_aJ_t^{\alpha+\beta} h(t) $$

$$ (ii)\ {}_aJ_t^\alpha\ {}_aJ_t^\beta h(t) = {}_aJ_t^\beta\ {}_aJ_t^\alpha h(t) $$

$$ (iii)\ {}_aJ_t^\alpha (t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (t-a)^{\alpha+\gamma} $$

**Definition 3.** The Caputo fractional derivative $\left( {}_aD_t^\alpha \right)$ $of\ h(t)$ is defined as

$$ {}_aD_t^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau) d\tau \quad (1) $$

$$ for\ n-1 < \alpha \leq n, n \in N, t \geq a \geq o\ and\ h \in C_{-1}^n $$

Hence, we have following properties:

$(i) If\ n-1 < \alpha \leq n, n \in N, and\ h \in C_\mu^n, \mu \geq -1, then\ {}_aD_t^\alpha\ {}_aJ_t^\alpha h(t) = h(t)$ and

$${}_aJ_t^{\alpha}\ {}_aD_t^{\alpha}h(t)=h(t)-\sum_{k=0}^{n-1}h^{(k)}(a)\frac{(t-a)^k}{k!} \qquad (2)$$

$(ii) let\ h(t)\in C_{-1}^{n}, n\in N. Then\ {}_aD_t^{\alpha}h, 0\le\alpha\le n\ is\ well\ defined\ and\ {}_aD_t^{\alpha}h\in C_{-1}$

**3. Analysis of the method**

In this section, we extend the application of the MHPM to the fractional order differential equations in the following form:

$$\begin{aligned}
{}_aD_t^{\alpha_1}y_1(t)&=f_1(t,y_1,y_2,y_3,\dots y_n),\\
{}_aD_t^{\alpha_2}y_2(t)&=f_2(t,y_1,y_2,y_3,\dots y_n)\\
{}_aD_t^{\alpha_3}y_3(t)&=f_3(t,y_1,y_2,y_3,\dots y_n)\\
&\vdots\\
{}_aD_t^{\alpha_n}y_n(t)&=f_n(t,y_1,y_2,y_3,\dots y_n)
\end{aligned} \qquad (3)$$

Subject to the following initial condition: $y_i(a)=c_i, i=1,2,3\dots n$

where $0<\alpha_i\le 1, t\ge a\ge 0, f_i$ is an arbitrary linear or nonlinear function.

**3.1. Solution by HPM**

The HPM is a universal one which can be applied to various kinds of linear and nonlinear equations. It usually needs only a few iterations to lead to the active approximate analytical solutions for a given system. In view of the HPM, we construct a homotopy for the Equations which satisfies the following relations:

$${}_aD_t^{\alpha_n}y_i(t)=pf_i(t,y_1,y_2,y_3,\dots y_n) \qquad (4)$$

where $i=1,2,3\dots n, p\in[0,1]$ is an embedding parameter.

When $p=0$, equation (4) becomes linear ${}_aD_t^{\alpha_n}y_i=0$, and when $p=1$, equation (4) turns out to be the original equation in equations. The basic assumption is that the solution of above equation can be expanded as power series in *p*.

$$y_i(t)=y_{io}+py_{i1}+p^2y_{i2}+p^3y_{i3}+\cdots \qquad (5)$$

And the initial conditions are taken as

$$y_{i0}(a)=y_i(a)=c_i, y_{ik}(a)=0, k=1,2,3\dots$$

where $y_{ij}(t), j=0,1,2,3\dots$ are the functions to be determined later. Substituting equation (5) into equation (4), collecting the terms of the same powers of *p*, we obtain

$P^0$: ${}_0D_t^{\alpha_i}y_{i0}=0$

$P^1$: ${}_0D_t^{\alpha_i}y_{i1}=f_{i1}(t,y_{10,}y_{20}y_{30},\dots.y_{n0})$

$P^2$: ${}_0D_t^{\alpha_i}y_{i2}=f_{i2}(t,y_{10,}y_{20}y_{30},\dots.y_{n0},y_{11,}y_{21},y_{31,}\dots y_{n1})$

$P^3$: ${}_0D_t^{\alpha_i}y_{i3}=f_{i3}(t,y_{10,}y_{20}y_{30},\dots.y_{n0},y_{11,}y_{21},y_{31,}\dots y_{n1},y_{12},y_{22},y_{32},\dots y_{n2})$

.where $f_{i1}, f_{i2}\dots$ satisfy the following equation

$f_i(t,y_{10}+py_{11}+p^2y_{12}+\cdots,y_{n0}+py_{n1},p^2y_{n2}+\cdots)$

$$= f_{i1}(t, y_{10,} y_{20} y_{30}, \dots . y_{n0}) + p f_{i2}(t, y_{10,} y_{20} y_{30}, \dots . y_{n0}, y_{11,} y_{21}, y_{31,} \dots y_{n1})$$
$$+ p^2 f_{i3}(t, t, y_{10,} y_{20} y_{30}, \dots . y_{n0}, y_{11,} y_{21}, y_{31,} \dots y_{n1}, y_{12}, y_{22}, y_{32}, \dots y_{n2}) + \cdots$$

Applying the integral operator ${}_aJ_t^{\alpha_i}(.)$ on both sides of the above fractional order equation in (7), which considering the initial condition (6), by using the properties of the Caputo fractional derivative, we can determine the unknown function $y_{ij}(t)$.
By setting $p = 1$ in (5), the HPM series solutions to equations (1)-(3) are given as

$$y_i(t) = \sum_{j=o}^{\infty} y_{ij}(t) \quad (6)$$

where $i = 1,2,3 \dots n$. The N-term approximation of the HPM series can be expressed as
$$y_i(t) \approx \emptyset_{iN}(t) = \sum_{j=o}^{N-1} y_{ij}(t)\,, i = 1,2,3 \dots n. \quad (7)$$

**3.2. Solutions by MHPM**

The approximate solutions generally, as shall be shown in the numerical experiments of this paper, not valid for large t. The HPM treated as an algorithm in a sequence of intervals for finding accurate approximate solution to the equations .The modified HPM, i.e. the MHPM, can give the valid solutions for a long time.

The time interval $[a, t)$ can be divided into a sequence of subintervals $[t_0, t_1), [t_1, t_2), [t_2, t_3) \dots [t_{j-1}, t_j)$, in which$t_0 = a, t_j = t$. Without loss of generality, the subintervals can be chosen as the same length $\Delta t$. i.e. .$\Delta t = t_l - t_{l-1}, (l = 1,2,3 \dots j)$. Furthermore, the equations in (7) can be solved by HPM in every sequential interval $[t_{l-1}, t_1), l = 1,2,3 \dots,$ , Choosing the initial approximations as

$$y_{io}(t^*) = y_i(t^*) = c_i^*, y_{ik}(t^*) = 0, i = 1,2,3 \dots, n, k = 1,2,3 \dots$$

where $t^*$is the left-end point of each sub interval, but in general, we only having the initial values at the point $t^* = t_0 = a$ . A simple way to obtain the other necessary values could be by means of the previous N-terms approximate solutions $\emptyset_{iN(t)}, i = 1,2,3 \dots n$ of the preceding subinterval $[t_{l-2}, t_{l-1}), (l = 2,3, \dots j), i.e$

$$y_{i0}^*(t^*) = \emptyset_{iN(t^*)}.$$

Finally, the unknown functions $y_{ij}(t), i = 1,2,3 \dots \dots n, j = 0,1,2,3 \dots \dots ..$ can be obtained by the fractional integral operator

$${}_{t^*}J_t^{\alpha_i} y_{ij}(t) = \frac{1}{\Gamma(\alpha_i)} \int_{t^*}^{t} (t-\tau)^{\alpha_i - 1} y_{ij}(\tau) d\tau$$

**4.1. MHPM usage in the fractional order Chen system**

The fractional order Chen system described as

$${}_0D_t^{\alpha_1} x(t) = \alpha(y - x)$$

$${}_0D_t^{\alpha_1} y(t) = (c - a)x - xz + cy$$

$${}_0D_t^{\alpha_1} z(t) = xy - bz$$

Subject to the initial conditions $x(0) = c_1, y(0) = c_2, z(0) = c_3$

where $0 < \propto_i < 1, i = 1,2,3, x, y, z$ are state positive parameters.

$$_0D_t^{\alpha_1} x(t) = P\big(\alpha(y - x)\big)$$

$$_0D_t^{\alpha_1} y(t) = P((c - a)x - xz + cy)$$

$$_0D_t^{\alpha_1} z(t) = P(xy - bz)$$

where $P \in [0,1]$ is an embedding parameter.

$$P^0\text{: } {_0D_t^{\alpha_1}} x_0 = 0$$

$$P^1\text{: } {_0D_t^{\alpha_1}} x_1 = ay_0 - ax_0$$

$$P^2\text{: } {_0D_t^{\alpha_1}} x_2 = ay_1 - ax_1$$

$$P^3\text{: } {_0D_t^{\alpha_1}} x_3 = ay_2 - ax_2$$

$$P^4\text{: } {_0D_t^{\alpha_1}} x_4 = ay_3 - ax_3$$

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$$P^0\text{: } {_0D_t^{\alpha_2}} y_0 = 0$$

$$P^1\text{: } {_0D_t^{\alpha_2}} y_1 = (c - a)x_0 - x_0 z_0 + cy_0$$

$$P^2\text{: } {_0D_t^{\alpha_2}} y_2 = (c - a)x_1 - x_0 z_1 - x_1 z_0 + cy_1$$

$$P^3\text{: } {_0D_t^{\alpha_2}} y_3 = (c - a)x_2 - x_0 z_2 - x_1 z_1 - x_2 z_2 + cy_2$$

$$P^4\text{: } {_0D_t^{\alpha_2}} y_4 = (c - a)x_3 - x_0 z_3 - x_1 z_2 - x_2 z_1 - x_3 z_0 + cy_3$$

$$P^5\text{: } {_0D_t^{\alpha_2}} y_5 = (c - a)x_4 - x_0 z_4 - x_1 z_3 - x_2 z_2 - x_3 z_1 - x_4 z_0 + cy_3$$

. . . .

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$$P^0\text{: } {_0D_t^{\alpha_3}} z_0 = 0$$

$$P^1\text{: } {_0D_t^{\alpha_3}} z_1 = x_0 y_0 - bz_0$$

$$P^1\text{: } {_0D_t^{\alpha_3}} z_1 = x_0 y_1 - x_1 y_0 - bz_1$$

$$P^2\text{: } {_0D_t^{\alpha_3}} z_2 = x_0 y_2 - x_1 y_1 - x_2 y_0 - bz_1$$

$$P^3\text{: } {_0D_t^{\alpha_3}} z_3 = x_0 y_3 - x_1 y_2 - x_2 y_1 - x_3 y_0 - bz_2$$

$$P^4\text{: } {_0D_t^{\alpha_3}} z_4 = x_0 y_4 - x_1 y_3 - x_2 y_2 - x_3 y_1 - x_4 y_0 - bz_3$$

$$P^5\text{: } {_0D_t^{\alpha_3}} z_5 = x_0 y_5 - x_1 y_4 - x_2 y_3 - x_3 y_2 - x_4 y_1 - x_5 y_0 - bz_4$$

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The initial Condition $x_0 = x_0(0) = c_1 = 0$

$$y_0 = y_0(0) = c_2 = 0$$

$$z_0 = z_0(0) = c_3 = 0, a = 35, b = 3, c = 28$$

Using the initial conditions we get the following results

$$x_1 = (ac_2 - ac_1).\frac{t^{\alpha_1}}{\Gamma(\alpha_1 + 1)}$$

$$y_1 = (cc_1 - ac_1 + cc_2 + c_1c_3).\frac{t^{\alpha_2}}{\Gamma(\alpha_2 + 1)}$$

$$z_1 = (c_1c_2 - bc_3)\frac{t^{\alpha_3}}{\Gamma(\alpha_3 + 1)}$$

$$x_2 = \left(acc_1 - a^2c_1 + acc_2 - ac_1c_3\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)}\right) - (a^2c_2 - a^2c_1)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1 + 1)}$$

$$y_2 = (acc_2 - acc_1 - a^2c_2 + a^2c_1 - ac_2c_3 + ac_1c_3)\frac{t^{\alpha_1+\alpha_2}}{\Gamma\alpha_1 + \alpha_2 + 1}$$

$$+(c^2c_1 - acc_1 + c^2c_2 - cc_1c_3)\frac{t^{2\alpha_2}}{\Gamma 2\alpha_2 + 1} - (c_1^2c_2 - bc_1c_3)\frac{t^{\alpha_2+\alpha_3}}{\Gamma\alpha_2 + \alpha_3 + 1}$$

$$z_2 = (cc_1^2 - ac_1^2 + cc_1c_2 - c_1^2c_3)\frac{t^{\alpha_2+\alpha_3}}{\Gamma\alpha_2 + \alpha_3 + 1} + (ac_2^2 - ac_1c_2)\frac{t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1 + \alpha_3 + 1)}$$

$$- (bc_1c_2)\frac{t^{2\alpha_3}}{\Gamma(2\alpha_3 + 1)}$$

**5.1.2. Numerical solution of fractional order Chen system**

The approximate series solutions of system are expressed given bellow

$$x(t) = -10 + 352.926\, t^{0.98} + 1633.703\, t^{1.96} - 46144.635\, t^{2.94} + 59962.710\, t^{3.92} + 4346980.875\, t^{4.90} + \cdots$$

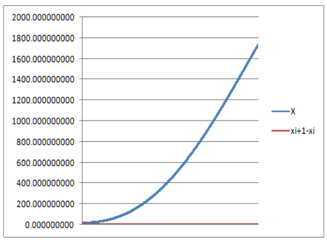
$$y(t) = 443.679\, t^{0.98} - 2173.084\, t^{1.96} - 39592.798\, t^{2.94} + 650719.273\, t^{3.92} - 1741346.262\, t^{4.90} + \cdots$$

$$z(t) = 37 - 111.928\, t^{0.98} - 2109.292\, t^{1.96} + 63948.645\, t^{2.94} + 42356.398\, t^{3.92} - 9383166.519\, t^{4.90} + \cdots$$

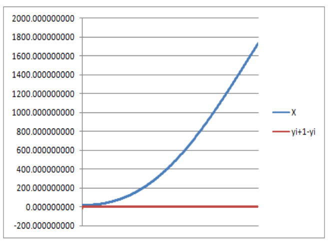
By taking values for $t \in (0.0001, 0.5)$ and using equations, the sample values obtained out of nearly 5000 values are tabulated for reference. In the power series analysis difference in the values of ‘x, y and z’ are of the system variable is calculated as reference and are plotted as shown below

| Sl. No. | X | Y | Z | $x_{i+1}$-$x_i$ | $y_{i+1}$-$y_i$ | $z_{i+1}$-$z_i$ |
|---|---|---|---|---|---|---|
| 1 | 15.804211022 | -17.52948430 | 35.55132056 | 0.0021954238 | -0.0239830667 | -0.0432005619 |
| 2 | 15.892334871 | -18.12227334 | 34.43662147 | 0.0044349490 | -0.0202958407 | -0.0400730446 |
| 3 | 15.901289022 | -18.16276187 | 34.35653955 | 0.0046035051 | -0.0200903177 | -0.0399456849 |
| 4 | 16.003233463 | -18.52771167 | 33.60752949 | 0.0062158216 | -0.0182617154 | -0.0388756244 |
| 5 | 17.014645465 | -19.91864004 | 30.04571304 | 0.0148469950 | -0.0107259293 | -0.0347045938 |
| 6 | 18.011449279 | -20.42896913 | 28.12907800 | 0.0202676853 | -0.0072164402 | -0.0325138541 |
| 7 | 19.021126681 | -20.70022775 | 26.70401241 | 0.0247333409 | -0.0048427567 | -0.0307778900 |
| 8 | 20.004206891 | -20.84804101 | 25.59150515 | 0.0285268096 | -0.0031372059 | -0.0293095826 |
| 9 | 21.000951407 | -20.92965511 | 24.64590190 | 0.0320014580 | -0.0017963415 | -0.0279491352 |
| 10 | 22.007689978 | -20.96761412 | 23.82600872 | 0.0352337529 | -0.0007208981 | -0.0266577440 |
| 11 | 23.035662668 | -20.97560904 | 23.09649073 | 0.0383127457 | 0.0001621605 | -0.0253957575 |
| 12 | 24.026972750 | -20.96306636 | 22.47564676 | 0.0411118047 | 0.0008538569 | -0.0242156457 |
| 13 | 25.001247235 | -20.93715678 | 21.93108122 | 0.0437279519 | 0.0014107867 | -0.0230801712 |
| 14 | 26.036120329 | -20.89926623 | 21.41319953 | 0.0463828418 | 0.0018925634 | -0.0218922685 |
| 15 | 27.034678746 | -20.85566493 | 20.96471862 | 0.0488402372 | 0.0022675532 | -0.0207579948 |

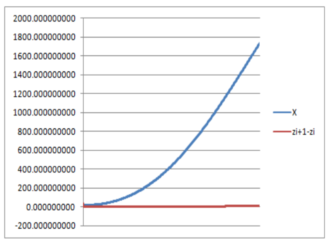
### 5.1.3. Graphical representation of fractional order Chen system



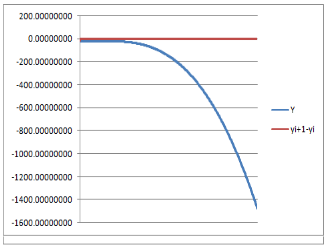

$X$ with $X_{i+1} - X_i$



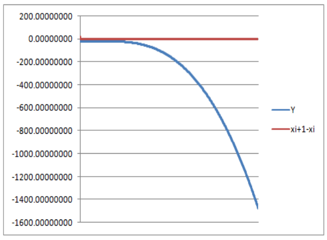

$X$ with $Y_{i+1}Y_i$



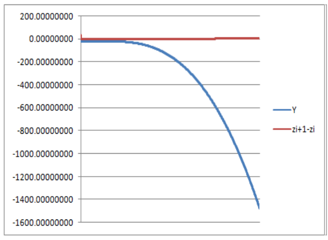

$X$ with $Z_{i+1} - Z_i$



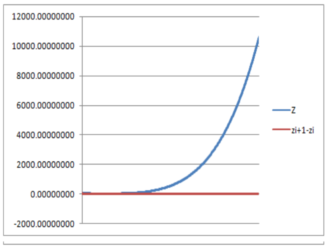
$Y\ with\ X_{i+1} - X_i$



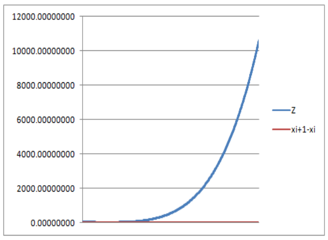
$Y\ with\ Y_{i+1}Y_i$



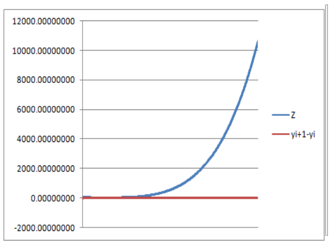
$Y\ with\ Z_{i+1} - Z_i$



$Z\ with\ X_{i+1} - X_i$



$Z\ with\ Y_{i+1} - Y_i$



$Z\ with\ Z_{i+1} - Z_i$

## 6. Conclusion

The behavior of the Chen system of fractional order is analyzed with the various possibilities of system variables x, y, z along there mutually perpendicular directions using Multi stage Homotophy Perturbation method, we conclude that the graphs corresponding to the solution of Hybrid dynamical system growth is in only one direction and not in the other directions.

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